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Wallman-type compaerifications and function lattices

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Topologia. — *Wallman-type compactifications and function lattices* (*). Nota di ALESSANDRO CATERINO e MARIA CRISTINA VIPERA (**), presentata (***) dal Socio G. ZAPPA.

ABSTRACT. — Let $F \subset C^*(X)$ be a vector sublattice over $\mathbb{R}$ which separates points from closed sets of X. The compactification $e_F X$ obtained by embedding X in a real cube via the diagonal map, is different, in general, from the Wallman compactification $\omega(Z(F))$. In this paper, it is shown that there exists a lattice F_z containing F such that $\omega(Z(F)) = \omega(Z(F_z)) = e_F X$. In particular this implies that $\omega(Z(F)) \geqslant e_F X$. Conditions in order to be $\omega(Z(F)) = e_F X$ are given. Finally we prove that, if αX is a compactification of X such that $Cl_{\alpha X}(\alpha X \setminus X)$ is 0-dimensional, then there is an algebra $A \subset C^*(X)$ such that $\omega(Z(A)) = e_A X = \alpha X$.

KEY WORDS: Compactifications; Normal bases; Function lattices; Zero-sets.

RIASSUNTO. — *Compattificazione di tipo Wallman e reticoli di funzioni.* — Sia $F \subset C^*(X)$ reticolo ed $\mathbb{R}$-spazio vettoriale che separa i punti dai chiusi. La compattificazione $e_F X$, ottenuta immergendo X in un cubo reale mediante l'applicazione diagonale e_F, è in generale diversa dalla compattificazione di Wallman $\omega(Z(F))$. In questa nota si dimostra che esiste un reticolo F_z contenente F tale che $\omega(Z(F)) = \omega(Z(F_z)) = e_{F_z} X$. Ciò implica in particolare che $\omega(Z(F)) \geqslant e_F X$. Si danno condizioni necessarie e sufficienti affinché valga l'uguaglianza. Infine si dimostra che, se αX è una compattificazione di X tale che $\mathrm{Cl}_{\alpha X}(\alpha X \setminus X)$ è zero-dimensionale, allora esiste un'algebra A di funzioni continue limitate definite su X tale che $\omega(Z(A)) = e_A X = \alpha X$.

1. INTRODUCTION

Let X be a Tychonoff space and let F be a subset of $C^*(X)$, the ring of all bounded continuous real functions on X.

If the diagonal map $e_F: X \to \mathbb{R}^F$ is an embedding, in particular if F separates points from closed sets, then we denote by $e_F X$ the compactification $\overline{e_F(X)}$ of X.

One has another compactification naturally associated with F, when $Z(F)$, the family of the zero-sets of the elements of F, is a normal base: the Wallman-type compactification $\omega(Z(F))$ (see [10]). These two compactifications can be very different: for instance it is known that, if X a metrizable non-compact locally compact space and $F = \{f \in C^*(X) | \lim_{x \to \infty} f(x) = r_f \in \mathbb{R}\}$ then $Z(F)$ is a normal base and $\omega(Z(F)) = \beta X \neq e_F X = X \cup \{\infty\}$.

More generally, if αX is a T_2-compactification of X, and C_α is the ring of the real continuous functions which extend to αX, then $Z(C_\alpha)$ is a normal base and $\omega(Z(C_\alpha)) \geqslant \alpha X = e_{C_\alpha} X$ (see [7]). In this paper it is shown that, for every compactification $\omega(Z(F))$, where $F \subset C^*(X)$ satisfies suitable conditions to guarantee that $Z(F)$ is a normal base, there exists a set $G \subset C^*(X)$ such that $\omega(Z(F)) = \omega(Z(G)) = e_G X$.

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More precisely, if F is a vector sublattice of $C^*(X)$ such that $\mathbb{R} \subset F$ and e_F is an embedding, then we remark that $Z(F)$ is a normal base and we show how to enlarge F to a lattice F_z such that $Z(F_z) = Z(F)$ and $\omega(Z(F_z)) = e_{F_z}X$. (This implies in particular $\omega(Z(F)) \geqslant e_F X$).

