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Meccanica dei solidi e delle strutture. — *Extremum theorems for finite-step backward-difference analysis of elastic-plastic nonlinearly hardening solids.* Nota di Giulio Maier e Giorgio Novati (*), presentata (**) dal Corrisp. G. Maier.

Dedicated to the unfading memory of Professor Michele Capurso.

Abstract. — For the finite-step, backward-difference analysis of elastic-plastic solids in small strains, a kinematic (potential energy) and a static (complementary energy) extremum property of the step solution are given under the following hypotheses: each yield function is the sum of an equivalent stress and a yield limit; the former is a positively homogeneous function of order one of stresses, the latter a nonlinear function of nondecreasing internal variables; suitable conditions of «material stability» are assumed. This communication anticipates results to be presented elsewhere in an extended version. Therefore, proofs of the statements and various comments are omitted.

Key words: Computational Plasticity; Extremum theorems; Hardening.

Riassunto. — *Teoremi di minimo per l'analisi a passi finiti olonomi di solidi elastoplastici ad incrudimento nonlineare.* Per l'analisi evolutiva a passi-finiti di sistemi elastoplastici in regime di piccole deformazioni, una proprietà estremale cinematica ed una statica vengono dimostrate in base ai seguenti assunti sulle leggi costitutive: le funzioni di snervamento sono somme di funzioni omogenee del primo ordine nelle tensioni e di limiti di snervamento; questi sono funzioni nonlineari di variabili interne non decrescenti e danno luogo a funzioni energia soggette ad opportune condizioni di convessità. Questa comunicazione presenta risultati da pubblicare in altra sede in forma estesa: qui si omettono le dimostrazioni dei risultati e vari commenti.

1. Introduction

Approximate evolutive analysis by finite steps is currently the object of a considerable amout of research work in computational elastoplasticity (see e.g. [1-3]). The so-called «backward difference» time-integration of the (nonlinear, differential) relationships governing incremental elastoplasticity is attracting special attention for its favourable properties of convergence and stability [2, 4]. Mechanically interpreted, this procedure basically amounts to: *a*) replace within each finite step the actual incremental constitutive law or «material model» by a suitably generated «holonomic» counterpart; *b*) update between two successive steps all history-dependent variables, so that the irreversible nature of plasticity be allowed for.

Backward-difference elastoplastic analysis in the above sense was frequently performed mostly by quadratic programming, in the context of plasticity theory with piecewiselinearized yield surface and linear hardening under the labels of «step-wise holonomic» or «multistage» analysis (e.g. in [5-7]; see survey in [8]).

Two finite-step estremum properties earlier established in [9-11] as a foundation

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for piecewiselinearized «engineering plasticity» theory, are generalized here to a class of material models with nonlinear yield functions and nonlinear hardening.

2. Formulation of the finite-step elastoplastic boundary value problem

In the usual (Cartesian tensor) description, the elastic-plastic response of the solid considered to a given history of external actions (body forces $b_i(t)$, imposed (e.g. thermal) strains $\theta_{ij}(t)$, tractions $T_i(t)$ and imposed displacements $U_i(t)$, all functions of space coordinates x_h), are governed by the following relation set:

(1a, b) $$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad \text{in } V, \qquad u_i = U_i(t) \quad \text{on} \quad S_u,$$

(2a, b) $$\sigma_{ij,i} + b_j(t) = 0 \quad \text{in } V, \qquad \sigma_{ij} n_i = T_j(t) \quad \text{on} \quad S_T,$$

(3) $$\sigma_{ij} = E_{ijrs}(\varepsilon_{rs} - \varepsilon_{rs}^p - \theta_{rs}(t)),$$

(4) $$\phi_\alpha = \frac{\partial \Phi_\alpha}{\partial \sigma_{ij}}(\sigma_{rs})\sigma_{ij} - Y_\alpha(\lambda_\beta) \leqq 0 \quad (\alpha = 1,\dots, y),$$

(5, a, b, c) $$\dot{\varepsilon}_{ij}^p = \frac{\partial \Psi_\alpha}{\partial \sigma_{ij}}(\sigma_{rs})\dot{\lambda}_\alpha, \qquad \dot{\lambda}_\alpha \geqq 0, \qquad \phi_\alpha \dot{\lambda}_\alpha = 0 \quad (\alpha = 1,\dots, y).$$

Here index summation convention holds; V denotes the region occupied by the solid; S its boundary with n_j unit outward normal (S_u with kinematic essential, S_T with static natural boundary conditions). Equation (3) is Hooke's law with the usual positive definiteness and symmetries in the elastic tensor, ε_{ij}^p being the plastic strain tensor. The yield criterion (4) encompasses y yield modes (for instance: $y = 1$ in Mises material model with isotropic hardening; in Martin-Resende model $y = 3$, namely Drucker-Prager, «cap» and «cut-off» modes [2]).

