Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Gianfranco Cimmino

A conjecture on minimal surfaces

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **82** (1988), n.4, p. 639–644.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1988_8_82_4_639_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1988.

Atti Acc. Lincei Rend. fis.
(8), LXXXII (1988), pp. 639-644

Analisi matematica. — *A conjecture on minimal surfaces.* Nota (*) del Socio GIANFRANCO CIMMINO.

ABSTRACT. — Simple computations support the conjecture that a small spherical surface with its center on a minimal surface cannot be divided by the minimal surface into two portions with different area.

KEY WORDS: Minimal surfaces.

RIASSUNTO. — *A proposito di una congettura sulle superficie minime.* Semplici calcoli per avvalorare la congettura che una piccola superficie sferica col centro su una superficie minima non può esser divisa da questa in due porzioni di differenti aree.

Is it possible that a minimal surface divides a small spherical surface with the centre on it in two portions of different areas? Many years ago, I conjectured (1) that the answer should be «no». If the question is still open, the following calculations can be of some interest (2).

1. Let $z(x,y)$ be $=0$ for $x=y=0$, with the Taylor expansion

$$(1)\qquad z(x,y)=p_0x+q_0y+\frac{1}{2}(r_0x^2+2s_0xy+t_0y^2)+\frac{1}{6}(\lambda_0x^3+3\mu_0x^2y+ \\ +3\nu_0xy^2+\tau_0y^3)+\frac{1}{24}(\alpha_0x^4+4\beta_0x^3y+6\gamma_0x^2y^2+4\delta_0xy^3+\varepsilon_0y^4)+\ldots,$$

(*) Presentata nella seduta del 22 giugno 1988.

(1) G. CIMMINO. *Sulla curvatura media delle superficie.* «Rend. Circ. Mat. Palermo», LVI, 281-288 (1932).

(2) Postilla (di Giuseppe Scorza Dragoni). Gianfranco Cimmino è morto il 30 di maggio del 1989. Un maggio nel quale sono andato spesso a trovarLo in Bologna. Durante quello che poi sarebbe stato il nostro ultimo colloquio Egli mi manifestò il timore di non riuscire forse a correggere le bozze della Nota che ora vede la luce (e di un'altra, che la vedrà prossimamente). Immediata e spontanea l'offerta: Se del caso, me ne sarei occupato io. All'offerta Egli replicò avvertendomi che per dare agilità alle formule aveva fatto uso di scritture quali «cφ», «cθ», «c$^2\varphi$», «c$^2\theta$», «sφ», ... in luogo delle consuete «cos φ», «cos θ», «cos$^2\varphi$», «cos$^2\theta$», «sen φ», ... Se lo desiderava, dissi, ero pronto a far rientrare il Suo scritto nell'ortodossia. Me lo vietò deciso: «Oh no! qualcuno capirà bene! capirà bene qualcuno!» Già, ma con me aveva preferito andare sul sicuro, Gli feci osservare, desideroso di concludere con un sorriso quella che in fondo era una disposizione testamentaria (con l'assicurazione che certi Suoi scrupoli erano assolutamente fuori di luogo avevo qualche mese prima concluso un altro nostro colloquio: la nostra più che sessantennale amicizia Lo aveva indotto a parlarmi di un altro testamento, quello vero, e di certi dubbi che Lo assillavano). E naturalmente ho rispettato il suo desiderio. Il testo corrisponde all'originale a meno di due aggiunte, volute da questa nostra Accademia per uniformità di composizione: il titolo in italiano ed il riassunto in inglese sono *de mon cru.* G.S.D.

or, in polar coordinates $x = \rho s\theta c\varphi$, $y = \rho s\theta s\varphi$, $z = \rho c\theta$, $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \varphi \leqslant 2\pi$,

$$c\theta = \omega_1 s\theta + \frac{1}{2}\rho\omega_2 s^2\theta + \frac{1}{6}\rho^2\omega_3 s^3\theta + \frac{1}{24}\rho^3\omega_4 s^4\theta + \dots, \tag{2}$$

$$\omega_1 = p_0 c\varphi + q_0 s\varphi, \quad \omega_2 = r_0 c^2\varphi + 2s_0 c\varphi s\varphi + t_0 s^2\varphi, \quad \omega_3 = \lambda_0 c^3\varphi + 3\mu_0 c^2\varphi s\varphi + \tag{2'}$$
$$+ 3\nu_0 c\varphi s^2\varphi + \tau_0 s^3\varphi, \quad \omega_4 = \alpha_0 c^4\varphi + 4\beta_0 c^3\varphi s\varphi + 6\gamma_0 c^2\varphi s^2\varphi + 4\delta_0 c\varphi s^3\varphi + \varepsilon_0 s^4\varphi, \dots$$