The above result allows us to get some equivalent conditions for $\omega(Z(F))$ to be equal to $e_F X$.

Finally, we prove that, if αX is a compactification such that $\mathrm{Cl}_{\alpha X}(\alpha X \setminus X)$ is 0-dimensional, then there exists a subring A of C_α such that $\alpha X = e_A X = \omega(Z(A))$.

2. Preliminary results

All spaces considered are Tychonoff.

We recall that, for a given (Hausdorff) compactification αX of X, the map $f \mapsto f^\alpha$ (where f^α is the extension of f to αX) is an algebra-isomorphism and a lattice-isomorphism between C_α and $C(\alpha X)$ which is also a homeomorphism with respect to the uniform convergence topology (u.c. topology, for short).

Following [3], we say that $F \subset C^*(X)$ *generates* a compactification αX if αX is equivalent to $e_F X$ (in the usual sense). In this case, we have $F \subset C_\alpha$. If $F \subset G \subset C^*(X)$ and F, G generate $\alpha_1 X$, $\alpha_2 X$ respectively, then $\alpha_1 X \leqslant \alpha_2 X$.

From now on, we do not distinguish between equivalent compactifications.

We wish to recall a result about normal bases, which will be used later. (See [10], for instance, for the definitions of normal bases and Wallman-type compactifications).

Let αX be a compactification of X, $\mathcal{L}$ a family of closed subsets of X. Then we put $\overline{\mathcal{L}} = \{\mathrm{Cl}_{\alpha X}(S) | S \in \mathcal{L}\}$.

Proposition 1. Let αX be a compactification of X and let $\mathcal{L}$ be a lattice of closed subsets of X. Then the following conditions are equivalent:

(i) $\mathcal{L}$ is a normal base for X and $\alpha X = \omega(\mathcal{L})$;

(ii) $\overline{\mathcal{L}}$ is a base for the closed subsets of αX, and disjoint elements of $\mathcal{L}$ have disjoint closures in αX.

This proposition is a slight modification of Theorem 1 in [4] (there ascribed to Shanin). Its condition (b) (that is $\overline{A \cap B} = \overline{A} \cap \overline{B}$ for all A, $B \in \mathcal{L}$) can be replaced by the requirement that disjoint elements of $\mathcal{L}$ have disjoint closures in αX, in view of Lemma 2.3 in [5].

3. Normal bases of zero-sets

Let X be any (Tychonoff) space. Then one has:

Proposition 2. Let $F \subset C^*(X)$ be a lattice such that $\mathbb{R} + F \subset F$, $\mathbb{R}F \subset F$ and which separates points from closed sets. Suppose that, $\forall f, g \in F$ such that $Z(f) \cap Z(g) = \emptyset$, we have $\mathrm{Cl}_{e_F X}(Z(f)) \cap \mathrm{Cl}_{e_F X}(Z(g)) = \emptyset$. Then $Z(F)$ is a normal base and $\omega(Z(F)) = e_F X$.

PROOF. Since $Z(F)$ is a lattice, in view of proposition 1, we only have to prove that $\{\mathrm{Cl}_{\alpha X}(Z(f)) \mid f \in F\}$ is a base for the closed subsets of $\alpha X = e_F X$. By [6], prop. 1, we have that F is a dense subset of C_α with respect to the u.c. topology. Hence $F^\alpha = \{f^\alpha \mid f \in F\}$, is dense in $C(\alpha X)$, therefore F^α separates points from closed subsets of αX. Now, let A be a closed subset of αX, $y \in \alpha X \setminus A$ and let $f^\alpha \in F^\alpha$ be such that $f^\alpha(y) \notin f^\alpha(A)$.

