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# Rheologies quasi wave number independent in a sphere and splitting the spectral line

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**Geofisica.** — *Rheologies quasi wave number independent in a sphere and splitting the spectral lines.* Nota (*) del Socio MICHELE CAPUTO.

ABSTRACT. - The solution of the equations which govern the slow motions (for which the inertia forces are negligible) in an elastic sphere is studied for a great variety of rheological models and surface tractions with rotational symmetry (Caputo 1984a).

The solution is expressed in terms of spherical harmonics and it is shown that its time dependent component is dependent on the order of the harmonic. The dependence of the time component of the solution on the order of the harmonic number is studied.

The problem of causality is then discussed showing that the rheological models defined by strees-strain relations of the generalized Maxwell type (Caputo 1984b), which contain derivatives of real order, are causal.

It is also seen that the rheological model based on stress strain relations of the generalized Maxwell type multiplies the number of spectral lines of the free modes of a spherical shell. The same applies also to the rheologies of Voigt, Maxwell and of the standard linear solid.

KEY WORDS: Splitting; Eigenvalues; Rheology; Wavenumber; Quasistatic.

RIASSUNTO. - *Reologie quasi indipendenti dal numero d'onda che causano moltiplicazione degli autovalori.* Si studiano le deformazioni quasistatiche di una sfera anelastica per una vasta classe di reologie. Si trova che esse sono reologie quasi indipendenti dal numero d'onda. Si discute il problema della causalità definendo una classe di reologie causali. Si trova infine che le classi di reologie più studiate causano moltiplicazioni delle righe spettrali di uno strato sferico.

## INTRODUCTION

In a recent note (Caputo 1987), the dependence of rheological model on the wave number has been discussed for the quasistatic deformation of a sphere. Additional properties have been found for these rheologies requiring an additional discussion.

It had been found that in some rheological models the deformation of a sphere subject to surface tractions (Caputo 1987) is dependent on the wave number. Among those models one may list that used to fit the laboratory data of polycrystalline halite (Caputo 1983, 1986) which has, in the one dimensional case, the creep function

$$(1) \quad -(a_0\alpha + b_0\beta)^{-1} - (a_0 + b_0)^{-1}t + \frac{a_0 b_0(\alpha - \beta)^2}{(a_0 + b_0)^2\alpha\beta(\alpha a_0 + \beta b_0)}\left(1 - \exp\left[-\frac{\alpha\beta(a_0 + b_0)t}{\alpha a_0 + \beta b_0}\right]\right)$$

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where $\alpha$, $\beta$, $a_0$, $b_0$ are the constants depending on the medium, and also the rheological model with derivatives of real order z ($0 < z < 1$) in the stress strain relation which has, in the one dimensional case, the creep function

$$H_1(t)/\mu + t^z/\eta\Gamma(1+z) \tag{2}$$

where $\mu$, $\eta$, z are the constants depending on the medium, $H_1(t)$ the Heaviside function.

The steady state solution in the rheological models is of the greatest interest in the solution of theoretical and applied geophysical problems.

However, one may not use laboratory data (Caputo 1986) unless they are observed after the cracks naturally existing in the sample are closed and the sample may be considered representative of the physical state of layers of the Earth involved in the phenomenon studied. Limits to these data seem to arise also from the non-linearity of the deformation at the stress when it is considered that the steady state is achieved. Another important problem existing is that associated to causality which is neglected in most of the theoretical rheological studies currently made.

In the dynamic case, the free modes of the sphere, the phase velocity and the $Q^{-1}$ depend on the frequency which is determined by the mode type, order and overtone and therefore they are associated also to the wave number. When we consider deformations so slow that the inertia forces are negligible, then the decomposition of the motion in its Fourier coordinates is physically artificial and the reference to the frequencies is not meaningful. It is, therefore, not meaningful to discuss the rheology in terms of frequency. However, we may make reference to the wave number and discuss the rheology in terms of wave numbers.

In this paper we shall discuss, in spherical coordinates, the rheology of a sphere, considering the quasistatic case and associating a creep (or relaxation) function to each wavenumber. These functions depend on the coordinate system and the results are related to them; they are of help in understanding the phenomena of the media studied in this coordinate system.

In a sphere with generalized Maxwell rheology, this is defined by one function only; but the creep function changes with the wave number, then it is useful to discuss the deformation considering a creep function for each wave number.

As in the uniaxial laboratory experiments (with the exception of the case with zero lateral strain), each point of the sphere rigorously has a different creep function. For the points aligned with the centre of the sphere however, it is only a matter of a factor.

## The solution for an elastic sphere

Let us consider an elastic sphere with radius $r_0$, elastic parameters $\lambda$ and $\mu$, and with stress strain relations (Caputo 1986)

$$h * \dot{\tau}_{ij} + \mu(\tau_{ij} - \delta_{ij}\tau_{rr}/3) = \lambda\delta_{ij}h * \dot{\epsilon}_{rr} + 2\mu h * \dot{\epsilon}_{ij} \tag{3}$$

where r, $\theta$ (colatitude) and $\psi$ (longitude) are polar coordinates, h(t) defines the rheology and $\tau_{ij}$ and $\epsilon_{ij}$ are the stress and the strain tensor respectively.

The Laplace Transforms (LT) $S_1$ and $S_2$, of the displacement components $s_1$ and $s_2$, when the sphere is subject to a surface stress on $r = r_0$ with axial symmetry and $\bar{D}_n = D_n d(t)$

$$\begin{aligned} \tau_{11} &= \sum_0^\infty \bar{D}_n P_n(\cos\theta) \\ \tau_{12} &= 0 \end{aligned} \tag{4}$$

($P_n(\cos\theta)$ is the Legendre polynomial of order n) is, assuming $d(t) = \delta(\tau)$,

$$A = \mu\left(\lambda + \frac{2}{3}\mu\right)$$

$$\begin{aligned} S_1 = \sum_0^\infty {}_n \Bigg\{ &- \frac{n(n+1)\left[A + pH\left(\lambda + \frac{n-2}{n}\mu\right)\right]}{(2n^2+4n+3)\left[A + pH\left(\lambda + \frac{2n^2+2n+2}{2n^2+4n+3}\mu\right)\right]}\left(\frac{r}{r_0}\right)^2 + \\ &- \frac{n^{2(n}+2)\left[A + pH\left(\lambda + \frac{n(n^2+n-1)}{n^2(n-2)}\mu\right)\right](1-\delta_{1n})}{(n-1)(2n^2+4n+3)\left[A + pH\left(\lambda + \frac{2n^2+2n+2}{2n^2+4n+3}\mu\right)\right]}\Bigg\} \cdot \\ &\cdot \left(\frac{r}{r_0}\right)^{n-2} \frac{rD_n(\mu + pH)}{2\mu pH} Y_{n0} \end{aligned}$$

