Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Irena Lasiecka, Roberto Triggiani

Sharp regularity theory for second order hyperbolic equations of Neumann type

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **83** (1989), n.1, p. 109–113.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1989_8_83_1_109_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1989.

Atti Acc. Lincei Rend. fis.
(8), LXXXIII (1989), pp. 109-113

Equazioni a derivate parziali. — *Sharp regularity theory for second order hyperbolic equations of Neumann type* (*). Nota (**) di IRENA LASIECKA e ROBERTO TRIGGIANI, presentata dal Corrisp. R. CONTI.

ABSTRACT. — This note provides sharp regularity results for general, time-independent, second order hyperbolic equations with non-homogeneous data of Neumann type.

KEY WORDS: Hyperbolic partial differential equations.

RIASSUNTO. — *Regolarità delle soluzioni di equazioni differenziali iperboliche del secondo ordine con dati al contorno di tipo Newmann.* Si danno risultati di regolarità delle soluzioni del problema misto per equazioni a derivate parziali del secondo ordine di tipo iperbolico, con dato non omogeneo sulla frontiera di tipo Neumann.

1. REGULARITY PROBLEM, PRELIMINARIES, AND STATEMENT OF MAIN RESULTS

Let $x > 0$ be a scalar positive variable, t be a real variable, and $y = [y_1, \dots, y_{n-1}]$ be an $(n-1)$-dimensional vector with real components. In symbols: $x \in R^1_{x^+}$; $t \in R^1_t$; $y \in R^{n-1}_y$. Let

$$\Omega \equiv R^1_{x^+} \times R^{n-1}_y, \qquad \Gamma \equiv R^{n-1}_y = \Omega|_{x=0} \tag{1.1}$$

be, respectively, an n-dimensional half-space Ω with boundary Γ. On Ω we consider the second order differential operator

$$P(x, y; D_t, D_x, D_y) \equiv -aD_t^2 + \sum_{i,j=1}^{n-1} a_{ij} D_{y_i} D_{y_j} + 2 \sum_{j=1}^{n-1} a_{nj} D_{y_j} D_x + D_x^2 \tag{1.2}$$

with space-dependent, but time-independent coefficients

$$a \equiv a(x, y), \qquad a_{ij} \equiv a_{ij}(x, y), \qquad i = 1, \dots, n; \ [x, y] \in \Omega, \ j = 1, \dots, n-1 \tag{1.3}$$

satisfying the symmetricity conditions $a_{ij} = a_{ji}$, $i, j = 1, \dots, n-1$. Here and throughout we use the notation

$$D_t \equiv \frac{1}{\sqrt{-1}} \frac{\partial}{\partial t}; \quad D_x \equiv \frac{1}{\sqrt{-1}} \frac{\partial}{\partial x}; \quad D_{y_j} \equiv \frac{1}{\sqrt{-1}} \frac{\partial}{\partial y_j} \quad \text{etc.}$$

On Γ, the boundary of the half-space Ω, we consider the first order operator

$$B(y; D_x, D_y) \equiv D_x + \sum_{j=1}^{n-1} b_j D_{y_j} \qquad \text{on } x = 0 \tag{1.4}$$

with space-dependent, but time-independent coefficients

$$b_j \equiv b_j(y), \qquad y \in \Gamma. \tag{1.5}$$

(*) Pervevuta all'Accademia l'11 agosto 1988.
(**) Il contenuto di questa Nota è stato presentato in una serie di seminari alla Scuola Normale Superiore, Pisa, nel luglio 1985; e, inoltre, all'IFIP Conference, Santiago di Campostela, Spagna, nel luglio 1987.

The present paper investigates *regularity properties* of the solution $u(t,x,y)$ of the following second order hyperbolic mixed problem wich Neumann boundary conditions

$$(1.6)\qquad \begin{cases} (a)\quad P(x,y;D_t,D_x,D_y)\,u=f(t,x,y) & \text{on } \Omega,\ t>0, \\ (b)\quad B(y;D_x,D_y)\,u=g(t,y) & \text{on } \Gamma,\ t>0, \\ (c)\quad u|_{t=0}=u_0;\ D_t u|_{t=0}=u_1 & \text{on } \Omega,\ t=0, \end{cases}$$

at least for a few specific fundamental function spaces for f and g. Other classes of function spaces will be examined in a subsequent paper [13]. Generally, we are interested in the continuity of the map from the data (u_0,u_1,f,g) in preassigned function spaces (possibly, subject to compatibility conditions) into the solution u, $u_t, \ldots$ and possibly its trace $u|_\Gamma, \ldots$ in suitable (optimal) function spaces. Throughout the paper, problem (1.6) will be subject to the following *assumptions*:

(i) the coefficients a, a_{ij}, a_{nj} of P and b_j of B are assumed real, time independent, sufficiently smooth in the space variables, and constant outside a compact set $\mathcal{K}_{xy}$ of $R^1_{x^+}\times R^{n-1}_y=\Omega$;

