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Equazioni a derivate parziali. — *Fourth-order nonlinear elliptic equations with critical growth* (*). Nota di DAVID E. EDMUNDS, DONATO FORTUNATO e ENRICO JANNELLI (**), presentata (***) dal Corrisp. A. AMBROSETTI.

ABSTRACT. — In this paper we consider a nonlinear elliptic equation with critical growth for the operator Δ^2 in a bounded domain $\Omega \subset \mathbb{R}^n$. We state some existence results when $n \geqslant 8$. Moreover, we consider $5 \leqslant n \leqslant 7$, expecially when Ω is a ball in $\mathbb{R}^n$.

KEY WORDS: Biharmonic operator; Critical exponent; Sobolev embeddings.

RIASSUNTO. — *Equazioni ellittiche non lineari del quarto ordine a crescita critica.* In questa nota si studia un'equazione ellittica non lineare a crescita critica per l'operatore Δ^2 in un aperto limitato $\Omega \subset \mathbb{R}^n$. Vengono enunciati alcuni teoremi di esistenza di soluzioni non banali per questa equazione quando $n \geqslant 8$. Si considerano, inoltre, le dimensioni $5 \leqslant n \leqslant 7$, con particolare riguardo al caso in cui Ω è una sfera di $\mathbb{R}^n$.

1. INTRODUCTION AND STATEMENTS OF THE RESULTS

In this paper we are concerned with the problem

$$(1) \qquad \begin{cases} \Delta^2 u - u|u|^{8/(n-4)} - \lambda u = 0 & \text{in } \Omega, \\ u = \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n \geqslant 5$.

We search for non-trivial solutions of problem (1), which, after suitable stretching, are the critical points, with positive critical values, of the functional

$$(2) \qquad F(u) = \frac{1}{2}\int_\Omega |\Delta u|^2\,dx - \frac{\lambda}{2}\int_\Omega u^2\,dx$$

on the manifold

$$V = \left\{ u \in H_0^2(\Omega) \,\middle|\, \int_\Omega |u|^{2n/(n-4)}\,dx = 1 \right\}.$$

We shall confine ourselves to the case $\lambda < \lambda_1$, where λ_1 is the first eigenvalue of the problem

$$(3) \qquad \begin{cases} \Delta^2 u - \lambda u = 0 & \text{in } \Omega, \\ u = \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega; \end{cases}$$

therefore we indeed are searching for the minima of $F(u)$ on V.

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This analysis is complicated by lack of compactness due to the fact that H^2 is not compactly embedded in $L^{2n/(n-4)}$; therefore, the manifold V is not weakly closed in the H^2 topology. We shall carry out an analysis of problem (1) closely related to the methods developed by Brezis-Nirenberg in studying nonlinear second order elliptic problems whose nonlinear terms have critical growth (see [1]). We remark that works on problems like (1), involving powers of the Laplacian, were initiated by P. Pucci and J. Serrin (see [7, 8]), who formulated various results and conjectures.

The steps of our analysis will be the following ones:

I) The quantity

$$K = \min_{\substack{u \in H^2_{\mathrm{loc}}(\mathbb{R}^n) \\ \int_{\mathbb{R}^n} |u|^{2n/(n-4)}\,dx = 1}} \int_{\mathbb{R}^n} |\Delta u|^2\,dx \tag{4}$$

exists and the minimum is obtained *only* when

$$u(x) = \frac{[(n-4)(n-2)\,n(n+2)\,\varepsilon^2]^{(n-4)/8}}{(\varepsilon + |x-x_0|^2)^{(n-4)/2}} \tag{5}$$

for any $x_0 \in \mathbb{R}^n$ and any $\varepsilon > 0$. Moreover, for any *bounded domain* Ω in $\mathbb{R}^n$,

$$\inf_{u \in V} \int_{\Omega} |\Delta u|^2\,dx = K \tag{6}$$

and the infimum is *not* attained.

II) $F|_V(u)$ satisfies the Palais-Smale (P-S) condition in $]-\infty, K[$.

III) $\inf_{u \in V} F(u) < K$, if $0 < \lambda < \lambda_1$ and $n \geq 8$.

Obviously I), II) and III) imply the following

THEOREM 1. *If* $0 < \lambda < \lambda_1$ *and* $n \geq 8$, *problem* (1) *has a nontrivial solution.*

REMARK 1. The bound on the dimension n (≥ 8) in Theorem 1 seems to be optimal: indeed, Pucci and Serrin have shown (see [8]) that, if Ω is a ball and $5 \leq n \leq 7$, problem (1) has no non-trivial *radial* solutions for $\lambda \in [0, \mu]$, where μ is sufficiently close to 0 (however, for the case $5 \leq n \leq 7$ see also Theorem 2 and Theorem 3 in the present note).

REMARK 2. Theorem 1 may be extended with slight modifications to the case of all $\lambda > 0$: of course, when $\lambda > \lambda_1$ we search for critical points of F which are no longer minima (for analogous results in the case of the Laplace operator, see [2]).