$$\begin{aligned} S_2 = \sum_1^\infty {}_n \Bigg\{ &- \frac{n(n+3)\left[A + pH\left(\lambda + \frac{n+5}{n+3}\mu\right)\right]}{(2n^2+4n+3)\left[A + pH\left(\lambda + \frac{2n^2+2n+2}{2n^2+4n+3}\mu\right)\right]}\left(\frac{r}{r_0}\right)^2 + \\ &- \frac{n(n+2)\left[A + pH\left(\lambda + \frac{n^2+n-1}{n^2(n-2)}\mu\right)\right](1-\delta_{1n})}{(n-1)(2n^2+4n+3)\left[A + pH\left(\lambda + \frac{2n_2+2n+2}{2n^2+4n+3}\mu\right)\right]}\Bigg\} \cdot \\ &\cdot \left(\frac{r}{r_0}\right)^{n-2} \frac{rD_n(\mu + pH)}{2\mu pH} \frac{\partial Y_{n0}}{\partial\theta} \end{aligned} \tag{5}$$

where H = LT(h). It has been observed that, for n sufficiently large, these functions are weakly dependent on n. In fact (5) may be written

$$\begin{aligned}(6)\qquad S_1 = &\{[-n(n+1)(r/r_0)^2 - n^2(n+2)(1-\delta_{1n}n)/(n-1)]\Phi(p) + \\ &[(2n^2+7n+1)(n+1)r^2/2(2n^2+4n+3)r_0^2 + (1-\delta_{1n})(n^3+5n^2+3n)/ \\ &2(n-1)(2n^2+4n+3)][1+\mu^2(f(n)-2/3)\Omega(n,p)/(\lambda+\mu f(n))]/(\lambda+\mu f(n))\}\cdot \\ &\cdot r D_n Y_{n0}(r/r_0)^{n-2}/(2n^2+4n+3) \\ S_2 = &\{[-(n+3)(r/r_0)^2/(2n^2+4n+3) - n(n+2)(1-\delta_{1n})/(n-1)(2n^2+4n+3)]\cdot \\ &\cdot\Phi(p) + [-(6n^2+15n+9)(r/r_0)^2/(2n^2+4n+3)^2 - (1-\delta_{1n})\cdot \\ &\cdot(-n^2-5n-3)/(n-1)(2n^2+4n+3)^2][1+\mu^2(f(n)-2/3)\Omega(n,p)/ \\ &(\lambda+\mu f(n))]/[\lambda+\mu f(n)]\}rD_n(r/r_0)^{n-2}(\partial Y_{n0}/\partial\theta)\end{aligned}$$

$$\begin{aligned}\Phi(p) &= 1/2\,pH + 1/2\mu \\ \Omega(n,p) &= \{pH + A/(\lambda+\mu f(n))\}^{-1} \\ A &= \mu(\lambda+2\mu/3) \\ f(n) &= \frac{2n^2+2n+2}{2n^2+4n+3}\end{aligned}$$

If $\lim_{p\to\infty} pH = h(0) = \infty$ and $D_n$ is proportional to $p^{-1}$, which is the case of a surface load constant in time, since

$$\lim_{p\to\infty} pS_i = s_i(t=0), \tag{7}$$

then the ratio

$$(s_1/s_2)_{t=0} \tag{8}$$

is equal to that of the static case and one may rigorously compute the initial trajectory of the motion of the points from the solution of the static case.

One may also see that in (6), when $D_n = D(p)$ for all n, which implies the same time history for the tractions at the points on the boundary, there are terms with coefficient $(1/\mu + 1/pH)$ (let us call them $\Phi$ terms) which are independent of n, terms (let us call them B terms) which are independent of p and terms with coefficient $1 / ((\mu(\lambda + 2\mu/3)) / ((\lambda + \mu f(n)) + pH))$ (let us call them $\Omega$ terms), which depend weakly on n; the B and $\Omega$ terms have a factor proportional to $1/n$ in $S_1$ and proportional to $1/n_2$ in $S_2$ for n large.

The solution for the static case is obtained for $pH\to\infty$. If H is such that we may neglect the B and $\Omega$ terms, then $S_1$ and $S_2$ have the function $(1/\mu + 1/pH)$ as factor. This may be seen also in (6) if H is such that the terms in braces are independent of p because the factors of $\mu$ are all nearly unity, which is the case for large n. This implies that $s_1$ and $s_2$ have the function $LT^{-1}(1/\mu + 1/pH)$ as factor, which in turn implies that

$$s_1/s_2 \tag{9}$$

is independent of time and equal to the ratio $s_1/s_2$ of the static case for all values of

t. In this case of quasi wave number independent rheology, the shape of the initial deformation does not change with time but only varies in amplitude.

If H is such that we may neglect the $\Omega$ terms, which are weakly dependent on n, then in $s_1$ and $s_2$ the time functions are independent of n, (9) is not valid any more but (8) is still valid. In this case the rheology is quasi wave number independent, but the shape of the initial deformation, in general, changes with time.

We note also, however, that t has an upper limit in the value which causes non-linear strain and that it is not always allowed to consider the limit for $t \rightarrow \infty$ which would give the steady state.

One may see that, as already noted by Caputo (1984a, 1987) rigorously, in general, one may not introduce the relaxation time of the medium because the time functions of the terms of the series depend on n. This is true also in the case of the Maxwell rheology as discussed by Caputo (1987). In fact, the relaxation time for the spherical harmonic component of order n is $\eta(\lambda + \mu f(n))/((\lambda + 2\mu/3)\mu)$ (Caputo 1984a, 1986) where $\eta$ is the Maxwell viscosity.

Concerning the $\Omega$ terms, let us note that, the dependence on n of their time function is defined by the term $(\lambda + 2\mu/3)/(\lambda + \mu f(n))$. An inspection of this term may be tentatively made setting $\lambda/\mu = 1$, an assumption generally accepted in Earth science, which gives

$$(\lambda + 2\mu/3)/(\lambda + \mu f(n)) = 5/3(1 + f(n)) = \frac{5(2n^2 + 4n + 3)}{3(4n^2 + 6n + 5)} \tag{10}$$

which is non-increasing; it is represented in Fig. 1 and it departs from the average (11/12) by 1/12.

The assumptions $\lambda/\mu = 1$ is conservative because, in general, in the Earth $\lambda/\mu > 1$, which makes the variation of (10) smaller. For $\lambda = 2\mu$, the function (10) would vary from 1 (for $n = 0$) to 8/9 (for $n = \infty$), thus reducing the departure from the average (17/18) to 1/18.

The wave number dependence of the solution is due to the variation of (10) with n; there is no rigorously wave number independent rheology in a sphere, but the departure from the case of wave number independence, in some cases, is limited.

Many Earth features have dimensions which may be approximated by adding Fourier wavelengths shorter than 1000 km; these correspond approximately to Legendre polynomials of order larger than 10. We may see in fig. 2 that the dependence of the rheology on the wave number, for $n > 10$, is limited to less than 5% in terms of variation of the fundamental parameter, which appears in the time function, as function of wave numbers n for n larger than 10. Forces constant for a long time may not significantly vary their shape with time depending on the rheology.