(ii) the boundary Γ $(x=0)$ is non-characteristic for P and P is «regularly hyperbolic with respect to t»; i.e. the characteristic polynomial of P,

$$(1.7a)\qquad p(x,y;\tau,\xi,\eta)\equiv -a\tau^2+\sum_{i,j=1}^{n-1}a_{ij}\eta_i\eta_j+2\xi\sum_{j=1}^{n-1}a_{nj}\eta_j+\xi^2$$

$$(1.7b)\qquad \equiv -a\tau^2+\left(\xi+\sum_{j=1}^{n-1}a_{nj}\eta_j\right)^2+\sum_{i,j=1}^{n-1}a_{ij}\eta_i\eta_j-\left(\sum_{j=1}^{n-1}a_{nj}\eta_j\right)^2$$

has *two real* and *distinct roots* in τ, for $(x,y)\in\Omega$ and (ξ,η) on the unit sphere $\xi^2+|\eta|^2=1$, where $|\eta|^2+\sum_{j=1}^{n-1}\eta_j^2$. If we consider $\eta=0$ and $\xi=1$, this requirement yields the condition

$$(1.8)\qquad \min a(x,y)>0 \qquad \text{in } \Omega;$$

moreover, if we consider the points of the unit sphere in (ξ,η) which lie also on the hyperplane $\xi+\sum_{j=1}^{n-1}a_{nj}\eta_j=0$, this requirement yields the necessary condition, which is plainly also sufficient, that the quadratic form in η

$$(1.9)\qquad d(x,y;\eta)\equiv a^2(x,y)\left\{\sum_{i,j=1}^{n-1}a_{ij}(x,y)\,\eta_i\eta_j-\left(\sum_{j=1}^{n-1}a_{nj}(x,y)\,\eta_j\right)^2\right\}$$

(independent of ξ) be positive definite

$$(1.10)\qquad d(x,y;\eta)>0 \qquad (x,y)\in\Omega,\ |\eta|^2\neq 0;$$

(iii) the first order operator $\widetilde{D}_x$ defined by

$$(1.11)\qquad \widetilde{D}_x=D_x+\sum_{j=1}^{n-1}a_{nj}(x,y)\,D_{y_j}$$

restricted on the boundary Γ, coincides with B; i.e.

$$B \equiv \widetilde{D}_x|_{x=0}; \quad \text{i.e.} \quad b_j(y) \equiv a_{nj}(0, y), \qquad j = 1, \dots, n-1. \tag{1.12}$$

The following results are known and provide the a-priori regularity needed in the subsequent development.

LEMMA 1.I. Let $u_0 = u_1 = 0$ in (1.6*c*) and let $0 < T < \infty$.

a) Let $g \equiv 0$ and $f \in L_1(0, T; L_2(\Omega))$ in (1.6). Then

$$\begin{matrix} u \in C([0, T]; H^1(\Omega)) \\ u_t \in C([0, T]; L_2(\Omega)) \end{matrix} \qquad (\text{a fortiori } u \in H^1([0, T] \times \Omega))$$

continuously.

b) Let $f \equiv 0$ and $g \in L_2(0, T; L_2(\Omega))$ in (1.6). Then [1]

$$\begin{matrix} u \in C([0, T]; H^{1/2}(\Omega)) \\ u_t \in C([0, T]; H^{-1/2}(\Omega)) \end{matrix} \qquad (\text{a fortiori } u \in H^{1/2}([0, T] \times \Omega)) \quad \square$$

Trace theory aspplied to Lemma 1.1*a*) then gives

$$\left.\begin{matrix} f \in L_1(0, T; L_2(\Omega)) \\ g = 0 \\ u_0 = u_1 = 0 \end{matrix}\right\} \rightarrow u|_\Sigma \in C([0, T]; H^{1/2}(\Gamma)) \text{ continuously}. \tag{1.13}$$

A main goal of the present paper is to show the following

MAIN THEOREM 1.2. Let $g = 0$ and $u_0 = u_1 = 0$ and let $f \in L_2(Q_+)$, $Q_+ = R^1_{t+} \times \Omega$. Then,

a) If $\Sigma_+ = R^1_{t+} \times \Gamma$, the trace $u|_\Sigma$ of the solution to (1.6) satisfies $u|_\Sigma \subset H^{3/5}(\Sigma_+)$ continuously: there is a constant $\mathcal{C} > 0$ independent of f such that

$$\|u|_\Sigma\|_{H^{3/5}(\Sigma_+)} \leq \mathcal{C} \|f\|_{L_2(Q_+)}. \tag{1.14}$$

b) In the special cases where the coefficients a_{ij}, $i, j = 1, \dots, n-1$; a_{nj}, $j = 1, \dots, n-1$ either do *not* depend on x, or else do *not* depend on y, then $u|_\Sigma \in H^{2/3}(\Sigma_+)$ continuously: there is a constant $\mathcal{C} > 0$ independent of f such that

$$\|u|_\Sigma\|_{H^{2/3}(\Sigma_+)} \leq \mathcal{C} \|f\|_{L_2(Q_+)}. \tag{1.15}$$

REMARKS 1.1. (i) The general case (1.14) represents an improvement by «1/10» $(1/2 + 1/10 = 3/5)$ in the space regularity of the trace over (1.13).

