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On oscillation of a kind of integro-differential equation with delay

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Equazioni funzionali. – *On oscillation of a kind of integro-differential equation with delay.* Nota di BINGGEN ZHANG, presentata (*) dal Corrisp. R. CONTI.

ABSTRACT. – This note contains a criterion for the oscillation of solution of a kind of integro-differential equations with delay.

KEY WORDS: Integro-differential equations; equations with delay; oscillation.

RIASSUNTO. – *Sull'oscillatorietà di un tipo di equazioni integro differenziali con ritardo.* Viene data una condizione di oscillatorietà per un tipo di equazioni integro-differenziali con ritardo.

1. INTRODUCTION

There is a large number of papers on oscillation of solutions to first order differential equations with delay [1], but there is no significant result on oscillation of integro-differential equations with delay, which very often appear in Ecological models [2].

In this note we consider a kind of integro-differential equation, with delay of the form

$$x'(t) - h(t) + A(t) f_1(x(t)) + \int_0^t D(t-s) f_2(x(s-\tau))\,ds = 0, \tag{1.1}$$

where τ is a positive number.

Our purpose is to provide conditions which guarantee oscillation of solutions of (1.1). We show that the idea used for differential equations with delay can be also used for integro-differential equations with delay.

DEFINITION 1.1 *By a solution of* (1.1) *we mean a function* $x(t) \in C'[0,\infty)$ *which satisfies* (1.1) *on* $[0,\infty), x(s) = \phi(s)$ *on* $[-\tau, 0]$ *and* $\sup_{t \geqq t_1} |x(t)| > 0$ *for any* $t_1 \geq 0$ *where* $\phi(s)$ *is given, bounded and intregrable on* $[-\tau, 0]$.

(*) Nella seduta del 14 marzo 1987.

DEFINITION 1.2 *A solution* $x(t)$ *is said to be oscillatory if it has arbitrarily large zeros and to be non-oscillatory otherwise.*

2. MAIN RESULTS

LEMMA 1. *Assume that*

i) $A \in C(R_+, R_+), R_+ = [0, \infty)$;

ii) $f_1, f_2 \in C(R, R), z f_i(z) > 0$ for $z \neq 0$, f_i is nondecreasing, $i = 1, 2$ and

$$\lim_{z \to 0} \frac{f_2(z)}{z} = M > 0;$$

iii) $D \in C(R_+, R_+)$ *and*

$$\lim_{t \to \infty} \int_{t-\tau}^{t} \int_{0}^{\infty} D(s - s_1)\, ds_1\, dt > 1/(eM). \tag{2.1}$$

Then

$$a) \quad x'(t) + A(t) f_1(x(t)) + \int_0^t D(t-s) f_2(x(s-\tau))\, ds \leq 0 \tag{2.2}$$

has no eventually positive solution;

$$b) \quad x'(t) + A(t) f_1(x(t)) + \int_0^t D(t-s) f_2(x(s-\tau))\, ds \geq 0 \tag{2.3}$$

has no eventually negative solution;

c) every solution of the equation

$$x'(t) + A(t) f_1(x(t)) + \int_0^t D(t-s) f_2(x(s-\tau))\, ds = 0, \tag{2.4}$$

is oscillatory.

Proof. To prove (*a*), assume that there exists a positive solution $x(t)$ of (2.2), for $t \geq T$, then $x'(t) \leq 0$ for $t \geq T + \tau$. According to assumptions (i) and (ii), from (2.2), we have

$$x'(t) + A(t)f_1(x(t)) + f_2(x(t-\tau)) \int_0^t D(t-s)\,ds \le 0 \tag{2.5}$$

for $t \ge T + \tau$.

Dividing (2.5) by $x(t)$ and integrating it from $t-\tau$ to t, we have

$$Ln\frac{x(t)}{x(t-\tau)} + \int_{t-\tau}^{t} \frac{f_2(x(t_1-\tau))}{x(s_1)} \int_0^{s_1} D(s_1-s)\,ds\,ds_1 \le 0 \tag{2.6}$$

for $t \ge T + \tau$.

Setting

$$W(t) = \frac{x(t-\tau)}{x(t)} \ge 1$$

from (2.6), we get

$$Ln\,W(t) \ge W(\zeta) \int_{t-\tau}^{t} \frac{f_2(x(t_1-\tau))}{x(t_1-\tau)} \int_0^{s_1} D(s_1-s)\,ds\,ds_1 \tag{2.7}$$

for $t \ge T + \tau$, where $\zeta \in (t-\tau, t)$.