In case $\lambda = \mu$, the term depending on the wave number is

$$\left(\frac{(\lambda + 2\mu/3)\mu}{\lambda + \mu f(n)} + pH\right)^{-1} = \left(\frac{5\mu}{3(1 + f(n))} + pH\right)^{-1} = \left(\mu\frac{2n^2 + 4n + 3}{4n^2 + 6n + 5}\,\frac{5}{3} + pH\right)^{-1} \tag{11}$$

Considering the case $H = \eta p^{z-1}$ we have

$$\left(\eta\left[\frac{5\mu}{3\eta}\frac{2n^2+4n+3}{4n^2+6n+5}+p^z\right]\right)^{-1} \tag{12}$$

Let $\psi(t)$ be the $LT^{-1}$ of (12), we have (Caputo 1984)

$$\psi(t) = \frac{\sin \pi z}{\eta \pi}\int_0^\infty \frac{r^z e^{-rt}\partial r}{r^{2z}+2Br^z\cos\pi z + B^2}; \quad B = \frac{5\mu}{3\eta}\frac{2n^2+4n+3}{4n^2+6n+5} \tag{13}$$

and setting $r = (Bu)^{1/z}$ we obtain the integral

$$\psi(t) = \frac{\sin \pi z}{\eta B^{1-1/z}\pi z}\int_0^\infty \frac{u^{1/z}[e^{-(Bu)^{1/z}t}]du}{u^2+2u\cos\pi z+1} \tag{14}$$

The creep may be obtained from the convolution of $\psi(t)$ with a box of duration T which implies that $D_n$ is a box of the same duration, it is

$$\psi_1 = \frac{\sin \pi z}{B\eta\pi z}\int_0^\infty \frac{(-1+\exp((Bu)^{1/z}T))\exp(-(Bu)^{1/z}t)}{u^2+2u\cos\pi z+1}du, t > T \tag{15}$$

$$\psi_2 = \frac{\sin \pi z}{B\eta\pi z}\int_0^\infty \frac{(+1-\exp(-(Bu)^{1/z}t)}{u^2+2u\cos\pi z+1}du, t < T \tag{16}$$

The integral in $\psi(t)$ has been computed for $T = (\eta/\mu)^{1/z}$, for several values of z and of n and is shown in fig. 4 where t is measured in units of $(\eta/\mu)^{1/z}$. One may see that since B is decreasing with n increasing, the integral in $\psi_2$ is decreasing with n increasing.

Instead of the relaxation time, since we have the creep $\psi_2(t)$ (assuming $T = \infty$), we may consider the time to reach a given percent x of the asymptotic value of $\psi_2(t)$ which is obtained from (16)

$$1/B\eta = 3(4n^2+6n+5)/5\mu(2n^2+4n+3) \tag{17}$$

Let $t_{0z}$ be this time for $n = 0$; the value $t_{nz}$ corresponding to n, as it may be seen from (16), must satisfy the relation

$$(\mu/\eta)^{1/z}t_{0z} = (5\mu(2n^2+4n+3)/3(4n^2+6n+5))^{1/z}t_{nz} \tag{18}$$

or

(19) $$t_{nz} = t_{0z}(3(4n^2 + 6n + 5)/5(2n^2 + 4n + 3))^{1/z}$$

which is increasing with n.

This implies that, concerning the wave number dependent time component, the displacements of the points in the Earth with shorter wave length, caused by a force constant in time, such as gravity force, will take longer time to occur than those with long wave length and that according to (17), these displacement are limited in size also when t becomes infinite. In fig. 3a and 3b, we see $t_{nz}$ for several values of z. The other time function of $s_1$ and $s_2$ is $LT^{-1}(2Hp)^{-1}$, which is rigorously wave number independent, with $H = \eta p^{z-1}$ is

$$LT^{-1}(2\eta p^z)^{-1} = t^{z-1}/2\eta\Gamma(z)$$

while the creep is

$$t^z/2\eta\Gamma(1 + z)$$

which implies that displacements of the points of the sphere actually increase indefinitely, although with a rate decreasing to zero.

## THE CONSTRAINTS OF CAUSALITY AND OF THE $Q^{-1}$

It is well known that a pulse propagating in a perfectly elastic, homogeneous medium keeps rigorously its shape and that all the frequency components have the same velocity and do not change their relative phases and amplitude. But, if the medium has a rheology described by the stress-strain relations (3), then the pulse changes shape due to attenuation and dispersion of its frequency component; not only, but also it is seen that it is not necessarily true that the causality principle is respected because in some models (Caputo 1969) at any time $t > 0$, after the pulse originates at $x = 0$ and $t = 0$, and at any distance $x > 0$, there is always the arrival of some energy.

Another important phenomenon to be remembered is that the attenuation factor $Q^{-1}$ is approximately constant over the range of seismic interest.

In order to be acceptable a rheology must, therefore, satisfy the two above mentioned physical requirements.

A simple model of acceptable rheology seems that with

(20) $$h(t) = \eta t^{-z}/\Gamma(1\text{-}z), \qquad H = \eta p^{z-1}$$

Caputo (1986) has shown that in the one dimensional case, when $\mu/\eta$ is small relative to $\omega^{-z}$, the complex index of refraction $n(p)$ and the $Q^{-z}$ are

$$[n(p)]^2 = (p^z + \mu/\eta)/(p^z + \mu(\lambda + 2\mu/3)/(\lambda + 2\mu)\eta)$$

$$Q^{-1} = 4\mu^2\omega^{-z}(\sin D\pi z/2)/3\eta(\lambda + 2\mu), \qquad p = iw \tag{21}$$

$$i^{-z} = e^{-iz\pi(1+4K)/2} = e^{-iz\pi D/2}, \qquad D = 1 + 4K, \qquad K = \text{integer}.$$