(ii) Addition of a *first* order differential operator to P does not affect the results. $\square$

A second main result of this paper is the following

[1] Lions-Magenes, vol. II, p. 120 provide only $L_2(0, T; \cdot)$; but this can be improved to $C([0, T]; \cdot)$ with the same space regularity, as e.g. in [3], [11].

MAIN THEOREM 1.3. Let $f = 0$, $u_0 = u_1 = 0$, and $g \in L_2(\Sigma_+)$. Then, continuously, for any $\varepsilon > 0$:

$$a) \begin{cases} v \in H^{3/5-\varepsilon}(Q_+) \left(\text{improvement by almost } «\frac{1}{10}» \text{ over Lemma } 1.1b\right) & (1.16) \\ AND \\ u|_\Sigma \in H^{1/5-\varepsilon}(\Sigma_+)\,. & (1.17) \end{cases}$$

b) In the special case where the coefficients a_{ij}, a_{nj}, $i, j = 1, \ldots, n-1$ *either do not depend on x*, or *else do not depend on y*, then

$$\begin{cases} u \in H^{2/3}(Q_+) & (1.18) \\ AND \\ u|_\Sigma \in H^{1/3}(\Sigma_+) \quad \square & (1.19) \end{cases}$$

REMARKS 1.2. (i) For $\dim \Omega \geq 2$ and the Laplacian case, one can show that $u \notin H^{3/4+\varepsilon}(Q)$, $\forall \varepsilon > 0$ [5], [12].

(ii) Result (1.17 is a *regularity result*. Trace theory applied to interior regularity (1.16) gives only $H^{3/5-1/2=1/10}(\Sigma)$, a result worse than (1.17) by «1/10». Similarly, trace theory applied to (1.18) gives $H^{2/3-1/2=1/6}(\Sigma)$, a result worse than (1.19) by «1/6».

iii) The regularity in (1-18)-(1.19) coincides with that proved *directly*, by eigenfunction expansions, for the Laplacian Δ on a *sphere* $= \Omega$ [3].

iv) Direct computations, by eigenfunction expansions, with the Laplacian on a parallelepiped Ω produce $u \in H^{3/4-\varepsilon}(Q)$, $\varepsilon > 0$ and $u|_\Sigma \subset H^{1/2-\varepsilon}(\Sigma_+)$ [3].

The proofs of Theorems 1.2 and 1.3 are very lengthy and technical and are to be found in [12].

REFERENCES

[1] LIONS J. L., private communication, May 1984.

[2] LIONS J. L. and MAGENES E., 1972. *Nonhomogeneous boundary value problems and applications.* Vols. I, II, Springer-Verlag, Berlin-Heidelberg-New York.

[3] LASIECKA I. and TRIGGIANI R., 1981. *A cosine operator approach to modelling $L_2(0,T; L_2(\Gamma))$-boundary input hyperbolic equations.* Applied Mathem. & Optimiz., 7: 35-93.

[4] LASIECKA I. and TRIGGIANI R., 1983. *Regularity of hyperbolic equations under $L_2(0,T;\ L_2(\Gamma))$-boundary terms.* Applied Mathem. & Optimiz., 10: 275-286.

[5] LASIECKA I. and TRIGGIANI R., 1989. *Trace regularity of the solutions of the wave equations with homogeneous Neumann boundary conditions.* J.M.A.A., 141: 49-71.

[6] LASIECKA I., LIONS J. L. and TRIGGIANI R., 1986. *Nonhomogeneous boundary value problems for second order hyperbolic operators.* J. de Mathématiques Pures et Appliquées, 65: 149-192.

[7] MIYATAKE S., 1973. *Mixed problems for hyperbolic equations of second order.* J. Math. Kyoto University, 13: 435-487.

[8] SAKAMOTO R., 1970. *Mixed problems for hyperbolic equations. I, II.* J. Math. Kyoto University, 10-2: 343-373 and 10-3: 403-417.

[9] SYMES W. W., 1983. *A trace theorem for solutions of the wave equation and the remote determination of acoustic sources.* Mathematical Methods in the Applied Sciences, 5: 131-152.

[10] TRIGGIANI R., 1978. *A cosine operator approach to modeling $L_2(0, T; L_2(\Gamma))$-boundary input problems for hyperbolic systems.* In «Proceedings 8th IFIP Conference on Optimization Techniques, University of Würzburg, West Germany 1977», Springer-Verlag, Lecture Notes CIS M6: 380-390.

[11] LASIECKA I. and TRIGGIANI R., 1988. *A lifting theorem for the time regularity of solutions to abstract equations with unbounded operators and applications to hyperbolic equations.* Proceedings Americ. Mathem. Soc., 104: 745-755.

[12] LASIECKA I. and TRIGGIANI R., *Sharp regularity theory for second order hyperbolic equations of Neumann type.* Part I: L_2 non-homodeneous data. Annali di Matematica Pura e Applicata, to appear.

[13] LASIECKA I. and TRIGGIANI R., 1989. *Regularity theory of hyperbolic equations with non-homogeneous Neumann boundary conditions.* Part II: *General boundary data*. J. Differ. Eqts., to appear.