REMARK 3. If Ω is starshaped, Pucci and Serrin have shown (see [7]) that problem (1) has no non-trivial solutions for $\lambda < 0$. The case $\lambda = 0$ is completely open.

When $5 \leq n \leq 7$, the following results hold:

THEOREM 2. *Let* $5 \leq n \leq 7$; *let* ψ *be an eigenfunction corresponding to the eigenvalue* λ_1 *of (3) such that*

$$\int_{\Omega} |\psi|^{2n/(n-4)}\,dx = 1\,.$$

Let

$$\lambda_* = \lambda_1 - \frac{2K}{\int_\Omega \psi^2\,dx}. \tag{7}$$

where K is defined in (4) (*obviously* $0<\lambda_*<\lambda_1$). *Then problem* (1) *has a non-trivial solution for* $\lambda_*<\lambda<\lambda_1$.

Theorem 2 follows by I), II) and from the fact that III) holds, independently from the dimension n, if $\lambda_*<\lambda<\lambda_1$.

When Ω is a ball in $\mathbb{R}^n$ and $5\leqslant n\leqslant 7$, Theorem 2 may be improved. In order to explain this situation, we define two auxiliary functions $\delta_n(t)$, $\eta_n(t)$ as follows:

$$\delta_n(t) = \det\begin{pmatrix} I_{n/2-1}(t) & J_{n/2-1}(t) \\ I_{n/2-2}(t) & J_{n/2-2}(t) \end{pmatrix} \tag{8}$$

$$\eta_n(t) = \begin{cases} \det\begin{pmatrix} I_{n/2-1}(t) - J_{n/2-1}(t) & I_{1-n/2}(t) - J_{1-n/2}(t) \\ I_{n/2-2}(t) - J_{n/2-2}(t) & I_{2-n/2}(t) - J_{2-n/2}(t) \end{pmatrix} & \text{for } n=5,7, \\ \det\begin{pmatrix} I_{n/2-1}(t) - J_{n/2-1}(t) & K_{n/2-1}(t) + \frac{1}{2}Y_{n/2-1}(t) \\ I_{n/2-2}(t) - J_{n/2-2}(t) & K_{n/2-2}(t) - \frac{1}{2}Y_{n/2-2}(t) \end{pmatrix} & \text{for } n=6, \end{cases} \tag{9}$$

where J_λ, I_λ, Y_λ, K_λ are Bessel functions (for their definition see the Appendix).

Let α, β the first positive roots of δ_n, η_n respectively (of course α, β depend on the dimension n). It is possible to prove that α, β indeed exist and that $\beta<\alpha$.

Now we are in the position to state the following.

THEOREM 3. *Let* $5\leqslant n\leqslant 7$, $B_n(0,R)$ *the ball (centred at the origin) of* $\mathbb{R}^n$ *of radius R and* λ_1 *the first eigenvalue of* (3) *on* $B_n(0,R)$. *Then problem* (1) *has a non-trivial radial solution for* $\lambda_*<\lambda<\lambda_1$, *where*

$$\lambda_* = \left(\frac{\beta}{R}\right)^4; \qquad \lambda_1 = \left(\frac{\alpha}{R}\right)^4. \tag{10}$$

REMARK 4. In order to prove that $\lambda_1=(\alpha/R)^4$ when $\Omega=B_n(0,R)$ one can reason just as in [6, Note F]. We remark that obviously the λ_* given by Theorem 3 is strictly less than the λ_* given by Theorem 2 when $\Omega=B_n(0,R)$.

Easy numerical calculations give the following estimations on λ_* in Theorem 3:

$$\frac{\lambda_*}{\lambda_1} = \begin{cases} 0.484\ldots & \text{for } n=5; \\ 0.220\ldots & \text{for } n=6; \\ 0.077\ldots & \text{for } n=7. \end{cases} \tag{11}$$

REMARK 5. Brezis and Nirenberg proved in [1] that the problem

$$\begin{cases} -\Delta u - u^5 - \lambda u = 0 & \text{in } B_3(0,R), \\ u=0 & \text{on } \partial B_3(0,R), \end{cases} \tag{12}$$

has a non-trivial radial solution *if and only if* $\lambda_* < \lambda < \lambda_1$, where

$$\lambda_* = \left(\frac{\beta}{R}\right)^2; \quad \lambda_1 = \left(\frac{\alpha}{R}\right)^2 \tag{13}$$

and $\alpha = \pi$ is the first positive root of the Bessel function $J_{1/2}(t) = \sqrt{2/\pi t}\sin(t)$, while $\beta = \pi/2$ is the first positive root of the Bessel function $J_{-1/2}(t) = \sqrt{2/\pi t}\cos(t)$. The analogy between (10) and (13) is transparent, and motivates the conjecture following which Theorem 3 should be indeed optimal, *i.e.* there should be no radial solutions for problem (1) under the hypotheses of Theorem 3 when $\lambda \leqslant \lambda_*$ (the already quoted results of Pucci and Serrin (see [8]) are a first step in this direction).

2. Sketch of the proof.

In this section we give a short sketch of the proof of Theorem 1; for the complete proof of all our assertions we refer to [4].

