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Meccanica. — *On Lagrangian systems with some coordinates as controls.* Nota di FRANCO RAMPAZZO, presentata (*) dal Corrisp. A. BRESSAN.

ABSTRACT. — Let Σ be a constrained mechanical system locally referred to state coordinates $(q^1, \ldots, q^N, \gamma^1, \ldots, \gamma^M)$. Let $(\tilde{\gamma}^1 \ldots \tilde{\gamma}^M)(\cdot)$ be an assigned trajectory for the coordinates γ^α and let $u(\cdot)$ be a scalar function of the time, to be thought as a control. In [4] one considers the control system $\Sigma_{\hat{\gamma}}$, which is parametrized by the coordinates $(q^1, \ldots, q^N)$ and is obtained from Σ by adding the time-dependent, holonomic constraints $\gamma^\alpha = \hat{\gamma}^\alpha(t) := \tilde{\gamma}^\alpha(u(t))$. More generally, one can consider a vector-valued control $u(\cdot) = (u^1, \ldots, u^M)(\cdot)$ which is directly identified with $\hat{\gamma}(\cdot) = (\hat{\gamma}^1, \ldots, \hat{\gamma}^M)(\cdot)$. If one denotes the momenta conjugate to the coordinates q^i by p_i, $i = 1, \ldots, N$, it is physically interesting to examine the continuity properties of the input-output map $\phi: u(\cdot) \to (q^i, p_i)(\cdot)$ associated with the dynamical equations of $\Sigma_{\hat{\gamma}}$, with respect to e.g. the C^0 topologies on the spaces of the controls $u(\cdot)$ and of the solutions $(q^i, p_i)(\cdot)$. Furthermore, in the theory of hyperimpulsive motions (see [4]), even discontinuous control are implemented. Then it is crucial to investigate the continuity of ϕ also with respect to topologies that are weaker than the C^0 one. In order that the input-output map ϕ exhibits such continuity properties, the right-hand sides of the dynamical equation for $\Sigma_{\hat{\gamma}}$ have to be affine in the derivatives $d\hat{\gamma}^1/dt, \ldots, d\hat{\gamma}^M/dt$. If this is the case, the system of coordinates (q^i, γ^α) is said to be M-*fit* (*for linearity*). In this note we show that, in the case of forces which depend linearly on the velocity of Σ, the coordinate system (q^i, γ^α) is M-fit if and only if certain coefficients in the expression of the kinetic energy are independent of the q^i. Moreover, if the forces are positional, for each 1-fit coordinate system (q'^i, γ') there exists a reparametrization (q^i, γ) such that $\partial\gamma/\partial q'^i = 0$ holds for every $i = 1, \ldots, N$ and the coordinates (q^i, γ) are locally geodesic.

KEY WORDS: Lagrangian systems; Impulsive controls; Kinetic metric.

RIASSUNTO. — *Sui sistemi lagrangiani in cui alcune coordinate fungono da controllo.* Sia Σ un sistema meccanico vincolato, riferito a coordinate $(q^1, \ldots, q^N, \gamma^1, \ldots, \gamma^M)$. Siano $(\tilde{\gamma}^1 \ldots \tilde{\gamma}^M)(\cdot)$ $\alpha = 1, \ldots, M$ delle preassegnate traiettorie per le coordinate γ^α e sia $u(\cdot)$ una funzione scalare del tempo, da assumersi come controllo. In [4] si considera il sottosistema $\Sigma_{\hat{\gamma}}$, parametrizzato dalle coordinate $(q^1, \ldots, q^N)$ e ottenuto da Σ mediante l'aggiunta di alcuni vincoli lisci espressi cinematicamente da $\gamma^\alpha = \hat{\gamma}^\alpha(t) := \tilde{\gamma}^\alpha(u(t))$, $\alpha = 1, \ldots, M$. Più in generale si può pensare ad un controllo vettoriale $u(\cdot) = (u^1, \ldots, u^M)(\cdot)$ direttamente identificato con $\hat{\gamma}(\cdot) = (\hat{\gamma}^1, \ldots, \hat{\gamma}^M)(\cdot)$. Denotati con p_i, $i = 1, \ldots, M$, i momenti coniugati alle coordinate q^i, è fisicamente importante stabilire quando il funzionale ingresso-uscita $\phi: u(\cdot) \to (q^i, p_i)(\cdot)$ associato alle equazioni dinamiche di $\Sigma_{\hat{\gamma}}$, sia continuo, per esempio rispetto alla topologia della convergenza uniforme sullo spazio dei controlli $u(\cdot)$ e delle soluzioni $(q^i, p_i)(\cdot)$. Inoltre, nella teoria del moto iperimpulsivo (v. [4]), si considerano controlli $u(\cdot)$ discontinui. Risulta perciò cruciale l'analisi della continuità del funzionale ϕ rispetto a topologie più deboli di quella della convergenza uniforme. Sulla base di alcuni recenti lavori su sistemi differenziali con controllo impulsivo risulta che, in ipotesi di equilimitatezza per la variazione totale dei controlli, la mappa ϕ presenta i suddetti caratteri di continuità se e solo se i secondi membri delle equazioni dinamiche per $\Sigma_{\hat{\gamma}}$ sono affini nelle derivate $d\hat{\gamma}^1/dt, \ldots, d\hat{\gamma}^M/dt$. Ciò avviene solo per una appropriata scelta del sistema di coordinate (q^i, γ^α), che in tal caso viene detto M-adatto. In questa nota si dimostra in particolare che, nel caso di forze dipendenti dalla velocità al più linearmente, il sistema di coordinate (q^i, γ^α) è M-adatto se e solo se certi coefficienti nell'espressione dell'energia cinetica non dipendono dalle q^i. Inoltre, data una parametrizzazione (q'^i, γ') 1-adatta, nell'ipotesi di forze posizionali viene provata l'esistenza di una riparametrizzazione (q^i, γ) che soddisfa $\partial\gamma/\partial q'^i = 0$ identicamente per ogni $i = 1, \ldots, N$ e tale che le (q^i, γ) sono coordinate localmente geodetiche.