It is seen (Nussenzveig 1972) that in order that the causality principle is satisfied, and that the $Q^{-1}$ is approximately constant, it is sufficient that the phase velocity $c(\omega)$ be such that the ratio of the phase velocities at two different frequencies $\omega_1$ and $\omega_2$ approximately satisfy the relation

$$c(\omega_1)/c(\omega_2) = 1 + (1/\pi Q)\ln(\omega_1/\omega_2) \tag{22}$$

When $H(p) = p^{-1+z}$ $(0 < z < 1)$, then for $\mu/\eta$ small relative to $\omega^{-z}$ and z small relative to 1, assuming $\lambda = \mu$ to simplify the formulas we have

$$c(\omega) = c(\infty)/\mathrm{Re}(1 + \mu(i\omega)^{-z}/\eta)^{\frac{1}{2}} = c(\infty)/(1 + (\mu\omega^{-z}/2\eta)\cos(D\pi z/2))$$

$$Q^{-1} = (4\mu/9\eta)\omega^{-z}\sin(D\pi z/2) \tag{23}$$

then since $Q^{-1}$ is slowly varying with $\omega$ (z is small relative to 1)

$$Q^{-1}(\omega_2) = Q^{-1}(\omega_1) + (\omega_2 - \omega_1)\left(\frac{\partial Q^{-1}}{\partial \omega}\right)_{\omega=\omega_1} = Q^{-1}(\omega_1)\,[1 - (\omega_2 - \omega_1)z/\omega_1] \tag{24}$$

and

(25)

$$c(\omega_1)/c(\omega_2) \cong (1 - (1/2)Q^{-1}(\omega^1)\mathrm{cotan}(D\pi z/2)/(1 - (1/2)Q^{-1}(\omega_2)\mathrm{cotan}(D\pi z/2)) \cong$$

$$\cong 1 + (((z/2)/Q(\omega_1))/(\tan(D\pi z/2)))\,(1 - \omega_2/\omega_1) \cong$$

$$\cong 1 + (((z/2)/Q(\omega_1))/(\tan(D\pi z/2)))\ln(\omega_1/\omega_2) \cong$$

and for small values of z and K such that $\sin(\pi z D/2) \sim 0$

$$\cong 1 + (1/\pi Q(\omega_1))\ln(\omega_1/\omega_2)$$

shows that in this case, in first order approximation, (3) satisfies the causality principle and gives an almost constant $Q^{-1}$.

When z and D are such that $\sin(\pi z D/2)$ is not small, then relation (19) is not rigorously satisfied; the question remains open.

## THE FREE MODES: THE $Q^{-1}$ AND THE SPLITTING CAUSED BY THE RHEOLOGY

In order to see the effect of the rheology on the $Q^{-1}$ of the free modes, we should study the complex frequencies $\omega$ of the free modes since the real part of $\omega$

gives the period of the mode and the imaginary part of $\omega$ gives the attenuation factor which allows us to compute the $Q^{-1}$.

We will examine the case of the free torsional modes of a spherical shell with a particular density distribution which gives a very simple solution of the equation which governs the torsional motion of the shell, and consider the case $H = \eta p^{z-1}$.

The LT of the equation which governs the free torsional oscillations of a spherical shell limited by the radii $r_1$ and $r_2$ ($r_1 < r_2$), reduced to the r component, is (e.g. Caputo 1963)

$$\mu\left[\frac{1}{r}\frac{d^2(rS)}{dr^2} - \frac{n(n+1)}{r^2}S\right] = \varrho p^2 S \tag{26}$$

It is not a limitation here to assume that the density is a function of r of the type

$$\varrho = \varrho_0/r^2 \tag{27}$$

The resulting model, contrary to the case of the Earth has the velocity of the shear waves increasing with depth. However it is seen that with an appropriate selection of $\varrho_0$ and $\mu$, that is of the average velocity of the shear wave in the layer, the frequencies of the free modes of low order may be reasonably close to those observed in the Earth.

If the rheology is described by $H = \eta p^{z-1}$, then in equation (23) $\mu$ is substituted by

$$\mu \leftrightarrow \frac{\mu\eta p^z}{\mu + \eta p^z} \tag{28}$$

and we obtain

$$\mu\eta p^z\left[\frac{1}{r}\frac{d^2(rS)}{dr^2} - \frac{n(n+1)}{r^2}S\right] = \varrho_0 p^2(\mu + \eta p^z)/r^2 \tag{29}$$

A solution of (26) is the following combination of powers of r (Caputo 1961)

$$\begin{gathered} S = C_1 r^{u_1} + C_2 r^{u_2} \\ u_{1,2} = -1/2 \pm \{\varrho_0 p^2(\mu + \eta p^z)/\mu\eta p^z + (n + 1/2)+\}^{\frac{1}{2}} \end{gathered} \tag{30}$$

where $C_1$ and $C_2$ are arbitrary constants depending on the initial conditions. We obtain the following condition

$$\det\begin{bmatrix} (u_1 - 1)r_2^{u_1-1} & (u_2 - 1)r_2^{u_2-1} \\ (u_1 - 1)r_1^{u_1-1} & (u_2 - 1)r_1^{u_2-1} \end{bmatrix} = 0 \tag{31}$$

which leads to find the solutions of the following equations

$$\varrho_0 p^2(\mu + \eta p^z)/\mu\eta p^z + (n + 1/2)^2 = 0 \tag{32}$$

$$\varrho_0 p^2(\mu + \eta p^z)/\mu\eta p^z + (n^2 + n - 2)^2 = 0 \tag{33}$$

For our purpose it is sufficient to find the solutions of (33). In fact the solutions of (32) give $u_1 = u_2$ and S in (30) is not a combination of independent solutions. Substituting $p = i\omega$, with $\omega$ the complex frequency of the modes, we obtain the equation

$$\omega^2 + \frac{\mu}{\eta}(\omega^{2-z})i^{-z} - E = 0; \quad \left(\nu^2 - \frac{9}{4}\right)\frac{\mu}{\varrho_0} = E,\ \nu = n + 1/2 \tag{34}$$

In general, $\mu\omega^{2-z}/\eta$ is small relative to $\omega^2$ and E, an approximate solution is then

$$\omega_0 = \sqrt{E} = (\mu(n^2 + n - 2)/\varrho_0)^{\frac{1}{2}} \tag{35}$$

and substituting in (34) gives

$$\omega = \omega_0\left\{1 - (\omega_0 i)^{-z}\frac{\mu}{2\eta}\right\} \tag{36}$$

For each n, however, to obtain $\omega$ we need $i^{-z}$ which is

$$i^{-z} = e^{-i(\pi/2 + 2K)z} = \cos\left(\frac{\pi z}{2} + 2\pi K z\right) - i\sin\left(\frac{\pi z}{2} + 2\pi K z\right) \tag{37}$$

where K assumes the value 0, $\pm 1$, $\pm 2$, ...

In general, for z real, we have an infinite number of values of $\omega$ solutions of (34).

Depending on the value of z, however, we may have different results.

For instance, if $z = 2/(4l + 1)$, $l$ = integer, there is always a value of K for which

$$\frac{z}{2} + 2Kz = \frac{1}{4l + 1} + \frac{4K}{4l + 1} = 1 + 4q;\ \text{integer}$$

In fact, the equation above may be written

$$1 + 4K = 4l + 1 + 4q(4l + 1)$$

which has solutions, because 4, $4l + 1$, $-1$ and 4, $4l + 1$ are sets of numbers relatively prime. Such q and k make the argument $(z/2 + 2Kz)\pi = \pi/2 + 2q\pi$ such that $(\omega_0 i)^{-z}\ \mu/2\eta$ assumes the value $\omega_0^{-z}\ \mu/2\eta$ and we have the frequency

$$\omega = \omega_0\left\{1 - (\omega_0)^{-z}\frac{\mu}{2\eta}\right\} \tag{38}$$

One may verify that

$$q = 1, \qquad K = 1$$

gives (38).