Since $x_1'(t) \le 0$ and $x(t) > 0$, so $\lim_{t\to\infty} x(t) = \alpha \ge 0$ exists. If $\alpha > 0$, from (2.7), we get

$$\lim_{t\to\infty} \int_{t-\tau}^{t} \int_0^{s_1} D(s_1-s)\,ds\,ds_1 \le 0$$

which contradicts (2.1). Therefore $\lim x(t) = 0$.

Now we show that $W(t)$ is bounded above. Indeed, from (2.1), there exists a $t^* \in (t-\tau, t)$ such that

$$\int_{t-\tau}^{t^*} \int_0^{s} D(s-s_1)\,ds_1\,ds > \frac{c}{2} \qquad \int_{t^*}^{t} \int_0^{s} D(s-s_1)\,ds_1\,ds > \frac{c}{2}$$

for large t, where $\overset{0}{C} > 1/e$.

Integrating inequality (2.2) from $t-\tau$ to t^*, we have

$$x(t^*) - x(t-\tau) + \int_{t-\tau}^{t^*} A(s)f_1(x(s))\,ds + \int_{t-\tau}^{t^*} \int_0^{s_1} D(s_1-s)f_2(x(s-\tau))\,ds\,ds_1 \le 0$$

so that, according to assumptions (i) and (ii), we get

$$-x(t-\tau)+f_2(x(t^*-\tau))\frac{c}{2}\leq 0$$

or

$$|x(t-\tau)\geq f_2(x(t^*-\tau))\frac{c}{2} \tag{2.8}$$

for large t.

On the other hand, integrating (2.2) from t^* to t, we can obtain the following inequality

$$-x(t^*)+\frac{c}{2}f_2(x(t-\tau))\leq 0\,. \tag{2.9}$$

Combining (2.8) and (2.9), we have

$$x(t^*)\geq\frac{c}{2}f_2(x(t-\tau))=\frac{c}{2}\cdot\frac{f_2(x(t-\tau))}{x(t-\tau)}x(t-\tau)$$

$$\geq\left(\frac{c}{2}\right)^2\frac{f_2(x(t-\tau))}{x(t-\tau)}\cdot\frac{f_2(x(t^*-\tau))}{x(t^*-\tau)}\cdot x(t^*-y)\,.$$

Thus, for large t, we have

$$W(t^*)=\frac{x(t^*-\tau)}{x(t^*)}\leq\left(\frac{2}{c}\right)^2\frac{x(t-\tau)}{f_2(x(t-\tau))}\cdot\frac{x(t^*-\tau)}{f_2(x(t^*-\tau))}\,. \tag{2.10}$$

Since $\lim\limits_{t\to\infty}x(t)=0$ and $\lim\limits_{z\to 0}\frac{f_2(z)}{z}=M>0$, so $W(t^*)$ is bounded above and t^* is arbitrary. Hence we proved that $w(t)$ is bounded above.

Setting

$$\lim_{t\to\infty}W(t)=r\geq 1$$

then r is finite.

We take inferior limit on both sides of (2.7), so that

$$\text{L}n\,r\geq rM\lim_{t\to\infty}\int\limits_{t-\tau}^{t}\int\limits_{0}^{s}D(s_1-s)\,ds_1\,ds\,.$$

Since $\max\limits_{r\geqq 1}\frac{\text{L}n\,r}{r}=\frac{1}{e}$, from the preceding inequality we get

$$\frac{1}{(eM)} \geq \lim_{t\to\infty} \int_{t-\tau}^{t} \int_{0}^{s} D(s_1 - s)\, ds_1\, ds,$$

which contradicts assumption (2.1) and (*a*) is proved.

The proof of (*b*) is similar. Combining (*a*) and (*b*) the conclusion (*c*) is obtained. The lemma is proved.

We are now in a position to state the Theorem.

THEOREM 1. *Let the assumptions of Lemma* 1 *hold, and there exists a function* $H \in C'(R_+, R)$, *constants* q_1, q_2 *and sequences* $\{t'_m\}$, $\{t''_m\}$ *such that*

$$H'(t) = h(t),$$

$$q_1 \leq H(t) \leq q_2,$$

$\lim_{m\to\infty} t'_m = \infty$, $\lim_{m\to\infty} t''_m = \infty$ *and* $H(t'_m) = q_1$, $H(t''_m) = q_2$. *Then every solution of* (1.1) *is oscillatory.*

Proof. Assume that the conclusion of the Theorem is false; without loss of generality, we assume that $x(t) > 0$ is a solution for $t \geq T$.

Setting

$$y(t) \equiv x(t) - H(t)$$

then

$$y'(t) = x'(t) - h(t) = -A(t)A(t)f_1(x(t)) - \int_{0}^{t} D(t-s) f_2(x(s-\tau))\, ds \leq 0$$

for $t \geq T + \tau$.