(*) Nella seduta del 22 giugno 1988.

1. Introduction

Recently, some results ([8], [6], [2]) have been achieved on the continuity of the input-output map $\Phi : u \to x(u, \cdot)$ associated with a Cauchy problem of the form

$$\begin{cases} \dot{x}(t) = f(x(t), u(t), t) + g(x(t), u(t), t) \cdot \dot{u}(t) \\ x(0) = \bar{x}, \end{cases} \tag{1.1}$$

where x belongs to an open subset Ω of $\mathbb{R}^n$, f and g are regular vector fields from $\Omega \times \times \mathbb{R} \times \mathbb{R}$ into $\mathbb{R}^n$, and as usual, the dot denotes differentiation with respect to time. In particular, a definition of solution for (1.1) is given in [8] for the case of a continuous control u, by means of approximating solutions $x(u_n, \cdot)$ corresponding to a sequence $(u_n)_{n \in N}$ of controls which approximate u in the C^0 norm. In [2] this procedure is extended to integrable controls.

Relying on the analytical framework supplied by the above results, Bressan [4] developed a theory where, starting from an arbitrary Lagrangian system Σ referred to local coordinates (q^i, γ^α) $(i = 1, \ldots, N,\ \alpha = 1, \ldots, M)$, a control is considered in the following way. Let $T(t, q, dq/dt, \gamma, d\gamma/dt)$ denote the kinetic energy of Σ and let $p_i = \partial T / \partial \dot{q}^i$ be the first N conjugate momenta. Given M functions $\tilde{\gamma}^\alpha :]S_1, S_2[\to \mathbb{R}$ and a scalar control $u : [0, T] \to]S_1, S_2[$, let the evolutions of the coordinates γ^α be predesignated by $\hat{\gamma}^\alpha(\cdot) = \tilde{\gamma}^\alpha(u(\cdot))$, and let $\Sigma_{\hat{\gamma}}$ be the subsystem — referred to the coordinates q^i — which arises from Σ by the addition to some frictonless constraints represented by $\gamma = \hat{\gamma}(\cdot)$. By setting $x = (q^i, p_i)$, one can transform any (second order) Cauchy problem for Σ into a first order Cauchy problem of the form

$$\begin{cases} \dot{x} = F(t, x, u(t), \dot{u}(t)) \\ x(0) = \bar{x} \end{cases} \tag{1.2}$$

In order to check physically the correctness of the theory, it is important that the solution $x(u, \cdot) = \phi(u)(\cdot)$ of (1.2), regarded as a function of $u = u(\cdot)$, are continuous e.g. with respect to the C^0 norms, at least when

a) one restricts the domain of ϕ to a family of controls with equibounded total variations.

Indeed, such continuity of ϕ means that (sufficiently) small changes in the values of $u(\cdot)$ have the effect of producing (arbitrarily preassigned) small changes in the values of the corresponding $x(\cdot) = (q^i, p_i)(\cdot)$, uniformly in time. Let us notice that, because of the presence of du/dt on the right hand side of (1.2), this behaviour of ϕ is anything but obvious. In fact, the above continuity (specified in a certain way) occurs if — see [2] [8] — and only if — see [7] — (1.2) reduces to the special form (1.1), i.e. if and only if the right-hand side of (1.2) is affine in du/dt.

Moreover, under the hypothesis *a*), [3] and [7] allow one to state the same thing for the case of vector-valued controls $(u' \ldots, u^M)(\cdot)$. In other words, under the main assumption of equibounded total variation for the controls, the input-output map

$\phi : (u^1, \ldots, u^M) \to ((u^1, \ldots, u^M), \cdot)$ associated to a Cauchy problem of the form

$$\begin{cases} \dot{x} = G(t, x, (u^1, \ldots, u^M), (\dot{u}^1, \ldots, \dot{u}^M)) \\ x(0) = \bar{x} \end{cases} \tag{1.3}$$

turns out to be continuous with respect to C^0 norms if and only if G is affine in $(\dot{u}^1, \ldots, \dot{u}^M)$, i.e. if and only if (1.3) reduces to

$$\begin{cases} \dot{x} = f(t, x, (u^1, \ldots, u^M)) + \sum_{\alpha=1}^{M} g_\alpha (t, x, (u^1, \ldots, u^M))\, \dot{u}^\alpha \\ x(0) = \bar{x}. \end{cases} \tag{1.4}$$

Then, under hypothesis *a*) it is physically reasonable to use a vector-valued control $u(\cdot) = (u^1, \ldots, u^M)(\cdot)$ directly identified with $\hat{\gamma}(\cdot) = (\hat{\gamma}^1, \ldots, \hat{\gamma}^M)$ (see Section 2).