If $a = f^\alpha(y)$, we set $g = |f - a|$; then $g \in F$, $g^\alpha(y) = 0$ and, for some $b > 0$, $g^\alpha(z) \geq b$ $\forall z \in A$. We note that $Z = g^{-1}([b/2, +\infty[) = Z((g - b/2) \wedge 0) \in Z(F)$ and $y \notin (g^\alpha)^{-1}([b/2, +\infty[) \supset \mathrm{Cl}_{\alpha X}(Z)$. It remains only to show that $A \subset \mathrm{Cl}_{\alpha X}(Z)$. Let $t \in A$ and let U be a neighbourhood of t in αX. Since also $(g^\alpha)^{-1}([b/2, +\infty[)$ is a neighbourhood of t in αX and X is dense in αX, we have

$$\emptyset \neq X \cap (U \cap (g^\alpha)^{-1}([b/2, +\infty[)) = U \cap Z.$$

Hence $t \in \mathrm{Cl}_{\alpha X}(Z)$. ■

In the above proposition the hypothesis «F separates points from closed sets» can be replaced by «e_F is an embedding». In fact, these two conditions are equivalent when F is a lattice such that $\mathbb{R}F \subset F$ and $\mathbb{R} + F \subset F$.

If $f \in C^*(X)$, we set

$$S(f) = \{f^{-1}([a, b]) \mid a, b \in \mathbb{R}\}$$

and, if $F \subset C^*(X)$ we put $S(F) = \bigcup_{f \in F} S(f)$.

We note that $S(F) = Z(F)$ when F is a lattice such that $\mathbb{R} + F \subset F$, $\mathbb{R}F \subset F$. In fact

$$f^{-1}([a, b]) = f^{-1}([a, +\infty[) \cap f^{-1}(]-\infty, b]) = Z((f - a) \wedge 0) \cap Z((f - b) \vee 0) \in Z(F).$$

Under the same hypotheses on F, if we put

$$F_z = \{g \in C^*(X) \mid S(g) \subset Z(F)\},$$

then we have $F \subset F_z$ and $Z(F_z) = Z(F)$. Furthermore one can easily prove that F_z is a lattice and $\mathbb{R} + F_z \subset F_z$, $\mathbb{R}F_z \subset F_z$.

THEOREM 3. Let $F \subset C^*(X)$ be a vector sublattice over $\mathbb{R}$, which contains all constant functions and separates points from closed sets. Then $Z(F)$ is a normal base and $\omega(Z(F)) = \omega(Z(F_z)) = e_{F_z} X$. Therefore $\omega(Z(F)) \geq e_F X$.

PROOF. Put $\alpha X = e_{F_z} X$. Let f, $g \in F$ be such that $Z(f) \cap Z(g) = \emptyset$. If

$$h = \frac{|f|}{|f| + |g|},$$

one easily sees that $h \in F_z$. Then h has a continuous extension h^α to αX, hence $(h^\alpha)^{-1}(0)$, $(h^\alpha)^{-1}(1)$ are disjoint closed subsets of αX containing respectively $Z(f)$, $Z(g)$. Since $Z(F_z) = Z(F)$, then F_z satisfies all the hypoteses of Proposition 2. ■

Proposition 4. Let $F \subset C^*(X)$ be a vector sublattice over $\mathbb{R}$ which contains all constant functions and separates points from closed sets. Then the following are equivalent:

(i) $\omega(Z(F)) = e_F X$;

(ii) For every $f,\ g \in F,\ Z(f) \cap Z(g) = \emptyset$ implies

$$\mathrm{Cl}_{e_F X}(Z(f)) \cap \mathrm{Cl}_{e_F X}(Z(g)) = \emptyset;$$

(iii) $F_z \subset C_{e_F}$;

(iv) F is a dense subset of F_z with respect to u.c. topology.

Proof. (i) $\Rightarrow$ (ii) is a consequence of well known facts about normal bases (see also prop. 1).

(ii) $\Rightarrow$ (iii). If $h \in F_z$ and $a,\ b \in \mathbb{R},\ a > b$, then $h^{-1}([a, +\infty[),\ h^{-1}(]-\infty, b])$ are disjoint sets belonging to $Z(F)$. So they have disjoint closures in αX. Then, by [6], cor. 3, h has a continuous extension to $e_F X$.