The values of $\omega$ corresponding to the other values of K are in the range

$$\omega_0 \pm \omega_0 (\omega_0)^{-z} \frac{\mu}{2\nu}$$

The width of this range is $\omega_0^{1-z}\ \mu/\eta$.

In order to have a line coinciding with that of the purely elastic case, it should be

$$\frac{z}{2} + 2Kz = \frac{1}{4l+1} + \frac{4K}{4l+1} = \frac{1}{2} + q$$

or

$$8K - 2q(4l+1) = -4l - 1$$

which does not have integer solutions K, q since any integer l makes 8, 2(4l + 1), − 1 + 4l relatively prime, while 8, 2(4l + 1) are not.

If $z = 4/(4l+1)$ in order to have a line coinciding with that of the purely elastic case, one should have

$$\frac{2}{4l+1} + \frac{8K}{4l+1} = \frac{1}{2} + q$$

or

$$16K - 2q(4l+1) = 4l - 3$$

which has no solutions K,q because 16, 2(4l + 1), 4l − 3 is a set of relatively prime numbers, but 16, 2(4l + 1) is not.

However, for $z = 4/(4l+1)$ there is a line with $Q^{-1} = 0$, in fact, in order to obtain this we must have

$$\frac{2}{4l+1} + \frac{8K}{4l+1} = q, \qquad (q = \text{integer})$$

or

$$8K + (4l+1)q = 2$$

Which has solutions K,q because 2, 4l + 1, 2 and 2, 4l + 1 are sets of relatively prime numbers.

When $z = 3/(4l+1)$, in order to have the line coinciding with that of the purely elastic case, we should have

$$\frac{3}{2(4l+1)} + \frac{6K}{4l+1} = \frac{1}{2} + q, \qquad q = \text{integer}$$

or

$$6K - (4l + 1)q = 2l - 1$$

which has solutions q,K because 6, $-4l + 1 - 1$ and 6, $-(4l + 1)$ are sets of numbers relatively prime. However, there are no lines on the boundary of the interval $\omega_0(1 - \omega_0^{z}\mu/2\eta)$, $\omega_0(1 + \omega_0^{z}\mu/2\eta)$ because in such case it should be

$$\frac{3}{2(4l + 1)} + \frac{6K}{4l + 1} = q$$

or

$$12K - 2(4l + 1)q = -3$$

which has no solutions K,q since 12, $-2(4l + 1)$, $-3$ is a set of numbers relatively prime, but 12, $-2(4l + 1)$ is not.

The $Q^{-1}$ is usually defined as in (21) by one half the ratio of the imaginary to the real part of $\omega$.

For the free modes

$$Q^{-1} = \left(\mu\omega^{-z}\sin\left(\frac{\pi z}{2} + 2K\pi z\right)\right)/\eta$$

$$\omega = \sqrt{\mu(n^2 + n - 2)/\varrho_0}$$

then the energy dissipated per unit time is

$$\omega Q^{-1}/2\pi = (\mu/2\pi\eta)\,[\mu(n^2 + n - 2)/\varrho_0]^{(1-z/2)}\sin\left(\frac{\pi z}{2} + 2K\pi z\right)$$

which is a function of the wave number but also of K. This shows that in this model, for $0 < z < 1$, the high frequency modes, not necessarily, vanish faster than those with low frequencies. The energy present in each mode, sometime after they have been excited depends also on the energy distributed by the source exciting them. It is to be seen how much energy is input by the sources in the modes with $Q^{-1} = 0$.

In the Maxwell rheology, $z = 1$, the energy dissipated per unit time is independent of n and K. In two of the cases considered, $z = 2/(4l + 1)$, $z = 4/(4l + 1)$, there are values of $\omega$ whose imaginary part is nil, it therefore seems that there are modes with very small or nil dissipation.

One may verify that for $z = 1/(4l + 1)$ and $z = 3/(4l + 1)$, there is a line coinciding with that of the purely elastic case and none has zero $Q^{-1}$; the lines are then contained in the band of width $\omega_0^{1-z}\mu/\eta$.

The splitting of the torsional modes of an anelastic shell caused by the rheology, when the density is given by (27), may be seen also for the case when the density is constant. In fact formula (36) is valid also in this case assuming that $\omega_0$ is the approximate frequency of oscillation of the homogeneous model, which is obtained by solving an equation in which Bessel functions appear (Caputo 1963). In this note we have preferred the model with density given by (27) because it leads to the explicit values of the free modes without the need of the solution of a transcendental equation, although in both cases the computation of the splitting itself is obtained without the need of a computer by means of (38).

## THE FREE OSCILLATIONS OF THE LITHOSPHERE

We like to note that if the shell is relatively thin and the density $\varrho$ is constant and $r_1 = r_2\alpha$ with $1 - \alpha$ equal or less than 0.01 which is the case of the inner radius of the lithosphere, then

$$\varrho_0 r^{-2} = \varrho r^{-2}(r_1^2 + r_1 r_2 + r_2^2)/3$$

represents the density $\varrho$ with an approximation of $1 - \alpha$ and the average densities of the two models differ by less than $10^{-5}$.

With a relative approximation better that $10^{-5}$ the free frequencies of the homogeneous model may be written using the formula (35)

$$\omega_0 = (\mu(n^2 + n - 2)/\varrho_0)^{1/2} = (3\mu(n^2 + n - 2)/\varrho(r_1^2 + r_1 r_2 + r_2^2))^{1/2}$$

It is then verified that with $v_s = (\mu/\varrho)^{1/2} = 4.2712 \ 10^5$, $r_2 = 6.371 \ 10^8$, $r_1 = 6.291 \ 10^8$ (Stacey 1977) $n = 2$, the period of the free torsional oscillation of the lithosphere over the asthenosphere is $79^m$.

The period of $79^m$ has been observed (Bozzi Zadro and Caputo 1968) in the analysis of the records of the Chilean 1960 earthquake obtained in Trieste with the horizontal pendulums of Marussi (1960), and is much larger than any of the theoretically accepted free periods of the Earth. However Bozzi Zadro and Caputo (1968) observed the same period in the bispectral analysis of the record of the 1960 Chilean earthquake, apparently due to the interaction of the modes ${}_0S_{10}$ and ${}_2S_5$.

We have seen here that the fundamental period of the torsional oscillations of the lithosphere is also $79^m$; this could be due to a mere coincidence but could also be an explanation of the period found in free oscillations of the Earth due to the Chilean 1960 earthquake.

Since the $Q^{-1}$ of this mode, due to the friction in the boundary layer between the lithosphere and the asthenosphere, should be rather low, the analysis of the $Q^{-1}$ of this mode could throw more light on this problem. However the estimate of the $Q^{-1}$ is complicated by several interferences and a safer procedure already used succesfully to establish the validity of the identification of a mode (Caputo and Marcucci 1982) is the detection of the splitting of the mode itself due to the rotation of the Earth. This method led to the identification of the mode ${}_2S_1$ and of other modes (Caputo and Marcucci 1982).