The above considerations lead to the problem investigated in Sections 3 and 4 of the present paper: the characterization of those parametrizations (q^i, γ^α) for which, under suitable hypotheses on the forces, the right-hand sides of the differential equations governing $x = (q^i, p_i)$ depend linearly on the derivatives $d\gamma^1/dt, \ldots, d\gamma^M dt$. These coordinates are called M-*fit* (for linearity) in [4]. When the $d\gamma^1/dt, \ldots, d\gamma^M/dt$ do not appear at all in the equations governing the (q^i, p_i), the coordinates (q^i, γ^α) are called *strongly* M-*fit* (for linearity).

Bressan [4] characterizes both M- and strongly M-fit parametrizations, in terms of algebraic conditions on the zero-, first-, and second-order derivatives of certain kinematical and dynamical quantities. Here, in Section 3, another characterization is provided, which is of the same type, but in various situations is easier to handle. In particular, it is essential in order to derive the results in Section 4 concerning scalar controls. This case is particular, since it turns out that each Lagrangian system has infinitely many sets of 1-fit coordinates, e.g. the locally geodesic ones, which in fact are strongly M-fit. Actually, the main result (Theorem 4.1) of this work shows that each 1-fit system of coordinates (q'^1, γ') may be transformed into a locally geodesic system (q^i, γ) simply by means of a diffeomorphisms satisfying the identity $\partial\gamma/\partial q'^i = 0$, for every $i = 1, \ldots, N$.

2. LAGRANGIAN COORDINATES AS CONTROLS

In this section some basic concepts are recalled from Bressan's theory of *controllizable* coordinates for Lagrangian systems [4].

Let us consider a mechanical systems Σ with $D(< +\infty)$ degree of freedom, subjectyed to frictionless bilateral constrains (1). The motion of Σ on the constraint manifold $\boldsymbol{M}$ is governed by the Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial \boldsymbol{T}}{\partial \dot{\chi}^R}\right) - \frac{\partial \boldsymbol{T}}{\partial \chi^R} = Q_R, \qquad R = 1, \ldots, D, \tag{2.1}$$

where $\chi = (\chi^1, \ldots, \chi^D)$ denote a set of local coordinates defined on some open subset

(1) Since the topics to be discussed have a differential geometric character, throughout this paper we tacitly assume that the constraints and the functions under consideration have a suitable degree of smoothness, so that all the required differentiations can be actually performed.

$W \subseteq \boldsymbol{M}$, $\boldsymbol{T} = \boldsymbol{T}(t, \chi, d\chi/dt)$ denotes the kinetic energy, and $Q_R = Q_R(t, \chi, d\chi/dt)$ denotes the R-th Lagrangian component of the applied force. It is well-known that $\boldsymbol{T} = \boldsymbol{T}(t, \chi, d\chi/dt)$ has the form

$$(2.2) \qquad \boldsymbol{T} = \boldsymbol{T}(t, \chi, \dot{\chi}) = \frac{1}{2} \sum_{R,S=1}^{D} A_{RS}(t, \chi)\, \dot{\chi}^R \dot{\chi}^S + \sum_{R=1}^{D} A_R(t, \chi)\, \dot{\chi}^R + \frac{1}{2} A_0(t, \chi),$$

where, for all values of t and χ, the matrix $A_{RS}(t, \chi)$ is symmetric and positive definite.

Let us define the momenta π_R and the Hamiltonian $\boldsymbol{H}$ corresponding to the Lagrangian function $\boldsymbol{T}$ (i.e. the Legendre transform of $\boldsymbol{T}$) by

$$(2.3) \qquad \pi^R := \frac{\partial \boldsymbol{T}}{\partial \dot{\chi}^R}, \qquad R = 1, \ldots, D, \quad \text{and } \boldsymbol{H} := \sum_{R=1}^{D} \pi_R \dot{\chi}^R - \boldsymbol{T}.$$

By inverting (2.3) one can consider $\boldsymbol{H}$ and the Q_R as functions of t, χ, and π: then (2.1) is equivalent to the differential system, in «semi-Hamiltonian» form,

$$(2.4) \qquad \begin{cases} \dot{\chi}^R = \dfrac{\partial \boldsymbol{H}}{\partial \pi_R} \\ \dot{\pi}_R = -\dfrac{\partial \boldsymbol{H}}{\partial \chi^R} + Q_R \end{cases} \qquad R = 1, \ldots, D.$$

For $N < D$, let us set $(\chi^R) = (q^i, \gamma^\alpha)$ and $(\pi_R) = (p_i, \mathrm{p}_\alpha)$, with $R = 1, \ldots, D$, $i = 1, \ldots, N$, $\alpha = 1, \ldots, M$, $N + M = D$ (2). Moreover let $U \subseteq \mathbb{R}^N$ and $V \subseteq \mathbb{R}^M$ be open subsets such that $U \times V$ is contained in the range of the coordinate chart $(W, (q^i, \gamma^\alpha))$.

