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The importance of rational extensions

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<http://www.bdim.eu/item?id=RLINA_1988_8_82_4_623_0>

**Abstract.** — The rational completion $\overline{M}$ of an $R$-module $M$ can be characterized as a $\tau_M$-injective hull of $M$ with respect to a (hereditary) torsion functor $\tau_M$ depending on $M$. Properties of a torsion functor depending on an $R$-module $M$ are studied.

**Key Words:** Torsion-functor; Rational extension.

**Riassunto.** — *Funtore-torsione $\tau$ dipendente di un $R$-modulo $M$.* Si considerano le estensioni razionali e i completamenti razionali degli $R$-moduli. Il completamento razionale $\overline{M}$ di un $R$-modulo $M$ può essere considerato come l'inviluppo $\tau$-iniettivo $\overline{M} = M_e$ di $M$ per uno speciale funtore-torsione $\tau$ dipendente da $M$. Vengono investigate le proprietà di $\tau$.

**1. Introduction**

In the following a rational extension $M$ of a non-zero submodule $N$ (of $M$) will be the leading notion; $M$ is a *rational extension* of $N$ ($N \subset_{r} M$) if for any $m_1 \in M$, $0 \neq m_2 \in M$, $\exists r \in R$, such that $rm_1 \in N$, $rm_2 \neq 0$. We have the following equivalent statements: (i) $N \subset_{r} M$; (ii) $\text{Hom}_R (A/N; M) = 0$ whenever $N \subset A \subset M$; (iii) $\text{Hom}_R (M/N; \overline{M}) = 0$. The $R$-module $M$ is called *rationally complete* if $M$ has no proper rational extension. E.g. an injective $R$-module is rationally complete. Every $R$-module $M$ has a rational extension $\overline{M} = E_r (M)$ which is rationally complete; $\overline{M}$ is unique up to

isomorphism over $M$. We have the following representations of $\bar{M}$ (see [1]):

$$\bar{M} = \cap \{\ker f|f \in \text{End}_R(\hat{M}); \phi(M) = 0\} = \{x \in \hat{M}|\forall 0 \neq y \in \hat{M}, \exists r \in R, rx \in M, ry \neq 0\}.$$ 

The rational completion $\bar{M}$ of an $R$-module $M$ can be characterized as the $\tau$-injective hull $M_\tau$ (of $M$) for a special torsion functor (depending on $M$).

Any torsion functor $\tau$ to the category $\text{R-Mod}$ can be defined by means of a filter $L$ of left ideals of $R$ (see e.g. [2]). If $L$ is such a filter, then $L$ defines for every $R$-module $A$ a torsion submodule

$$(1) \quad \tau(A) = \tau_L(A) = \{a \in A|\text{Ann}_R(a) \in L\}.$$ 

conversely any torsion functor $\tau$ determines uniquely the corresponding filter $L_\tau$ by $L_\tau = \{I \in R|R/I \text{ is } \tau\text{-torsion}\}$. $A$ is called a $\tau_L$ torsion $R$-module if $\tau_L(A) = A$, and $A$ is $\tau_L$-torsionfree if $\tau_L(A) = 0$.

An $R$-module $A$ is called $\tau$-injective, if for every diagram

$$(*) \quad 0 \rightarrow C \rightarrow B \xrightarrow{f} A$$

with $B/C$ being $\tau$-torsion, any $R$-homomorphism $f: C \rightarrow A$ has an extension $f': B \rightarrow A$ making the diagram $(*)$ commutative.

1.1. If $A$ is any $R$-module, then $A$ has a minimal $\tau$-injective extension $A_\tau$, $A \subset A_\tau \subset \hat{A}$, uniquely determined by the properties: (i) $\hat{A}/A$ is $\tau$-torsionfree; (ii) $A_\tau/A \cong \tau(A/A)$; (iii) $A \subset \hat{A}$; (iv) $A_\tau = \{x \in \hat{A}|(A:x) \in L_\tau\}$. The minimal $\tau$-injective extension $A_\tau$ of $A$ is called the $\tau$-injective hull of $A$.