(iii) $\Rightarrow$ (i) The hypothesis implies $e_F X = e_{F_z} X$, which is equal to $\omega(Z(F))$ by Thm. 3.

Finally, since F is dense in C_α and C_α is a closed set in $C^*(X)$, we have (iii) $\Leftrightarrow$ (iv). ■

The following proposition establishes that, for every compactification αX whose remainder is sufficiently disconnected (in a sense to be made precise), there exists a subring A of C_α such that $\alpha X = e_A X = \omega(Z(A))$.

Proposition 5. Let αX be a compactification of X and let $A = \{f \in C_\alpha \mid \forall p \in \mathrm{Cl}_{\alpha X}(\alpha X \setminus X)$ there is a neighbourhood U of p such that $f^\alpha | U$ is constant$\}$.

Then $\alpha X = e_A X = \omega(Z(A))$ if and only if $\mathrm{Cl}_{\alpha X}(\alpha X \setminus X)$ is 0-dimensional.

Proof. First we show that disjoint elements of $Z(A)$ have disjoint closures in αX. Suppose that $Z(f) \cap Z(g) = \emptyset$ with $f,\ g \in A$.

Now, if $p \in \mathrm{Cl}_{\alpha X}(Z(f)) \cap \mathrm{Cl}_{\alpha X}(Z(g))$, let U_f and U_g be neighbourhoods of p such that $f^\alpha | U_f$ and $g^\alpha | U_g$ are constant.

Then $f(x) = g(x) = 0$ for every $x \in U_f \cap U_g \cap X$, which is non-empty. This is a contradiction, because $Z(f) \cap Z(g) = \emptyset$. Now, let $Y = \mathrm{Cl}_{\alpha X}(\alpha X \setminus X)$ be 0-dimensional. Since A is obviously a lattice and an $\mathbb{R}$-algebra, if we prove that A generates αX then, by applying Proposition 4, we obtain $\alpha X = e_A X = \omega(Z(A))$. Therefore it is sufficient to prove that A^α separates points of αX (see [3], thm. 2.3).

Let $x,\ y \in \alpha X,\ x \neq y$. First suppose that one of them, say x, does not belong to Y. Then choose a closed neighbourhood V of $Y \cup \{y\}$ not containing x. If $h\colon \alpha X \to \mathbb{R}$ is a continuous map such that $h(x) = 0$ and $h(V) = 1$, one has $h|X \in A$ and $h = (h|X)^\alpha$ separates x from y.

Now, suppose $x,\ y \in Y$. Since Y is 0-dimensional, there exist disjoint closed subsets C_1 and C_2 of Y (hence closed in αX) such that $x \in C_1$, $y \in C_2$ and $C_1 \cup C_2 = Y$. Let

V_1, V_2 be disjoint closed neighbourhoods in αX of C_1 and C_2 respectively, and let $k: \alpha X \to \mathbb{R}$ be a continuous map such that $k(C_1) = 0$ and $k(C_2) = 1$. As before $k|X \in A$ and $k(x) \neq k(y)$.

Conversely, suppose that $Y = \mathrm{Cl}_{\alpha X}(\alpha X \setminus X)$ is not 0-dimensional. Then there is a connected subset C of Y which is not a singleton. Since locally constant functions defined on a connected space are constant, then A^α does not separate points of C.

Moreover, it cannot happen that $\alpha X = \omega(Z(A))$. Indeed, since A^α does not separate points of αX, then $\overline{Z(A)}$ is not a base for closed subsets of αX, because $\mathrm{Cl}_{\alpha X}(Z(f)) = Z(f^\alpha)$ for all $f \in A$. ■

As a final remark, we point out that Brooks, in [7], used similar arguments to prove that, in the case of X locally compact, $\alpha X = \omega(Z(A))$ if and only if $\alpha X \setminus X$ is 0-dimensional.

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