At this point,

A) Bressan [4] considers the problem of predesignating the evolutions

$$(2.5) \qquad \hat{\gamma}^\alpha : [T_1, T_2] \to V, \qquad \alpha = 1, \ldots, M,$$

for the last M coordinates γ^α, by means of the laws $\hat{\gamma}^\alpha(\cdot) := \tilde{\gamma}^\alpha(u(\cdot))$, where the $\tilde{\gamma}^\alpha :]S_1, S_2[\to \mathbb{R}$ are fixed trajectories and $u : [0, T] \to]S_1, S_2[$ is a scalar function, assumed as control.

More generally — see also the Introduction — on the basis of [3],

B) one can consider a vector-valued control $(u^1, \ldots, u^M)(\cdot)$, to be directly identified with $(\hat{\gamma}^1, \ldots, \hat{\gamma}^M)(\cdot)$.

From the mechanical point of view, **A**) [or **B**)] is equivalent to the addition of forces whose Lagrangian components Q_R are given, a posteriori, by

$$(2.6) \qquad \mathrm{Q}_R(t) := \dot{\pi}_R + \frac{\partial H}{\partial \chi^R}(t, q(t), \hat{\gamma}(t), p(t), \mathrm{p}(t)) - Q_R(t, q(t), \hat{\gamma}(t), p(t), \mathrm{p}(t)), \qquad R = 1, \ldots, D.$$

It is clear that this is not sufficient to determine the evolutions of the q^i, unless further conditions are given on the additional forces which give rise to the $\hat{\gamma}^\alpha(\cdot)$. In fact, Bressan requires that

$$(2.7) \qquad \mathrm{Q}_h(t) = 0, \qquad h = 1, \ldots, N,$$

(2) Throughout this paper lower case Latin indexes run from 1 to N, lower case Greek indexes run from 1 to M, upper case Latin indexes run from 1 to D, and $N + M = D$.

hold, for any given $\hat{\gamma}(\cdot) = (\hat{\gamma}^1, \dots, \hat{\gamma}^M)(\cdot)$. Under the hypotheses (2.7), in the case **A**) [resp. **B**)], the evolution of $x = (q^i, p_i)$ is governed by a system of the form (1.2) [resp. (1.3)], $(n = 2N)$, as one can easily check by using $(2.3)_1$ to express to p_α as functions of t, q, p, γ and $d\gamma/dt$.

More precisely, the evolutions of the (q^i, p_i) satisfy the control-system

$$\begin{cases} \dot{q} = \dfrac{\partial \hat{H}}{\partial p_h} \\ \dot{p}_h = -\dfrac{\partial \hat{H}}{\partial q^h} + \hat{Q}_h \end{cases} \tag{2.8}$$

where we have set

$$\begin{aligned} &\hat{H}(t, q, p, \hat{\gamma}(t), \dot{\hat{\gamma}}(t)) = \boldsymbol{H}(t, q, \hat{\gamma}(t), p, \mathrm{p}(t, q, \hat{\gamma}(t), p, \dot{\hat{\gamma}}(t))) , \\ &\hat{Q}_h(t, q, p, \hat{\gamma}(t), \dot{\hat{\gamma}}(t)) = Q_h(t, q, \hat{\gamma}(t), p, \mathrm{p}(t, q, \hat{\gamma}(t), p, \dot{\hat{\gamma}}(t))) , \text{ and} \\ &\hat{\gamma}^\alpha(\cdot) := \tilde{\gamma}^\alpha(u(\cdot)) \quad \text{in the case } \mathbf{A}) , \\ &\hat{\gamma}^\alpha(\cdot) = u^\alpha(\cdot) \quad \text{in the case } \mathbf{B}) . \end{aligned} \tag{2.9}$$

In other words, the evolution problem for $x = (q^i, p_i)$, with $\hat{\gamma}(\cdot)$ as a control, has a deterministic character as soon as one imposes that above additional forces act as frictionless bilateral contraints with respect to the subsystem $\Sigma_{\hat{\gamma}}$ parametrized by the q^i. It easy to verify that, for each control $\hat{\gamma}(\cdot)$, such additional forces exist. Some examples of this are given in [4] and [5].

3. COORDINATES CHARTS FIT TO HYPERIMPULSES

Yet, the conditions (2.7) are not sufficient in order that the derivatives $d\hat{\gamma}^\alpha/dt$ appear linearly in (2.8), i.e. these conditions do not guarantee that in case **A**) [resp. **B**)] (2.8) has the form (1.1) [resp. (1.4)]. On the other hand (see the Introduction), under the assumption of equibounded total variations for the controls, the systems of the form (1.4) are the most general among the systems of type (1.2) for which the input-output map $\boldsymbol{\Phi}: u \to x(u, \cdot)$ is continuous with respect to e.g. the C^0 topologies. Furthermore, in case **A**) the linearity of (1.2) with respect to du/dt is shown in [7], to hold if and only if *hyperimpulsive motions* — where both positions and velocities suffer first order discontinuities, possibly to solve some optimal control problems (see [4]) — can be treated in a certain satisfactory way.