For more details about $\tau$-injectivity, see [2]. In connection with the theory of rationals we need a special torsion functor $\tau_M$ defined by means of the fixed chosen $R$-module $M$ and the corresponding filter

$$(2) \quad L_M = \{I \in R|\text{Hom}_R(R/I;\hat{M}) = 0\}.$$ 

Then the torsion functor $\tau_M$ belonging to (2) is given in (1):

$$(3) \quad \tau_M(A) = \{a \in A|\text{Hom}_R(R/\text{Ann}(a);\hat{M}) = 0\}.$$ 

This implies, that an $R$-module $A$ is $\tau_M$-torsion if and only if $\text{Hom}_R(A;\hat{M}) = 0$.

1.2. Let $\bar{M}$ be the rational completion of the $R$-module $M$; then (i) $\bar{M}$ is the $\tau_M$-injective hull of $M$; (ii) $\bar{M} = \{x \in \hat{M}|(M:x) \in L_M\}$.

Pr.: For the proof we use the equivalent statements: (1) $\bar{M}$ is a rational extension of $M$; (2) $M \subset \hat{M}$ and $\bar{M}/M$ is $\tau_M$-torsion. Now $\bar{M}$ satisfies the properties of the $\tau_M$-injective hull of $M$, and we have $\bar{M} \subset \hat{M}$. Let $K/\bar{M}$ be the $\tau_M$-torsion submodule of $\bar{M}/M$ then $K$ is a rational extension of $\bar{M}$; since $\bar{M}$ is rationally complete, $K = \bar{M}$, and $\bar{M}/M$ is $\tau_M$-torsionfree, and therefore $\bar{M}$ is the $\tau_M$-injective hull of $M$ (in $\hat{M}$). Then 1.1 (iv) learns that the rational completion $\bar{M}$ of $M$ is just the $\tau_M$-injective hull of the $R$-module $M$.

1.3 Corollary. If $M$ is a rationally complete $R$-module, then $M$ is a $\tau_M$-injective $R$-module.
2. The hereditary torsion functor $\tau_M$

We want to study the family of all torsion theories $\tau$ of $R$-Mod for which $\tau(M) = 0$ for a fixed chosen $R$-Module $M$. If $\tau(M) = 0$ and $M \subseteq M'$, then $\tau(M') = 0$.

Let $\Lambda(M)$ be the family of (hereditary) torsion functors $\tau$ with $\tau(M) = 0$; then for $M \subseteq M'$ we have $\Lambda(M) = \Lambda(M')$; consequently, for a given module $M$, the set $\Lambda(M)$ can be obtained by using for $M$ the injective hull $\hat{M}$ of $M$.

To avoid triviality we note that $\Lambda(0)$ is the collection of all torsion functors.

2.1. Let $M \neq 0$ be a fixed chosen $R$-module, $A$ any $R$-module,

\[ \tau_M(A) = \bigcap \{ \ker(\varphi|_A) : \varphi \in \text{Hom}_R(A; \hat{M}) \}, \]

then (i) $\tau_M$ is a hereditary torsion functor, $\tau_M(M) = 0$, (ii) $\rho \in \Lambda(M)$, if and only if $\rho \leq \tau_M$.

If we define that the functor $\tau_2$ is «stronger» than $\tau_1$ ($\tau_1 < \tau_2$), if $\tau_1(A) \subseteq \tau_2(A)$ for all $A \in \text{R-Mod}$, then this is equivalent with the property $L_{\tau_1} \subseteq L_{\tau_2}$ for the corresponding idempotent filters; we also say then: $\tau_1$ is «weaker» than $\tau_2$. Therefore the property (ii) expresses that, for a fixed chosen $R$-module $M$, $\tau_M$ is the strongest torsion functor $\rho$ of $R$-Mod with the property $\rho(M) = 0$. In other words: $\Lambda(M)$ has a «largest» element $(\tau_M)$, the torsion functor associated with $M$.