## Conclusions

Concerning the effect of rheology of the type $H = \eta p^{z-1}$ on the free modes, we may say:

1. The rheology causes a splitting of the spectral lines of the purely elastic case.
2. In the case $H = \eta p^{z-1}$, the split lines are spread over a band with width $\mu/\eta\omega^{-z}$.
3. The lines are located on both sides of the corresponding line of the purely elastic case.
4. The $Q^{-1}$ of the lines are different, they decrease with increasing separation of the line from that of the purely elastic case.
5. The $Q^{-1}$ of the lines on the border of the band is very small and the $Q^{-1}$ of the lines close to the centre of the band is $\mu\omega^{-z}/\eta$, then the observed width of the band does not give the $Q^{-1}$ of the line as it is sometimes computed.
6. The results obtained for $H = \eta p^{z-1}$ are not applicable exactly to other rheologies; however, if H is the ratio of two polynomials in p, which is the case of polycristalline halite (Caputo 1986), the rheology causes a splitting of the lines of the purely elastic case.
7. The broadening of the spectral lines observed is due to the broadening of the single lines split by the rheology but also to the splitting itself. Due to the small separation foreseen for the lines, it will be very difficult to resolve them in the free modes of the Earth.

It is clear that in many cases, since the form of the solutions of (5) is not necessarily exponential, we may not rigorously speak of "relaxation time" as is in the Maxwellian rheology.

We may also note that due to the form of equations (5), in general, the rheology is wavenumber dependent. Therefore the attempts of inferring the rheology of the mantle of the Earth from observations of the postglacial rebounds at different points and referring to different times does not give acceptable results without further verifications.

If z of the mantle is smaller than 0.1, the relaxation has a strong dependence on n, and this could be the cause the of observed variation of UT1 and of the variation of the node of LAGEOS supposedly induced by deglaciations occurred earlier than 1.8 10 years ago (e.g. Caputo 1985).

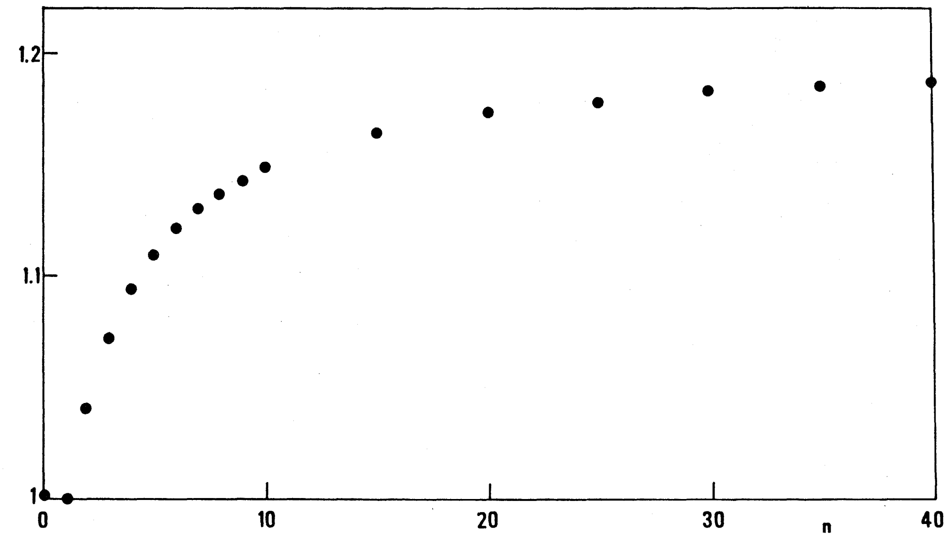


Fig. 1. - Dependence of the relaxation time $3\eta(4n^2 + 6n + 5)/5\mu(2n^2 + 4n + 3)$ on the wave number n for the rheology defined by $H = \eta$, (Maxwell modell) for $\lambda = \mu$ in units of $\eta/\mu$. The asymptotic value is 1.2. One may see that for wave numbers $n > 5$ the relaxation time scatters less than 5% of the average value.

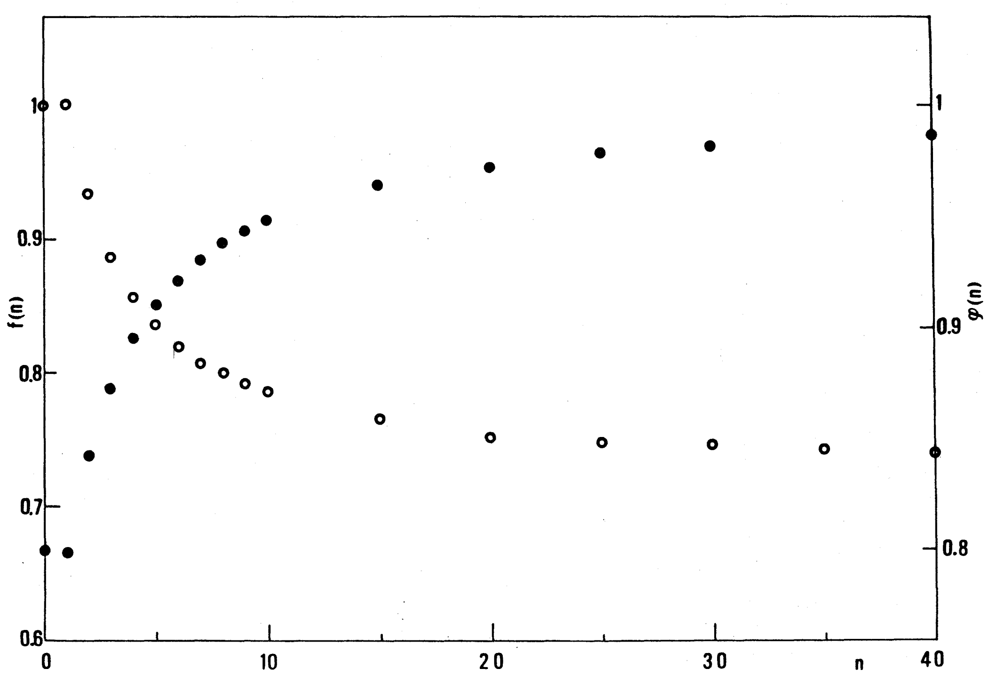


Fig. 2. - The function $f(n) = (2n^2 + 2n + 2)/(2n^2 + 4n + 3)$ (dots) which governs the dependence of the rheology on the wave number n. The asymptotic value is 1. The function $\psi(n) = 5(2n^2 + 4n + 3)/3(4n^2 + 6n + 5)$ (circles) gives the relaxation time dependence on n for the rheology defined by $H = \eta p^{z-1}$. The asymptotic value is 10/12.