The above considerations motivate the following definition.

DEFINITION 3.1 (see [4]). A coordinate chart $(W, (q^i, \gamma^\alpha))$ on $\boldsymbol{M}$, $i = 1, \dots, N$, $\alpha = 1, \dots, M$, will be called M-*fit* (*for linearity*) if, for every control $\hat{\gamma}(\cdot) = (\hat{\gamma}^1, \dots, \hat{\gamma}^M)(\cdot)$, the derivatives $d\hat{\gamma}^\alpha/dt$ appear linearly in the system (2.8). $(W, (q^i, \gamma^\alpha))$ will be called *strongly* M-*fit* (*for linearity*) if for each control $(\hat{\gamma}^1, \dots, \hat{\gamma}^\alpha)(\cdot)$ the derivatives $d\hat{\gamma}^\alpha/dt$ do not appear in (2.8).

In [4], Theorems 10.1-2, Bressan gives algebraic conditions on the quantities A_{RS}, A_S used in (2.2) (and on 1st and 2nd derivatives of them and Q_R), which characterize M-fit or strongly M-fit coordinate charts. In the following theorem another necessary and sufficient condition will be proved, which turns out to be easier to handle in various situations. Moreover this condition is essential to derive the results in Section 4, concerning the special case $M = 1$.

THEOREM 3.1. *Let $(W, (\chi^R) \equiv (q^i, \gamma^\alpha))$ be a coordinate chart for the constraint manifold $\boldsymbol{M}$, and let $A^{-1} = [A^{RS}(t,\chi)]$ denote the inverse of the matrix $A = [A_{RS}(t,\chi)]$ used in the expression of the kinetic energy. Moreover let $C = [C_{\alpha\beta}(t,\chi)]$ be the inverse matrix of $[A^{N+\alpha,N+\beta}(t,\chi)]_{\alpha,\beta=1,\dots,M}$. In addition, let the maps $d\gamma/dt \vdash Q_h(t,q,p,\gamma,\mathrm{p}(t,q,p,\gamma,d\gamma/dt))$ be twice differentiable and let $Q_{h\alpha\beta} = Q_{h\alpha\beta}(t,q,p,\gamma)$ denote their second derivatives w.r. to $d\gamma^\alpha/dt$ and $d\gamma^\beta/dt$, calculated for $d\gamma/dt = 0$.*

Then, the chart $(W, (q^i, \gamma^\alpha))$ is M-fit (for linearity) if and only if the following identities in t, q, p, γ

$$
(3.1) \qquad -\frac{\partial C_{\alpha\beta}}{\partial q^h} = Q_{h\alpha\beta}
$$

$$
\{2\} = 0
$$

hold, for every $\alpha, \beta = 1, \dots, M$ and for every $h = 1, \dots, N$.

In (3.1) the symbol $\{2\}$ is used to indicate the term of order $(|d\gamma/dt|^2)$ of the Taylor expansion of the map $d\gamma/dt \vdash Q_h(t,q,p,\gamma,\ \mathrm{p}(t,q,p,\gamma,d\gamma/dt))$ near $d\gamma/dt = 0$.

Proof. By definition, one has

$$
(3.2) \qquad \boldsymbol{H} = \boldsymbol{H}_2 + \boldsymbol{H}_1 + \boldsymbol{H}_0,
$$

where

$$
(3.3) \qquad \boldsymbol{H}_2 = \frac{1}{2}\sum_{RS=1}^{D} A^{RS}\pi_R\pi_S, \quad \boldsymbol{H}_1 = -\sum_{RS=1}^{D} A^{RS}\pi_R A_S \quad \text{and} \quad \boldsymbol{H}_0 = -\frac{1}{2}A_0 .
$$

Since, by $(2.3)_1$, one has

$$
(3.4) \qquad \pi_R = \sum_{S=1}^{D} A_{RS}\dot{\chi}^S + A_R ,
$$

it follows that the velocities $d\gamma^\alpha/dt (= d\chi^{N+\alpha}/dt)$ appear linearly in the expressions of the $\partial\boldsymbol{H}/\partial p_h$. Hence the non-linearity in the $d\gamma^\alpha/dt$ may affect only the second set of equations of the system (2.8). Let us examine separately the terms $\hat{Q}_h$ and $-\partial\hat{H}/\partial q^h$ of the right-hand sides of these equations. Let us note that:

I) if the maps $d\gamma/dt \vdash Q_h(t,q,p,\gamma,\ \mathrm{p}(t,q,p,\gamma,d\gamma/dt))$ are twice differentiable, then one consider the Taylor expansion

$$
(3.5) \quad Q_h(t,q,p,\gamma,\mathrm{p}(t,q,p,\dot{\gamma})) = Q_{h0}(t,q,p,\gamma) + \sum_{\alpha=1}^{M} Q_{h\alpha}(t,q,p,\gamma)\,\dot{\gamma}^\alpha + \\ + \frac{1}{2}\sum_{\alpha,\beta=1}^{M} Q_{h\alpha\beta}(t,q,p,\gamma)\,\dot{\gamma}^\alpha\dot{\gamma}^\beta + \{2\}
$$

where Q_{h0} and $Q_{h\alpha}$, $\alpha = 1, \dots, M$, are the zero-order and one-order coefficients, respectively, calculated in $d\gamma/dt = 0$;

II) the dependence of $-\partial\boldsymbol{H}/\partial q^h$ on the $d\gamma^\alpha/dt$ is related to the Riemannian structure induced on $\boldsymbol{M}$ by the metric $[A_{RS}(t,\chi)]$.