Example. Let $R$ be a commutative ring, $S$ a multiplicatively closed subset of $R$, then $S$ defines a torsion functor $[]^S$, where

\[ []^S(N) = \{ n \in N | \exists s \in S, s | n \} \]

If $P$ is a prime ideal of the (commutative) ring $R$, and $S = R \setminus P$, then $S$ is multiplicatively closed in $R$, and we write $\mu_P$ in stead of $\mu_{R \setminus P}$.

2.2. Let $P$ be a proper prime ideal of the commutative ring $R$; then $\tau_{R/P} = \mu_P$ (where $\mu_P$ means $\mu_S$ or $\mu_{R \setminus P}$).

Pr.: If $a \in R \setminus P$, then $ax \in P \Rightarrow x \in P$, hence $\mu_{R \setminus P}(R/P) = 0$, since $sa' = 0(a' \in R/P, s \in R/P)$ implies that $s = 0$. That implies, that $\mu_{R \setminus P} \leq \tau_{R/P}$. Now let $M$ be an ideal $M \in \bigwedge_{\mu_P}$, then $M$ contains no element of $R \setminus P$, i.e. $M \subseteq P$, and then $M$ annihilates all elements of $R/P$, i.e. $M \subseteq L_{\tau_{R/P}}$ and thus $\tau_{R/P} = \mu_P$.

2.3. If $\sigma$ is a hereditary torsion functor of $R$-Mod, then there exists an $R$-module $S$ such that $\sigma = \tau_S$; (see [3]).

(I.e. every hereditary torsion functor $\sigma$ is a $\tau_S$ for a suitable $S \in R$-Mod).

Let $X$ be a subset of all hereditary torsion functors, $M$ an $R$-module; define $\sigma(M) = \cap \rho(M)$; then $\rho$ is a hereditary torsion functor and $\sigma(M) \subseteq \rho(M)$ for all $\rho \in X$ and all $M \in \text{R-Mod}$. Furthermore we have: if $\tau < \rho$ for all $\rho \in X$, then $\rho < \sigma$. Thus it is reasonable to call $\sigma = \inf(X)$, and $\sigma(M) = \bigcap_{\rho \in X} \rho(M)$ the $\inf(\rho(M))$, $\rho \in X$.

2.4. Let $M = \bigoplus_{a \in A} M_a$, and $\tau_a = \tau_{M_a}$ ($a \in A$); then $\tau_M = \inf_\tau(\tau_{M_a})$.

Pr.: Since $M_a \subseteq M$, we have $\tau_M < \tau_{M_a}$ ($\forall a \in A$), i.e. $\tau_M \subseteq \bigcap_\tau \tau_M$.

If $\sigma < \tau_M (\forall a)$, then $\sigma < \tau_M$, and that implies that $\tau_M = \inf_\tau(\tau_{M_a})$. 
Moreover it implies that \( \tau_M(N) = \bigcap_{\sigma \in A} \tau_{M_i}(N)(\forall N \in R\text{-Mod}). \)

Let \( M \neq 0 \) be a uniform \( R \)-module, and suppose that we have for the associated torsion functor \( \tau_M = \rho \wedge \sigma \) for some pair of torsion functors \( \rho, \sigma \). Then \( 0 = \tau_M(M) = \rho(M) \cap \sigma(M) \). Since \( M \) is uniform we have \( \rho(M) = 0 \) or \( \sigma(M) = 0 \), and that implies that \( \rho \leq \tau_M \) or \( \sigma \leq \tau_M \); but from \( \tau_M \leq \rho \) (resp. \( \tau_M \leq \sigma \)) we conclude that \( \tau_M = \rho \) or \( \tau_M = \sigma \); i.e.