In the expression for the yield function ϕ_α the first addend Φ_α is the «equivalent stress» assumed to be positively homogeneous of order one and, hence, expressed according to Euler's theorem; the latter is the yield limit Y_α, which depends on the current values of the «internal variables» $\lambda_\alpha(t)$, $(\alpha = 1,\dots,, y)$, not on their past history. These nondecreasing variables λ_α can be interpreted as measures, at phenomenological level, of the irreversible rearrangements occuring at microscale level inside a material. The fact that all the above equations are homogeneous in the time derivatives reflects the «inviscid», time-independent nature of this kind of constitutions. The flow rule (5a) is linear in the sense that plastic strain rates are proportional to a tensor indipendent of rates. This tensor is the gradient of plastic potential Ψ_α, coincident with Φ_α in the associative case (normality rule), which will be considered henceforth.

Consider the functional of past evolution of the internal variables up to the current time t $(0 \leqq \tau \leqq t)$:

(6) $$\Pi(t, \lambda_\alpha(\tau)) \equiv \int_0^t Y_\alpha(\tau)\dot{\lambda}_\alpha(\tau)\, d\tau,$$

(7) $$\Pi_c(t, \lambda_\alpha(\tau)) \equiv Y_\alpha(t)\lambda_\alpha(t) - \Pi(t, \lambda_\alpha(\tau)).$$

We note that, for $\Phi_\alpha = \Psi_\alpha (\alpha = 1,..., y)$, the first and second variation of Π are, respectively:

$$\delta^{(1)} \Pi = \sigma_{ij}(t)\, \delta \varepsilon_{ij}^{p}(t) \tag{8}$$

$$\delta^{(2)} \Pi = \delta\lambda_\alpha \frac{\partial Y_\alpha}{\partial \lambda_\beta} \delta\lambda_\beta = \delta\sigma_{ij}\, \delta\varepsilon_{ij}^{p}\,. \tag{9}$$

These relations give to $\Pi(t)$ the clear mechanical meaning of plastic work. They easily flow from relations (5) and from the following consequence of (4):

$$\dot{\phi}_\alpha = \frac{\partial \Phi_\alpha}{\partial \sigma_{ij}} \dot{\sigma}_{ij} - \dot{Y}_\alpha (\lambda_\beta, \dot{\lambda}_\beta) \leqq O, \qquad \text{when } \phi_\alpha = 0 \tag{10}$$

Functional $\Pi(t, \lambda_\alpha(\tau))$ becomes a (path-independent) function $\Pi(\lambda_\alpha)$ if and only if:

$$\frac{\partial Y_\alpha}{\partial \lambda_\beta} = \frac{\partial Y_\beta}{\partial \lambda_\alpha} \tag{11}$$

and convex if and only if:

$$\lambda_\alpha \frac{\partial Y_\alpha}{\partial \lambda_\beta} \lambda_\beta \geqq 0, \qquad \text{for any } \lambda_1,..., \lambda_y\,. \tag{12}$$

Mechanically, condition (11) means «reciprocal» hardening. In view of eq. (9), condition (12), if combined with normality, means meterial stability in Drucker's sense and rules out softening.

Assume now that all the variables are known at the instant $\bar{t}$ of the (ordinative) «time»; at a later instant $t = \bar{t} + \Delta t$ the external actions ($b_i = \bar{b}_i + \Delta b_i$ etc.) are given and the response variables ($\sigma_{ij} = \bar{\sigma}_{ij} + \Delta\sigma_{ij}$ etc.) are sought.

Let us assume the following set of relationships as the «holonomic» substitute, over the interval Δt, of the original path-dependent plastic constitution (4) (5):

$$\phi_\alpha = \frac{\partial \Phi_\alpha}{\partial \sigma_{ij}} (\bar{\sigma}_{rs} + \Delta\sigma_{rs})(\bar{\sigma}_{ij} + \Delta\sigma_{ij}) - Y_\alpha (\bar{\lambda}_\beta + \Delta\lambda_\beta) \leqq 0 \tag{13a, b}$$

$$\Delta\varepsilon_{ij}^{p} = \frac{\partial \Phi_\alpha}{\partial \sigma_{ij}} (\bar{\sigma}_{rs} + \Delta\sigma_{rs}) \Delta\lambda_\alpha\,, \qquad \Delta\lambda_\alpha \geqq 0, \; \phi_\alpha \Delta\lambda_\alpha = 0 \tag{14, a, b, c}$$

The above step-wise path-independent constitution (13) (14) combined with the linear equations (1) (2) (3), written for the given finite increments of domain $(\Delta b_i, \Delta\theta_{ij})$ and boundary $(\Delta U_i, \Delta T_i)$ data, govern the finite-step b.v. problem formulated by the backward-difference, step-wise holonomic approach. This is the problem concerned by the statements given below.