By inverting (3.4) one obtains

$$\dot{\gamma}^{\rho} = \sum_{s=1}^{N} A^{N+\rho,s}(p_s - A_s) + \sum_{\alpha=1}^{M} A^{N+\rho,N+\alpha}(\mathrm{p}_\alpha - A_\alpha)$$

and hence

$$\mathrm{p}_\sigma = \sum_{\rho=1}^{N} C_{\sigma\rho}\dot{\gamma}^{\rho} - \sum_{\rho=1}^{M}\sum_{s=1}^{N} C_{\sigma\rho}A^{N+\rho,s}(p_s - A_s) + A_\sigma . \tag{3.6}$$

Then the term

$$\frac{1}{2}\sum_{\alpha,\beta=1}^{M} A^{N+\alpha,N+\beta}\,\mathrm{p}_\alpha\,\mathrm{p}_\beta \tag{3.7}$$

depends linearly and quadratically on the $d\gamma^\alpha/dt$, whereas the remaining part of $\boldsymbol{H}$, i.e.,

$$\boldsymbol{H}_0 + \boldsymbol{H}_1 + \frac{1}{2}\sum_{i,1=1}^{N} A^{i,1}p_i p_1 + \sum_{i=1}^{N}\sum_{\alpha=1}^{M} A^{i,N+\alpha}p_i\mathrm{p}_\alpha ,$$

is affine in the $d\gamma^\alpha/dt$.

By substituting the p's in (3.7) with the expression supplied by (3.6), one trivially checks that the quadratic dependence of $-\partial\boldsymbol{H}/\partial q^b$ on the velocities $d\gamma^\alpha/dt$ is given by the term

$$-\frac{1}{2}\sum_{\alpha,\beta,\rho,\delta=1}^{M}\left(\frac{\partial}{\partial q^b}A^{N+\alpha,N+\beta}\right)C_{\alpha,\rho}\,C_{\beta,\delta}\,\dot{\gamma}^{\rho}\,\dot{\gamma}^{\delta} . \tag{3.8}$$

Since, by $\sum_{\alpha=1}^{M} A^{N+\alpha,N+\beta}C_{\alpha\rho} = \delta_\rho^\beta$ [3], one has

$$\sum_{\alpha=1}^{M}\left(\frac{\partial}{\partial q^b}A^{N+\alpha,N+\beta}\right)C_{\alpha\rho} = -\sum_{\alpha=1}^{M} A^{N+\alpha,N+\beta}\,\frac{\partial C_{\alpha\rho}}{\partial q^b} ,$$

the expression in (3.8) coincides with

$$\frac{1}{2}\sum_{\alpha,\beta=1}^{M}\frac{\partial C_{\alpha,\beta}}{\partial q^b}\dot{\gamma}^{\alpha}\dot{\gamma}^{\beta} .$$

By (3.5) and (3.9) one concludes that the coefficient of $(d\gamma^\alpha/dt)\cdot(d\gamma^\beta/dt)$ in $-$ $-\,\partial\boldsymbol{H}/\partial q^b + Q_b - \{2\}$ is given by

$$\frac{1}{2}\left(\frac{\partial C_{\alpha,\beta}}{\partial q^b} + Q_{b\alpha\beta}\right). \tag{3.10}$$

This implies that the conditions (3.1) are equivalent to the linearity (in $d\gamma/dt$) of (2.8).

Q.E.D.

COROLLARY 3.1. *If the effective forces acting on the material points of Σ depend linearly on velocities, then the coordinate chart $(W, (q^i, \gamma^\alpha))$ is* M*-fit* (*for linearity*) *if and*

[3] We denote the Kronecker symbol indiffertly by δ_a^b, δ_{ab}, and δ^{ab}.

only if the $NM(M+1)/2$ identities (in t, q, and γ)

$$\frac{\partial A^{N+\alpha,N+\beta}}{\partial q_h} \equiv 0 \tag{3.11}$$

hold, for every $\alpha, \beta = 1, \ldots, M$ and every $h = 1, \ldots, N$.

Proof. When the effective forces depend linearly on velocities, one has $Q_{h\alpha\beta} \equiv 0$ and $\{2\} \equiv 0$. Then (3.1) reduces to

$$\frac{\partial C_{\alpha\beta}}{\partial q^h} \equiv 0\,, \tag{3.12}$$

which, because of the identity in (t, q, p)

$$\sum_{\alpha=1}^{M} A^{N+\rho,N+\alpha} C_{\alpha\beta} = \delta_\beta^\rho\,, \quad \rho, \beta = 1, \ldots, \mathrm{M}\,,$$

is equivalent to (3.11).