I) The existence of the minimum in (4) essentially follows from the arguments given by P. L. Lions in [5], who proves that the minimizing functions are (up to translations) radially symmetric and decreasing with their derivatives up to order two.

Now, one easily checks that the functions $u(x)$ in (5) satisfy the Euler equation regarding the minimization problem (4), and it is possible to prove that they are the only solutions of this equation which are (up to translations) radially symmetric and decreasing to zero at infinity.

Finally, rescaling arguments show that the infimum in (6) does not depend on Ω; this infimum is equal to the constant K in (4): in fact, it cannot be smaller than K (otherwise, for any Ω there exists $u \in H_0^2(\Omega)$ such that $I(u) = \int_\Omega |\Delta u|^2 dx < K$ and $|u|_{L^{2n/(n-4)}(\Omega)} = 1$; the extension of u by zero outside Ω leads to a contradiction), nor greater than K (in fact there exists a sequence $\{u_n\} \subset C_0^\infty(\mathbb{R}^n)$, $|u_n|_{L^{2n/(n-4)}} = 1$, such that $I(u_n) \to K$; see P. L. Lions [5]). This infimum is not achieved (on the contrary, let u, Ω such that $u \in H_0^2(\Omega$, $|u|_{L^{2n/(n-4)}(\Omega)} = 1$ and $I(u) = K$; extending u to zero outside Ω leads to a minimum of I on $\mathbb{R}^n$, but this minimizing function is not of the type (5)).

II) The proof is based on arguments analogous to those used by Brezis—Nirenberg [1] and Cerami-Fortunato-Struwe [3].

III) Let $\phi \in H_0^2(\Omega)$ and $n \geqslant 8$. Let

$$u_\varepsilon(x) = \frac{\phi(x)}{(\varepsilon + |x|^2)^{(n-4)/2}}. \tag{14}$$

Then one can prove the following asymptotic estimations (for $\varepsilon \to 0$):

$$\begin{cases} \|\Delta u_\varepsilon\|_{L^2(\Omega)} = C_1\,\varepsilon^{(4-n)/2} + O(1), \qquad |u_\varepsilon|^2_{L^{2n/(n-4)}(\Omega)} = C_2\,\varepsilon^{(4-n)/2} + O(1), \\ |u_\varepsilon|^2_{L^2(\Omega)} = \begin{cases} C_3\,\varepsilon^{(8-n)/2} + O(1) & \text{if } n \geqslant 9, \\ C_3 \log|\varepsilon| + O(1) & \text{if } n = 8, \end{cases} \end{cases} \tag{15}$$

where C_1, C_2, C_3 are positive constants and $C_1/C_2 = K$.

From (15) the conclusion of III) easily follows.

APPENDIX

In this appendix we shall briefly recall the definitions of those Bessel functions which have a role in our Theorem 3.

For any λ such that $-\lambda \notin \mathbb{N}$ the Bessel function J_λ is defined as

$$J_\lambda(z) = \sum_{h=0}^{\infty} (-1)^h \frac{(z/2)^{\lambda+2h}}{h!\,\Gamma(\lambda+h+1)} \tag{16}$$

while the modified Bessel function I_λ is defined as

$$I_\lambda(z) = \sum_{h=0}^{\infty} \frac{(z/2)^{\lambda+2h}}{h!\,\Gamma(\lambda+h+1)}. \tag{17}$$

Moreover, we use the Hankel function of second type Y_n which, for any $n \in \mathbb{N}$, is defined as

$$\begin{aligned} Y_n(z) = {} & 2(\gamma + \log(z/2))\,J_n(z) - \sum_{h=0}^{n-1} \frac{(n-h-1)!}{h!}\left(\frac{z}{2}\right)^{-n+2h} - \\ & - \sum_{h=0}^{\infty} (-1)^h \frac{(z/2)^{n+2h}}{h!(n+h)!}\left(1 + \frac{1}{2} + \dots + \frac{1}{h} + 1 + \frac{1}{2} + \dots + \frac{1}{n+h}\right) \end{aligned} \tag{18}$$

and the Macdonald function of second type K_n which, for any $n \in \mathbb{N}$, is defined as

$$\begin{aligned} K_n(z) = {} & (-1)^{n+1}(\gamma + \log(z/2))\,I_n(z) + \frac{1}{2}\sum_{h=0}^{n-1} (-1)^h \frac{(n-h-1)!}{h!}\left(\frac{z}{2}\right)^{-n+2h} + \\ & + \frac{1}{2}(-1)^n \sum_{h=0}^{\infty} \frac{(z/2)^{n+2h}}{h!(n+h)!}\left(1 + \frac{1}{2} + \dots + \frac{1}{h} + 1 + \frac{1}{2} + \dots + \frac{1}{n+h}\right) \end{aligned} \tag{19}$$

where $\gamma = 0.577215\ldots$ is the Euler constant.

We recall that, when λ is half of an odd integer, the functions J_λ and I_λ can be expressed in terms of elementary functions; therefore the functions $\delta_n(t)$ and $\eta_n(t)$ in (8) and (9) can also be expressed in terms of elementary functions when $n = 5$ or $n = 7$.

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