Let $\theta = \theta(\rho, \varphi)$ be the implicit function defined by (2), so that, for any little $\rho > 0$,

$$x = \rho s\theta(\rho,\varphi)c\varphi, \quad y = \rho s\theta(\rho,\varphi)s\varphi, \quad z = \rho c\theta(\rho,\varphi), \quad 0 \leqslant \varphi \leqslant 2\pi, \tag{3}$$

are parametric equations of the intersection curve of $z = z(x,y)$ with $x^2 + y^2 + z^2 = \rho^2$, on which we can assume that

$$\min_{0 \leqslant \varphi \leqslant 2\pi} s\theta > 0. \tag{4}$$

From (2), by derivating three times, we get

$$-\theta_\rho\left(s\theta + \omega_1 c\theta + \rho\omega_2 s\theta c\theta + \frac{1}{2}\rho^2\omega_3 s^2\theta c\theta + \dots\right) = \tag{4'}$$
$$= \frac{1}{2}\omega_2 s^2\theta + \frac{1}{3}\rho\omega_3 s^3\theta + \frac{1}{8}\rho^2\omega_4 s^4\theta + \dots,$$

$$-\theta_{\rho\rho}(s\theta + \omega_1 c\theta + \rho\omega_2 s\theta c\theta + \dots) - 2\theta_\rho(\omega_2 s\theta c\theta + \rho\omega_3 s^2\theta c\theta + \dots) - \tag{4''}$$
$$-\theta_\rho^2(c\theta - \omega_1 s\theta + \rho\omega_2(c^2\theta - s^2\theta) + \dots) = \frac{1}{3}\omega_3 s^3\theta + \frac{1}{4}\rho\omega_4 s^4\theta + \dots,$$

$$-\theta_{\rho\rho\rho}(s\theta + \omega_1 c\theta + \dots) - 3\theta_{\rho\rho}(\omega_2 s\theta c\theta + \dots) - 3\theta_\rho(\omega_3 s^2\theta c\theta + \dots) - \tag{4'''}$$
$$-3\theta_\rho\theta_{\rho\rho}(c\theta - \omega_1 s\theta + \dots) - 3\theta_\rho^2(\omega_2(c^2\theta - s^2\theta) + \dots) -$$
$$-\theta_\rho^3(-s\theta - \omega_1 c\theta - \dots) = \frac{1}{4}\omega_4 s^4\theta + \dots.$$

For $\rho = 0$, if we put $\theta^0 = \theta(0,\varphi)$, $\theta^0_\rho = \theta_\rho(0,\varphi)$, $\theta^0_{\rho\rho} = \theta_{\rho\rho}(0,\varphi)$, $\theta^0_{\rho\rho\rho} = \theta_{\rho\rho\rho}(0,\varphi)$, we deduce from (2), (4)

$$c\theta^0 = \frac{\omega_1}{\sqrt{\omega_1^2 + 1}}, \qquad s\theta^0 = \frac{1}{\sqrt{\omega_1^2 + 1}}, \tag{5}$$

and consequently from (4'), (4'')

$$\theta^0_\rho = -\frac{1}{2}\frac{\omega_2}{(\omega_1^2 + 1)^{3/2}}, \qquad \theta^0_{\rho\rho} = -\frac{1}{3}\frac{\omega_3}{(\omega_1^2 + 1)^2} + \frac{\omega_1\omega_2^2}{(\omega_1^2 + 1)^3}, \tag{5'}$$

whence finally, by means of (4'''),

$$\theta^0_{\rho\rho\rho} = -\frac{1}{4}\frac{\omega_4}{(\omega_1^2 + 1)^{5/2}} + \frac{5}{2}\frac{\omega_1\omega_2\omega_3}{(\omega_1^2 + 1)^{7/2}} - \frac{15}{4}\frac{\omega_1^2\omega_2^3}{(\omega_1^2 + 1)^{9/2}} + \frac{5}{8}\frac{\omega_2^3}{(\omega_1^2 + 1)^{9/2}}. \tag{5''}$$