4. Scalar controls

Througout this section it will be assumed that

i) $M = 1$, i.e. $N = D - 1$, and

ii) the constraints are time-independent.

Let us begin by recalling the definition of locally geodesic coordinate chart:

Definition 4.1. A coordinate chart $(W, (q^i, \gamma))_{i=1,\ldots,D-1}$ on a Riemannian manifold $\boldsymbol{M}$ is said to be *locally geodesic* if $A_{RD}(q^i, \gamma) \equiv \delta_{RD}$, $\forall R = 1, \ldots, D$, where $[A_{RS}(q^i, \gamma)]_{R,S=1,\ldots,D}$ is the representation of the metric tensor in the coordinates (q^i, γ).

In [3] Bressan notices that if, besides ii),

iii) the forces depend linearly on velocity, then each locally geodesic coordinates chart $((W, (q^i, \gamma))$ is 1-fit (for linearity). Actually the condition $A_{RD}(q^i, \gamma) \equiv \delta_{RD}$ is equivalent to $A^{RD}(q^i, \gamma) \equiv \delta_{RD}$, and hence the hypotheses (3.11) are satisfied.

Remark. Since there are infinitely many locally geodesic coordinate charts defined on suitable neighbourhoods of a point of $\boldsymbol{M}$, one may conclude that, *under the above hypotheses on the forces and on the constraints, there always exist infinitely many 1-fit coordinate charts.*

In the following theorem we characterize each 1-fit system of coordinates as the image of a (suitable) locally geodesic chart $(W, (q^i, \gamma))$, under a diffeomorphism which sends hypersurfaces $\gamma = k \in \boldsymbol{R}$ into hypersurfaces of the same type.

Theorem 4.1. *Let us assume the hypotheses* i), ii) and iii), *and let $(W', (q'^i, \gamma') \equiv (\chi'^R))$ be a 1-fit coordinate chart on $\boldsymbol{M}$, with range $U' \times I'$, $U' \subseteq \boldsymbol{R}^{D-1}$, $I' \subseteq \boldsymbol{R}$.*

Then, for each $c \in I'$, there exist an open subset $\Omega'_c \subseteq U' \times I'$ intersecting the hypersur-

faces $H_c = \{(q'^i, \gamma') \in U' \times I' \,|\, \gamma' = c\}$ *and a diffeomorfism*

$$f : \Omega'_c \to \Omega_c [\equiv f(\Omega'_c)]$$
$$(q'^i, \gamma') \to (q^i, \gamma) = (f^1, \ldots, f^N, f^D)(q'^i, \gamma')[\equiv f(q'^i, \gamma')]$$

such that:

i) f *induces the identity on* H_c;

ii) *one has*

$$\frac{\partial f^D}{\partial q'^i} = \frac{\partial \gamma}{\partial q'^i} = 0 \tag{4.1}$$

identically on Ω'_c, $\forall i = 1, \ldots, N (= D-1)$; *and*

iii) *the* $(q^i, \gamma)[\equiv (\chi^R)_{R=1,\ldots,D}]$ *are locally geodesic coordinates defined on a suitable open subset* W *of* W' *and taking values on* Ω_c.

Conversely, if $(W, (q^i, \gamma))$ *is a locally geodesic coordinate chart with range* $\Omega \subseteq \mathbf{R}^{D-1} \times \mathbf{R}$, *and* $g = (g^1, \ldots, g^N, g^D) : \Omega \to \Omega' \subseteq \mathbf{R}^{D-1} \times \mathbf{R}$, $g(q^i, \gamma) = (q'^1, \ldots, q'^N, \gamma')$ *is any diffeomorphism such that*

$$\frac{\partial g^D}{\partial q^i} = \frac{\partial \gamma'}{\partial q^i} = 0 \quad \forall i = 1, \ldots, N (= D-1), \tag{4.2}$$

then $(W, (q'^i, \gamma'))$ *is a 1-fit coordinate chart.*

Proof. The second part of the thesis is trivial. Indeed, let $g : \Omega \to \Omega'$ be a diffeomorphism satisfying (4.2), and let $A^{-1} = [A^{RS}(q^i, \gamma)]_{R,S=1,\ldots,D}$ be the inverse of the matrix $A = [A_{RS}(q^i, \gamma)]_{R,S=1,\ldots,D}$ representing the metric tensor in the coordinates $(q^i, \gamma)[\equiv (\chi^R)]$. Then, g transforms A^{-1} into the matrix $(A')^{-1}$, of components

$$A'^{RS}(q'^i, \gamma') = \sum_{P,Q=1}^{D} \frac{\partial \chi'^R}{\partial \chi^P} \frac{\partial \chi'^S}{\partial \chi^Q} A^{PQ}(g^{-1}(q'^i, \gamma')) . \tag{4.3}$$

Since the chart $(W, (q^i, \gamma))$ is locally geodesic, one has $A^{DQ} = \delta^{DQ}$ identically on Ω, for $Q = 1, \ldots, D$. Hence, by (4.2) and (4.3) one obtains

$$A'^{DD}(\gamma') = \left(\frac{\partial \gamma'}{\partial \gamma}\right)^2, \quad \frac{\partial A'^{DD}}{\partial q'^h} = 0 .$$

Then Corollary 3.1 allows to conclude that $(W, (q'^i, \gamma'))$is 1-fit.