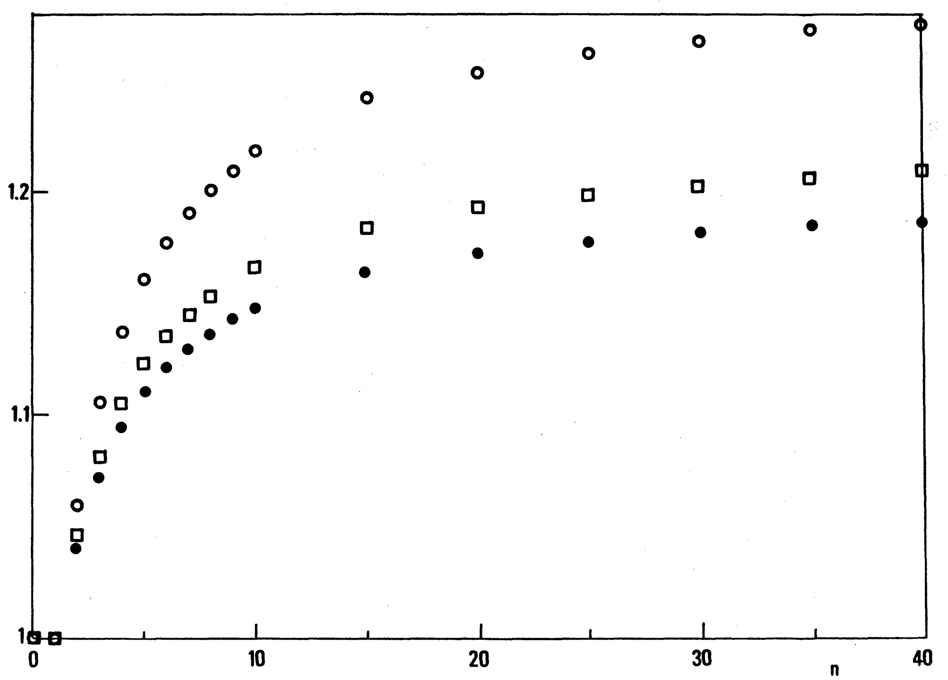


Fig. 3a. - Relaxation time of the rheology defined by $H = \eta p^{z-1}$, as function of n and for different values of z, measured in units of the relaxation time of n = 0. Circles z = 0.7, squares z = 0.9, dots z = 1. The relaxation time for n = 0 is measured in units of $(\eta/\mu)^{1/z}$.

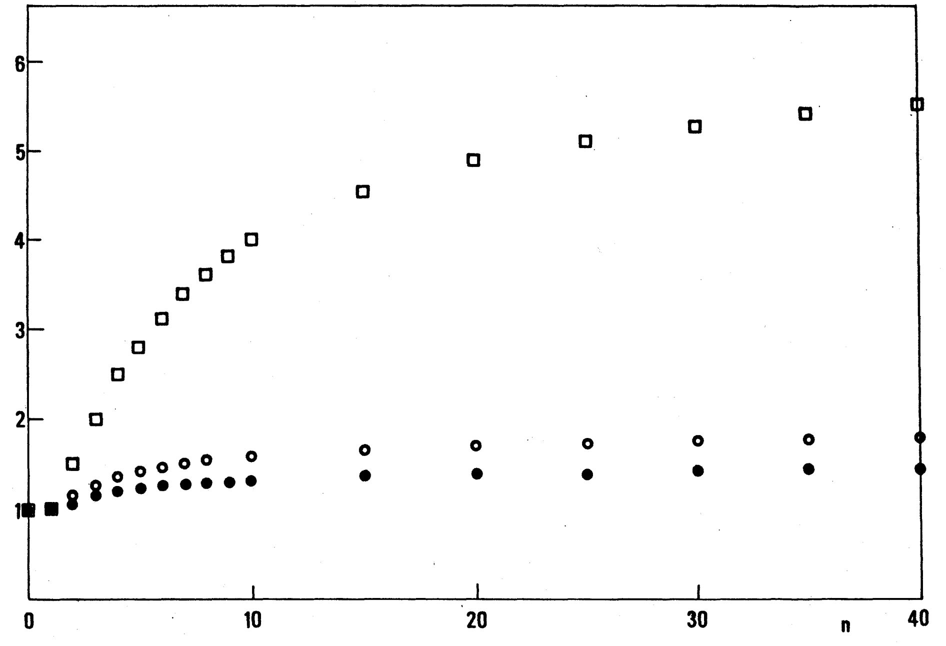


Fig. 3b. - As in fig 3a circles z = 0.3, squares z = 0.1, dots z = 0.5.

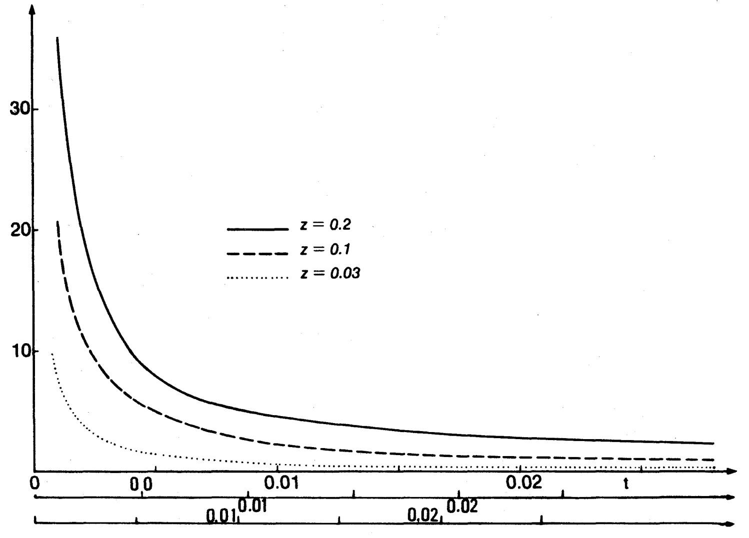


Fig. 4a. - The function $\psi(t)$ convolved with a box of duration T gives the component of the relaxation which depends on the wave number n (see formula (15)). t is measured in units of $(\eta/\mu)^{1/z}$. Top abscissa is for n = 0, median abscissa for n = 10, bottom abscissa for n = ∞. T = $(\eta/\mu)^{1/z}$. The integral appearing in $\psi(t)$ is shown.