3. KINEMATIC AND STATIC THEOREM FOR FINITE-STEP, BACKWARD-DIFFERENCE ANALYSIS

PROPOSITION 1. If in the material model the equivalent stresses Φ_α are order-one positively homogeneous convex functions of the stresses and the plastic work Π is a convex function of the internal variables λ_α (not a functional of their past history), then the solution of the backward-difference (step-holonomic) elastoplastic b.v. problem over $\bar{t} \leqq \tau \leqq \bar{t} + \Delta t = t$, coincides with the solution to the optimatization problem:

$$(15)\qquad \min_{\Delta u_i, \Delta\lambda_\alpha, \Delta\sigma_{ij}} \left\{ \Omega \equiv \int_V \frac{1}{2} \Delta\varepsilon_{ij}^e E_{ijhk} \Delta\varepsilon_{hk}^e \, dV + \int_V \Pi(\bar{\lambda}_\alpha + \Delta\lambda_\alpha)\, dV + \right.$$

$$\left. + \int_V \bar{\sigma}_{ij} \Delta\varepsilon_{ij}^e \, dV - \int_V (\bar{b}_i + \Delta\bar{b}_i)\Delta u_i \, dV - \int_{S_u} (\bar{T}_i + \Delta\bar{T}_i)\Delta u_i \, dS \right\},$$

subject to:

$$(16a,b)\qquad \Delta\varepsilon_{ij}^e = \frac{1}{2}(\Delta u_{i,j} + \Delta u_{j,i}) - \Delta\varepsilon_{ij}^p - \Delta\theta_{ij} \ \text{ in } V, \qquad \Delta u_i = \Delta U_i \quad \text{on } S_u$$

$$(17a,b)\qquad \Delta\varepsilon_{ij}^p = \frac{\partial\Phi_\alpha}{\partial\sigma_{ij}}(\bar{\sigma}_{ij} + \Delta\sigma_{ij})\Delta\lambda_\alpha, \ \Delta\lambda_\alpha \geqq 0 \quad (\alpha = 1,\ldots,y) \ \text{ in } V.$$

PROPOSITION 2. If in the material model the equivalent stresses Φ_α are convex order-one positively homogeneous functions of stresses, the yield limits Y_α are concave functions of the internal variables λ_β and the plastic work Π and the complementary plastic work Π_c are strictly convex and convex functions of them, respectively, then the solution to the elastoplastic backward-difference (step-holonomic) b.v. problem over $\bar{t} \leqq \tau \leqq \bar{t} + \Delta t = t$, coincides with the solution of the following convex optimization problem (C_{ijhk} denoting the elastic compliance tensor):

$$(18)\qquad \min_{\Delta\sigma_{ij}, \Delta\lambda_\alpha} \left\{ \Omega_c \equiv \int_V \frac{1}{2} \Delta\sigma_{ij} C_{ijhk} \Delta\sigma_{hk} \, dV + \int_V \Pi_c(\bar{\lambda}_\alpha + \Delta\lambda_\alpha)\, dV + \right.$$

$$\left. - \int_V Y_\alpha(\bar{\lambda}_\beta + \Delta\lambda_\beta)\bar{\lambda}_\alpha \, dV + \int_V \Delta\sigma_{ij}\Delta\theta_{ij} \, dV - \int_{S_u} \Delta\sigma_{ij} n_i \Delta U_j \, dS \right\},$$

subject to:

$$(19a,b)\qquad \Delta\sigma_{ij,i} + \Delta b_j = 0 \quad \text{in} \quad V, \qquad \Delta\sigma_{ij} n_i = \Delta T_j \quad \text{on} \quad S_T$$

$$(20)\qquad \phi_\alpha = \Phi_\alpha\{\bar{\sigma}_{ij} + \Delta\sigma_{ij}) - Y_\alpha(\bar{\lambda}_\beta + \Delta\lambda_\beta) \leqq 0 \qquad (\alpha, \beta = 1,\ldots,y).$$

4. CONCLUSIONS

The following links with earlier results are worth noting.

For perfectly plastic solids (constant Y_α) and $\bar{t} = 0$ ($\Delta t = t$: single step), Prop. 2 reduces to Haar-Kàrmàn theorem in «deformation theory» of plasticity. For linear yield functions and linear hardening, and for $\bar{t} = 0$ ($\Delta t = t$) Prop. 1 and Prop. 2 can be identified with those established by Maier in [9] for trusses and in [10] for continua. For $\Delta t = \delta t \to 0$, the two statements become the rate theorems due to Capurso [13] and Capurso-Maier [14], respectively. For no hardening (perfectly plastic) constitutions, Prop. 1 and 2 reduce to the statements derived by Franchi-Genna for discrete models [15].

The results of this paper fit in the more general theoretical framework established by Ponter-Martin on the basis of the optimal path concept [16, 17]. The present discussion and conclusions concern a fairly large but specific category of constitutions, adopt direct assumptions on constitutive dependences (rather than on the dissipation-function) and are intended to clarify the links with nonlinear constrained optimizations, as earlier results did with quadratic optimization under linear constraints in the restricted context of piecewise linearized structural plasticity.

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