Now let us prove the first part of the thesis. If $h : \Omega'_c \to \Omega_c$ is any diffeomorphism, $[(q^i, \gamma) \equiv](\chi^R) := h(\chi'^R)$, then the matrix $(A)^{-1} = [A'^{RS}(q^{i'}, \gamma')]_{R,S=1,\ldots,D}$ is transformed by h into the matrix $(A)^{-1} = [A^{RS}(q^i, \gamma)]_{R,S=1,\ldots,D}$, whose elements are given by

$$A^{RS}(q^i, \gamma) = \sum_{P,Q=1}^{D} \frac{\partial \chi^R}{\partial \chi'^P} \frac{\partial \chi^S}{\partial \chi'^Q} A'^{PQ}(h^{-1}(q^i, \gamma)) . \tag{4.4}$$

Using (4.4), one can rapidly check that

A) *if there exists a D-ple of maps* $(f^1, \ldots, f^N, f^D)$ *such that:*

a) $f^1, \ldots, f^N, f^D$ *are defined on an open subset* Ω'_c, *which intersects the hypersurface* H_c;

b) for each $i = 1, \ldots, N(= D - 1)$, f^i *solves the Dirichlet problem*

$$\left\{\begin{array}{l} \sum_{j=1}^{N} \dfrac{\partial f^i}{\partial q'^j} A'^{Dj}(q'^h, \gamma') + \dfrac{\partial f^i}{\partial \gamma'} A'^{DD}(\gamma') = 0 \\ f^i(q'^h, \gamma') = q'^i \quad \text{on } H_c; \end{array}\right. \tag{4.5}$$

c) f^D *satisfies*

$$\left\{\begin{array}{l} \dfrac{df^D}{d\gamma'} = |A'^{DD}(\gamma')|^{-1/2} \\ f^D(c) = c; \end{array}\right. \tag{4.6}$$

d) the Jacobian matrix $\partial(f^1, \ldots, f^N, f^D)/\partial(q'^i, \gamma')$ *has full rank on* Ω'_c;

then, setting $f := (f^1, \ldots, f^N, f^D) : \Omega'_c \to \Omega_c (:= f(\Omega'_c))$, *the diffeomorphism* f *satisfies i)-iii).*

Indeed, if the coordinates $(\chi^R) = (q^i, \gamma) := (f^1, \ldots, f^N, f^D)(q'^i, \gamma')$ satisfy *a)*, *b)*, *c)*, *d)* one has, for $j = 1, \ldots, N$,

$$\frac{\partial \gamma}{\partial q'^i} = 0, \quad \frac{\partial \gamma}{\partial \gamma'}\left(\sum_{j=1}^{N} A'^{Nj} \frac{\partial q^i}{\partial q'^j} + A^{NN} \frac{\partial q^i}{\partial \gamma'}\right) = 0, \quad \left(\frac{\partial \gamma}{\partial \gamma'}\right)^2 A^{NN} = 1, \tag{4.7}$$

and $f(q'^1, \ldots, q'^N, c) = (q'^1, \ldots, q'^N, c)$, $\forall (q^1, \ldots, q'^N, c) \in \Omega'_c$.

Since (4.4), $(4.7)_1$ imply that, $\forall i \in \{1, \ldots, N\}$, the right-hand side of $(4.7)_2$ [resp. $(4.7)_3$], coincide with A^{Di} [resp. A^{DD}], the (q^i, γ) turn out to be locally geodesic.

Let us observe that the possibility of restricting the choice of the transformations $(q, \gamma)(q', \gamma')$ to those satisfying $(4.7)_1$ is equivalent to the condition $A'^{DD} = A'^{DD}(\gamma')$, i.e. to the 1-fitness of the coordinates (q'^i, γ').

Then it remains to verify that a D-ple as in **A**) actually exists.

Since $(A')^{-1}$ is positive definite, the condition $A'^{DD} \neq 0$ is satisfied at any point of Ω'_c. This implies that the vector field of components $(A'^{DR})_{R=1,\ldots,D}$ (in the coordinates (q'^1, γ')) is transversal to H_c. Therefore (see e.g. [1]), for each $i = 1, \ldots, N$, the Dirichlet problem (4.5) has a local solution f^i. Moreover the (assumed) uniqueness for the solution of the differential system for the characteristic lines

$$\frac{dy^R}{ds} = A'^{DR}(y(s)) \, R = 1, \ldots, D \tag{4.8}$$

implies that the f^i's are independent, i.e. the rank of the $N \times D$ Jacobian matrix $(\partial f^i / \partial \chi'^R)$ equals N. Since $A'^{DD} \neq 0$, it follows that the functions $f^1, \ldots, f^N, f^D$ satisfy the required rank condition *d)*. This concludes the proof.

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