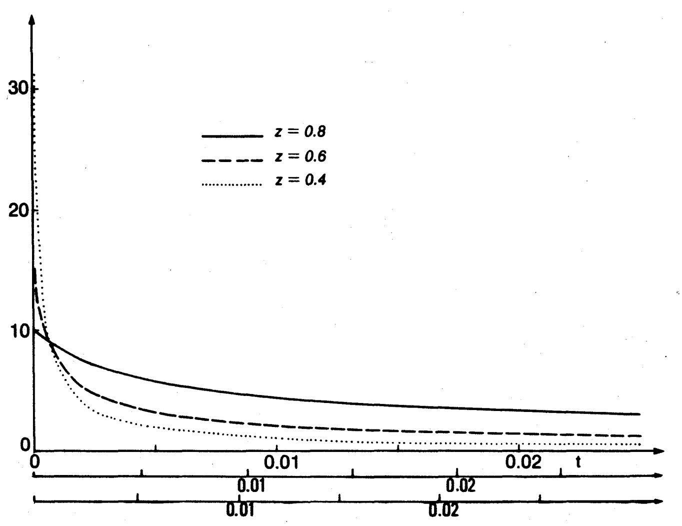


Fig. 4b. - As fig. 4a.

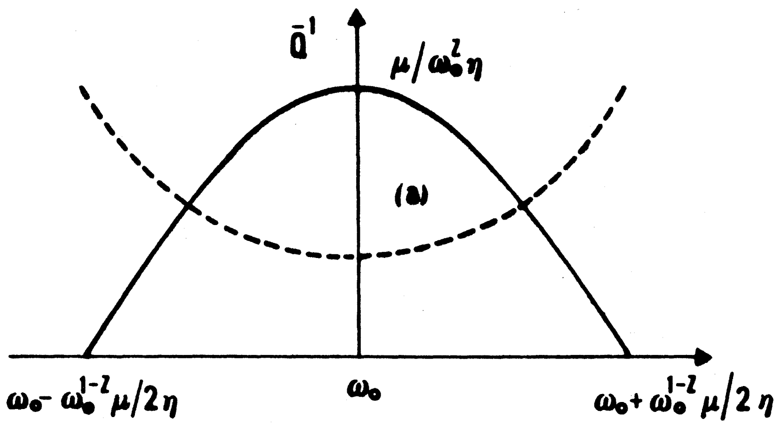


Fig. 5a. - Frequency band $\omega_0^{1-z}\mu/\eta$ of the frequencies split by the rheology defined by $H = \eta p^{z-1}$ and $Q^{-1}$ as function of the frequency. When z is rational, each line of the purely elastic case is split in a finite number of different lines, when z is irrational this number is infinite. The dashed line gives the percentage of energy after $\eta/\mu\omega_2^{1-z}$ cycles.

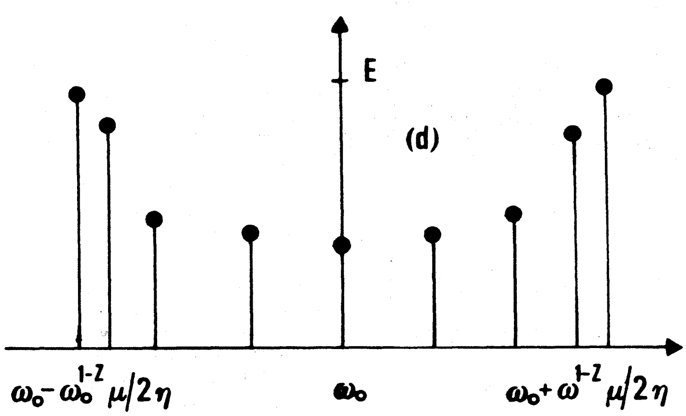


Fig. 5b. - As fig. 5a for z = 1/5 and z = 3/5. In both cases the splitting is in the same 5 lines.

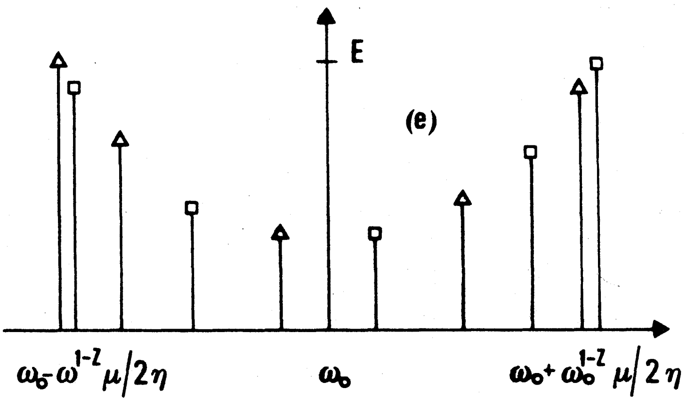


Fig. 5c. - As fig. 5b for z = 2/5 (squares) and z = 4/5 (triangles). In both cases the splitting is in 3 lines. The observation of any of these two splittings would allow the univoc determination of z.

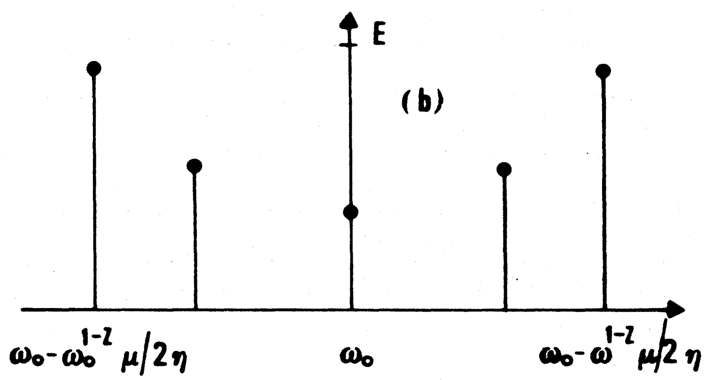


Fig. 5d. - As in fig. 5b for z = 1/9, z = 5/9, z = 7/9. For these values of z the splitting is the same in 9 lines symmetric with respect to $\omega_0$. The observation of this splitting, which is the same for all $\omega_0$, would not allow the univoc determination of z.

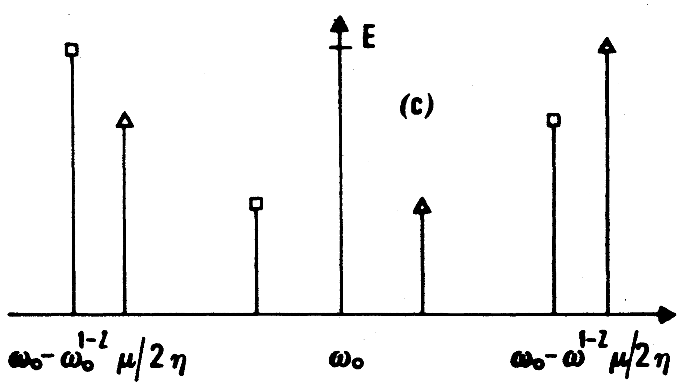


Fig. 5e. - As in fig. 5b for z = 4/9 and z = 8/9 (squares) and for z = 2/9 (triangles). For these values of z, the splitting is in 5 lines. The two sets of lines are symmetric with respect to $\omega_0$. The observation of this splitting would allow the univoc determination of z only in the case z = 2/9.

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