2.5. If \( M \neq 0 \) is a uniform \( R \)-module, then \( \tau_M \) is indecomposable in the sense that \( \tau_M = \rho \wedge \sigma \rightarrow \tau_M = \rho \) or \( \tau_M = 0 \).

If \( M \) is an injective uniform \( R \)-module, then \( M \) is an indecomposable injective \( R \)-module. Now suppose that \( R \) is a (left) Noetherian ring; then every injective \( R \)-module is a direct sum of indecomposable injective submodules. If now \( \tau \) is a hereditary torsion functor, then (by 2.3) there exists an \( R \)-module \( M \) such that \( \tau = \tau_M = \tau_M \).

Since \( \hat{M} = \bigoplus_{\alpha} M_{\alpha} \) is a direct sum of indecomposable injective submodules \( M_{\alpha} \), we have

\[ (** \quad \tau = \inf_{\alpha \in A} (\tau_{M_{\alpha}}), \]

and each \( \tau_{M_{\alpha}} \) is an indecomposable torsion functor in the sense of 2.5.: 2.6. If \( R \) is a (left) Noetherian ring, every hereditary torsion functor \( \tau \) of \( R \text{-Mod} \) is generated by all \( \tau_{M_{\alpha}} \), where \( M_{\alpha} \) is an indecomposable injective \( R \)-module, and where «generated» has the sense of (**).

We return to the hereditary torsion functor \( \tau_K \) associated with a fixed \( R \)-module \( K \), and

\[ \tau_K(M) = \cap (\ker(\phi)|\phi: M \rightarrow \hat{K}). \]

Then \( \tau_K(K) = 0 \), \( \tau_K \in \Lambda(K) \), while \( \rho \in \Lambda(K) \) iff \( \rho \leq \tau_K \).

For any \( R \)-module \( M \) we then have, if \( \rho \in \Lambda(K) \):

\[ \rho(M) \subseteq \tau_K(M)(\forall M \in R\text{-Mod}), \rho(M/\tau_K(M)) \subseteq \tau_K(M/\tau_K(M)) = 0 \text{ for all } \rho \in \Lambda(K) \]

and all \( M \in R\text{-Mod}, i.e. \rho(M/\tau(M)) = 0 \). Conclusion:

2.7. If \( K \neq 0 \) is a fixed \( R \)-module, \( \rho \in \Lambda(K) \), and \( M \in R\text{-Mod} \), then \( \tau_K(M) \) is, among the submodules \( \rho(M) \), the unique maximal submodule \( N \subseteq M \) with \( \rho(M/N) = 0 \). We have \( \tau_K(M) = M \) if and only if \( \text{Hom}_R(M; \hat{K}) = 0 \).

2.8. If \( K_1 \) and \( K_2 \) are complements of a submodule \( K \subseteq M \), then \( \tau_{K_1} = \tau_{K_2} \).

Pr.: If \( \pi: M \rightarrow M/K \), then \( K_i = \pi(K_i) \subseteq M/K(i = 1, 2) \); therefore \( \tau_{K_i} = \tau_{K_2} = \tau_{M/K} \).

If therefore \( K \neq 0 \) is a submodule of the uniform module \( M \), then \( \tau_K = \tau_M \).

In the following we consider a locally uniform \( R \)-module \( M \), i.e. every submodule \( 0 \neq N \subseteq M \) contains a uniform submodule. Let \( \{N_i| i \in I\} \) be a maximal independent set of uniform submodules \( \neq 0 \) of \( M \), i.e. \( \Sigma N_i = \bigoplus_{i \in I} N_i \subseteq M \).