On the spherical surface $x^2 + y^2 + z^2 = \rho^2$, the area of that portion in which z is $\geqslant z(x,y)$ is given by

$$\rho^2 \int_0^{2x} d\varphi \int_0^{\theta(\rho,\varphi)} s\theta d\theta = \rho^2 \int_0^{2x} [1 - c\theta(\rho,\varphi)]\, d\varphi. \tag{6}$$

It will be $= 2\pi\rho^2$ as soon as

$$\int_0^{2\pi} c\theta(\rho, \varphi)\, d\varphi = 0 . \tag{7}$$

If this happens to be true for all sufficiently small values of ρ, then it will be in particular

$$\begin{gathered}\left[\int_0^{2\pi} c\theta(\rho,\varphi)\,d\varphi\right]_{\rho=0} = 0\,, \qquad \left[\frac{d}{d\rho}\int_0^{2\pi} c\theta(\rho,\varphi)\,d\varphi\right]_{\rho=0} = 0\,,\\ \left[\frac{d^2}{d\rho^2}\int_0^{2\pi} c\theta(\rho,\varphi)\,d\varphi\right]_{\rho=0} = 0\,, \left[\frac{d^3}{d\rho^3}\int_0^{2\pi} c\theta(\rho,\varphi)\,d\varphi\right]_{\rho=0} = 0\,,\end{gathered} \tag{8}$$

i.e.

$$\begin{gathered}\int_0^{2\pi} c\theta^0\,d\varphi = 0, \qquad \int_0^{2\pi} \theta^0_\rho\, s\theta^0\,d\varphi = 0, \qquad \int_0^{2\pi} (\theta^0_{\rho\rho}\, s\theta^0 + (\theta^0_\rho)^2\, c\theta^0)\,d\varphi = 0\,,\\ \int_0^{2\pi} (\theta^0_{\rho\rho\rho}\, s\theta^0 + 3\theta^0_\rho\,\theta^0_{\rho\rho}\, c\theta^0 - (\theta^0_\rho)^3\, s\theta^0)\,d\varphi = 0\,.\end{gathered} \tag{9}$$

The first and the third of these equalities are verified because of (2′), (5), (5′), whereas the second and the fourth, by taking into account also (5″), are seen to be respectively equivalent to

$$\int_0^{2\pi} \frac{\omega_2}{(\omega_1^2+1)^2}\,d\varphi = 0\,, \tag{10}$$

$$\int_0^{2\pi}\left(\frac{1}{4}\,\frac{\omega_4}{(\omega_1^2+1)^3} - 3\frac{\omega_1\omega_2\omega_3}{(\omega_1^2+1)^4} - 6\frac{\omega_2^3}{(\omega_1^2+1)^5} + \frac{21}{4}\,\frac{\omega_2^3}{(\omega_1^2+1)^4}\right)d\varphi = 0\,. \tag{11}$$

2. The integrals we find in (10), (11) are of the type of the J_n defined by

(12) $J_n =$

$$= \int_0^{2\pi} \frac{A_0\, c^{2n}\varphi + A_1\, c^{2n-1}\varphi s\varphi + \ldots + A_n\, c^n\varphi s^n\varphi + A_{n+1}\, c^{n-1}\varphi s^{n+1}\varphi + \ldots + A_{2n}\, s^{2n}\varphi}{(ac^2\varphi + 2bc\varphi s\varphi + cs^2\varphi)^{n+1}}\,d\varphi\,,$$

for $n = 1, 2, 3, 4$.