We show that $y + q_1 > 0$ for large t. Suppose the contrary, since y is non-increasing, we have

$$y + q_1 \leq 0, \qquad \text{for } t \geq T_1.$$

Since $x(x) \equiv y(t) + H(t)$, so

$$x'(t'_m) = y(t'_m) + H(t'_m) = y(t'_m) + q_1 \leq 0$$

for $t'_m > T_1$, which contradicts the assumption $x(t) > 0$. Therefore

$$y(t) + q_1 > 0, \qquad \text{for } t \geq T_1. \tag{2.11}$$

Setting

$$z(t) = y(t) + q_1 \qquad \text{for } t \geq \mathrm{T}_1$$

then

$$z'(t) = y'(t) = -\mathrm{A}(t) f_1(x(t)) - \int_0^t \mathrm{D}(t-s) f_2(x(s-\tau))\, ds. \tag{2.12}$$

Since

$$f_i(x) = f_i(y + \mathrm{H}) \geq f_i(y + q_1) = f_i(z), \qquad i = 1, 2,$$

from (2.12), we have

$$z'(t) \leq -\mathrm{A}(t) f_1(z(t)) - \int_0^t \mathrm{D}(t-s) f_2(z(s-\tau))\, ds, \text{ for } t \geq \mathrm{T}_1 + \tau. \tag{2.13}$$

We note that $z(t) > 0$ for $t \geq \mathrm{T}_1$ so (2.13) contradicts the conclusion (a) of Lemma 1. The proof is completed.

Example. Consider the equation

$$x'(t) + tx(t) + \int_0^t tx\left(s - \frac{\pi}{2}\right) ds = \cos t \tag{2.14}$$

which satisfies all assumptions of Theorem 1, therefore every solution of (2.14) is oscillatory. Indeed, $x(t) = \sin t$ is a solution of (2.14).

Remark 1. Assume that $\tau = 0$ in (2.4), then (2.4) has no oscillatory solution due to uniqueness of solution [3]. Therefore oscillation here is caused by delay.

If (1.1) is linear equation of the form

$$x'(t) - h(t) + \mathrm{A}(t) x(t) + \int_0^t \mathrm{D}(t-s) x(s-\tau)\, ds = 0. \tag{2.15}$$

Then taking a transformation

$$y = x \exp \int_0^t \mathrm{A}(s)\, ds, \tag{2.16}$$

equation (2.15) becomes

$$y(t) - h(t) \exp \int_0^t A(s)\, ds + \tag{2.17}$$
$$+ \int_0^t D(t-s) (\exp \int_{s-\tau}^t A(s_1)(ds_1) y(s-\tau)\, ds = 0.$$

We can obtain the following results by the preceding argument.

LEMMA 2. *Assume that*

i) $A \in C(R_+, R)$;
ii) $D \in C(R_+, R_+)$ *and*

$$\lim_{t\to\infty} \int_{t-\tau}^t \int_0^s D(s-s_1)(\exp \int_{s_1-\tau}^s A(s_2)\, ds_2)\, ds_1\, ds > \frac{1}{e}. \tag{2.18}$$

Then

$$a) \quad x'(t) + A(t)x(t) + \int_0^t D(t-s)\, x(s-\tau)\, ds \le 0 \tag{2.19}$$

has no eventually positive solution;

$$b) \quad x_1(t) + A(t)x(t) + \int_0^t D(t-s)x(s-\tau)\, ds \ge 0 \tag{2.20}$$

has no eventually negative solution;
c) *every solution of the equation*

$$x'(t) + A(t)x(t) + \int_0^t D(t-s)x(s-\tau)\, ds = 0 \tag{2.21}$$

is oscillatory.

THEOREM 2. *Let the assumption of Lemma 2 hold, and let there exist a function* $H \in C'(R_+, R)$, *constants* q_1, q_2 *and sequences* $\{t_m'\}$, $\{t_m''\}$ *such that*

$$H'(t) = h(t) \exp \int_0^t A(s)\, ds$$

$$q_1 \leq H(t) \leq q_2$$

$\lim_{m\to\infty} t'_m = \infty$, $\lim_{m\to\infty} t''_m = \infty$ *and* $H(t'_m) = q_1$, $H(t''_m) = q_2$. *Then every solution of* (2.15) *is oscillatory.*

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References

[1] B.G. Zhang (1985) - *A survey of the oscillation of solution to first order differential equations with deviating arguments*, in «Trends in the Theory and Practice of Non-linear Analysis», Ed. V. Lakshmikantham, pp. 475-483.

[2] J.W. Cushing (1977) - *Integro-differential equations and delay models in population dynamics*, «Lecture notes in biomathematics» (Springer-Verlag).

[3] T.A. Burton (1983) - *Volterra Integral and Differential Equations* (Academic Press).