For each \( N_{i_0} \) (\( i_0 \in I \)) we choose a complement \( N_{i_0} \supseteq \bigoplus_{i \neq i_0} N_i \), then \( \bigcap_{i} N_i = 0 \).
and this is an irredundant intersection of essentially closed submodules of $M$. Identifying $M_i = M/N_i^j$, we have $N_i \subset M_i = M/N_i^j$, thus $M \cong \bigoplus_{i} x M_i = x M/N_i^j$.

According to the associated torsion functor $\tau_M$ of the locally uniform $R$-module $M$ we note that $N = \bigoplus_{i \in I} N_i \subset x M_i$, i.e. $\tau_M = \tau_N = \inf_{i \in I}(\tau_{N_i})$.

On the other hand $M = x M_i$ is an essential subdirect product in the sense that $M \cap M_i \subset M_i (i \in I)$. That implies, that $\tau_{M \cap M_i} = \tau_{M_i}$; since $M \cap M_i \subset M$, we have $\tau_M \subset \tau_{M \cap M_i} = \tau_{M_i} (\forall i \in I)$. We prove that if $\sigma < \tau_m (\forall i \in I)$, then $\sigma < \tau_M$. If $\sigma < \tau_m$, then $\sigma(M) \subset \tau_{M_i}(M) = \bigcap \{\ker \phi_i : M \to M_i\} \subset N_i^j$, therefore

\[
\pi_i \sigma(M) \subset \pi_i \tau_{M_i}(M) \subset \pi_i N_i^j = 0, \quad i.e. \quad \sigma(M) \subset \bigcap_i N_i^j = 0, \quad \sigma \leq \tau_M,
\]

and therefore

\[
\tau_M = \inf_{i \in I}(\tau_{M_i}).
\]

Summarizing we have:

2.9. If is a locally uniform $R$-module, $\{N_i | i \in I\}$ a maximal independent set of non-zero uniform submodules of $M$, $\{N_i^j | i \in I\}$ a set of complements of the $N_i$ in $M$, $N_i^j \supset i \bigoplus N_i$, then we have:

(i) $\bigoplus_{i \in I} N_i \subset M$;

(ii) $M \equiv x M/N_i^j$ is an irredundant essential subdirect product of the uniform modules $M_i = M/N_i^j (i \in I)$;

(iii) for the corresponding associated indempotent torsion functors $\tau_M$, $\tau_{N_i}$, $\tau_{M_i}$ we have the relations: $\tau_M = \inf_{i \in I}(\tau_{M_i}) = \inf_{i \in I}(\tau_{N_i})$.

The associated torsion functor $\tau_M$ of $M$ represented in 2.9 (iii) is independent of the representation (ii) of $M$. Therefore we define: if an essential submodule $N$ of $M$ is isomorphic with an essential submodule $N'$ of the $R$-module $M'$, then $M$ and $M'$, are called equivalent, and it follows that $\tau_M = \tau_{M'}$. Using the notations of 2.9 we suppose that $\{N_i | i \in I\}$ and $\{L_j | j \in J\}$ are two maximal independent sets of uniform submodules of $M$. Then there exists a $1 - 1 - \map \phi: I \to J$ such that $N_i$ and $L_{\phi(i)}$ are equivalent ($\forall i \in I$).

The corresponding representations of $M$ are

\[
M \equiv x M/N_i^j \quad \text{and} \quad M \equiv x M/L_j^j.
\]

Since $N_i \subset M/N_i^j$, $L_j \subset M/L_j^j$, the equivalence of $N_i$ and $L_{\phi(i)}$ implies that the representation 2.9 (ii) of $M$ as a subdirect product of uniform modules is unique up to equivalence of the components. From the equivalence of $N_i$ and $L_{\phi(i)}$ it follows moreover that $\tau_{N_i} = \tau_{L_{\phi(i)}} (\forall i)$. Therefore:

2.10. If $M$ is a locally uniform $R$-module, then the associated torsion functor $\tau_M$, as expressed in 2.9 (iii) is independent of the choice of a maximal independent set of non-zero uniform submodules of $M$. 


REFERENCES