In general, we can remark that

$$\begin{aligned} J_n = \frac{(-1)^n}{n!}\Bigg(&A_0\frac{\partial^n}{\partial a^n} + \frac{A_1}{2}\,\frac{\partial^n}{\partial a^{n-1}\partial b} + \ldots + \frac{A_n}{2^n}\,\frac{\partial^n}{\partial b^n} + \\ &+ \frac{A_{n+1}}{2^{n-1}}\,\frac{\partial^n}{\partial b^{n-1}\partial c} + \ldots + A_{2n}\frac{\partial^n}{\partial c^n}\Bigg)\int_0^{2\pi}\frac{d\varphi}{ac^2\varphi + 2bc\varphi s\varphi + cs^2\varphi}\,,\end{aligned} \tag{13}$$

where

$$(13') \qquad \int_0^{2\pi} \frac{d\varphi}{ac^2\varphi + 2bc\varphi s\varphi + cs^2\varphi} = 2\pi(ac - b^2)^{-1/2}.$$

The particular case $(A_0 c^{2n}\varphi + \ldots + A_{2n} s^{2n}\varphi) = (Ac^2\varphi + 2Bc\varphi s\varphi + Cs^2\varphi)^n$ is noteworthy, for in it one has

$$(14) \qquad J_n = -\frac{1}{n}\left(A\frac{\partial}{\partial a} + B\frac{\partial}{\partial b} + C\frac{\partial}{\partial c}\right)J_{n-1}$$

whence, for $n = 1, 2, 3$, by (13′),

$$(15) \qquad J_1 = \pi(cA - 2bB + aC)(ac - b^2)^{-3/2},$$

$$(15') \qquad J_2 = \pi\left[-(AC - B^2)(ac - b^2) + \frac{3}{4}(cA - 2bB + aC)^2\right](ac - b^2)^{-5/2},$$

$$(15'') \qquad J_3 = \pi\left[-\frac{3}{2}(AC - B^2)(cA - 2bB + aC)(ac - b^2)^{-5/2} + \right.$$

$$\left. + \frac{5}{8}(cA - 2bB + aC)^3\right](ac - b^2)^{-7/2}.$$

The integral (15), for

$$(16) \qquad A = r_0, \quad B = s_0 \quad C = t_0, \quad a = 1 + p_0^2, \quad b = p_0 q_0, \quad c = 1 + q_0^2,$$

is nothing other than the left hand side of (10). Thus equation (10) holds if, and only if, $z = z(x, y)$ verifies for $x = y = 0$ the equation

$$(17) \qquad (1 + q^2)r - 2pqs + (1 + p^2)t = 0$$

characterizing the minimal surfaces.

Furthermore, if this is the case, then, because of (15″), the fourth term at the left hand side of (11) will also be $= 0$. More generally, an immediate inductive argument based on (14) shows that

$$(18) \quad (1 + q_0^2)r_0 - 2p_0 q_0 s_0 + (1 + p_0^2)t_0 = 0 \Rightarrow \int_0^{2\pi} \frac{(r_0 c^2\varphi + 2s_0 c\varphi s\varphi + t_0 s^2\varphi)^n}{[1 + (p_0 c\varphi + q_0 s\varphi)^2]^{n+1}} d\varphi = 0$$

for every odd integer n.

3. Let us now suppose $z = z(x, y)$, $(x, y) \in$ open set $\subseteq R^2$, to be a minimal surface. We want to prove that not only (10) but also (11) will then be verified.

In (2′), the partial derivatives of $z(x, y)$ till the fourth order will now be taken not only for $x = y = 0$, but for all x, y sufficiently near to 0, with φ no more bound to x, y as in (3), but meaning an independent variable. Beside (15), (15′), (15″), where now $A = r$, $B = s$, $C = t$ and $cr - 2bs + at$ is the left hand side of (17), we have

$$(19)\quad \int_0^{2\pi} \frac{\omega_4}{(\omega_1^2+1)^3}\, d\varphi =$$

$$= \frac{1}{2}\left(\alpha \frac{\partial^2}{\partial a^2} + 2\beta \frac{\partial^2}{\partial a \partial b} + \frac{3}{2}\gamma \frac{\partial^2}{\partial b^2} + 2\delta \frac{\partial^2}{\partial b \partial c} + \varepsilon \frac{\partial^2}{\partial c^2}\right) \int_0^{2\pi} \frac{d\varphi}{ac^2\varphi + 2bc\varphi s\varphi + cs^2\varphi} =$$

$$= \frac{3\pi}{4}[\alpha c^2 - 4\beta bc + 2\gamma(ac+2b^2) - 4\delta ab + \varepsilon a^2](ac-b^2)^{-5/2},$$

$$(19')\quad \int_0^{2\pi} \frac{\omega_1\omega_2\omega_3}{(\omega_1^2+1)^4}\, d\varphi = \frac{1}{6}\left(p\lambda \frac{\partial^2}{\partial a^2} + \frac{q\lambda+3p\mu}{2}\frac{\partial^2}{\partial a\partial b} + 3\frac{q\mu+p\nu}{4}\frac{\partial^2}{\partial b^2} + \right.$$

$$\left. + \frac{3q\nu+p\tau}{2}\frac{\partial^2}{\partial b\partial c} + q\tau\frac{\partial^2}{\partial c^2}\right)\int_0^{2\pi}\frac{(rc^2\varphi + 2sc\varphi s\varphi + ts^2\varphi}{(ac^2\varphi + 2bc\varphi s\varphi + cs^2\varphi)^2}\, d\varphi =$$

$$= \frac{\pi}{2}\left[\left\{-p\lambda ct + \frac{q\lambda+3p\mu}{2}(bt+cs) - 3(q\mu+p\nu)\,bs + \right.\right.$$

$$\left. + \frac{3q\nu+p\tau}{2}(br+as) - q\tau ar\right\}(ac-b^2)^{-5/2} + \frac{1}{4}(cr-2bs+at)\Big\{5p\lambda c^2 -$$

$$\left. -5(q\lambda+3p\mu)\,bc + 3(q\mu+p\nu)(ac+4b^2) - 5(3q\nu+p\tau)\,ab + 5q\tau a^2\Big\}(ac-b^2)^{-7/2}\right],$$

$$(19'')\quad \int_0^{2\pi}\frac{\omega_2^3}{(\omega_1^2+1)^5}\, d\varphi = -\frac{1}{4}\left(\frac{\partial}{\partial a} + \frac{\partial}{\partial c}\right)\int_0^{2\pi}\frac{(rc^2\varphi + 2sc\varphi s\varphi + ts^2\varphi)^3}{(ac^2\varphi + 2bc\varphi s\varphi + cs^2\varphi)^4}\, d\varphi =$$

$$= \frac{\pi}{4}\left[\frac{3}{2}(r+t)(rt-s^2)(ac-b^2)^{-5/2} - \frac{15}{4}(a+c)(rt-s^2)(cr-2bs+at)(ac-b^2)^{-7/2} - \right.$$

$$\left. - \frac{15}{8}(cr-2bs+at)^2(r+t)(ac-b^2)^{-7/2} + \frac{35}{16}(cr-2bs+at)^3(ac-b^2)^{-9/2}\right].$$

Therefore, from the assumption $cr-2bs+at=0$ it follows that equation (11) becomes equivalent to

$$(20)\quad \frac{1}{8}[\alpha c^2 - 4\beta bc + 2\gamma(ac+2b^2) - 4\delta ab + \varepsilon a^2] + p\lambda ct - \frac{q\lambda+3p\mu}{2}(bt+cs) +$$

$$+3(q\mu+p\nu)bs - \frac{3q\nu+p\tau}{2}(br+as) + q\tau ar - \frac{3}{2}(r+t)(rt-s^2) = 0\,.$$

And, as one may easily check, the left hand side of (20) can also be written as

$$\left[\frac{1}{8}\left(c\frac{\partial^2}{\partial x^2} - 2b\frac{\partial^2}{\partial x \partial y} + a\frac{\partial^2}{\partial y^2}\right) + \frac{3}{4}\left((pt - qs)\frac{\partial}{\partial x} + \right.\right.$$

$$\left.\left. + (qr - ps)\frac{\partial}{\partial y}\right) - \frac{3}{2}(rt - s^2)\right](cr - 2bs + at) .$$

This completes the proof of the fact that minimal surfaces satisfy the equations (8), according to the conjecture that for them equation (7) should hold whenever ρ is sufficiently small.