Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **82** (1988), n.2, p. 291–297.

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<http://www.bdim.eu/item?id=RLINA_1988_8_82_2_291_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1988.

Atti Acc. Lincei Rend. fis.
(8), LXXXII (1988), pp. 291-297

Meccanica dei fluidi. – *On bounded channel flows of viscoelastic fluids.*
Nota di MARSHALL J. LEITMAN e EPIFANIO G. VIRGA, presentata(*) dal Corrisp. T. MANACORDA.

ABSTRACT. — We show that the smooth bounded channel flows of a viscoelastic fluid exhibit the following qualitative feature: Whenever the channel is sufficiently wide, any bounded velocity field satisfying the homogeneous equation of motion is such that if the flow stops at some time, then the flow is never unidirectional throughout the channel. We first demonstrate the qualitative property of the bounded channel flows. Then we show explicitly how a piecewise linear approximation of a relaxation function can admit non-zero bounded channel flows, even if the original function does not.

KEY WORDS: Viscoelastic fluids; Channel flow; Bounded flow.

RIASSUNTO. — *Sui flussi in un canale dei fluidi viscoelastici.* Mostriamo che il flusso in un canale di un fluido viscoelastico ha il seguente carattere: quando la larghezza del canale è più grande di un valore critico, ogni campo limitato di velocità che risolva l'equazione omogenea di moto è tale che, se il fluido è in quiete in qualche istante, quando si muove, il flusso non è mai unidirezionale in tutto il canale. Il risultato è illustrato con degli esempi, che, curiosamente, sussistono quando la funzione di rilassamento degli sforzi è l'interpolazione lineare di una funzione positiva integrabile, ma possono non sussistere, se la funzione di rilassamento è la funzione interpolata.

INTRODUCTION

We show that the smooth bounded channel flows of a viscoelastic fluid exhibit the following qualitative feature: Whenever the channel is sufficiently wide, any bounded velocity field satisfying the homogeneous equation of motion is such that if the flow stops at some time, then the flow is never unidirectional throughout the channel.

Now the class of non-zero bounded channel flows is quite small. For example, there are none if the relaxation function of the material is a decreasing exponential. We conjecture that this is the case whenever the relaxation function is positive, decreasing, integrable and *strictly* convex. Under these hypotheses, in the case of *finite* memory (*i.e.* when the relaxation function has bounded support), arguments following those of Hale [1] (Sects. 13, 14) can be used to show that zero is the on-

(*) Nella seduta del 20 novembre 1987.

ly globally bounded solution. We have not yet proved that this still holds true in the case of infinite memory. We will show, however, that piecewise linear interpolants of such relaxation functions can admit non-zero bounded solutions. Thus a close piecewise linear approximation of the relaxation function can induce a behaviour of the solution which is an artifact of the approximation scheme and is not physically meaningful.

This note is divided into two parts. In the first we demonstrate the qualitative property of the bounded channel flows. In the second we show explicitly how a piecewise linear approximation of a relaxation function can admit non-zero bounded channel flows.

1. A qualitative property for bounded flows

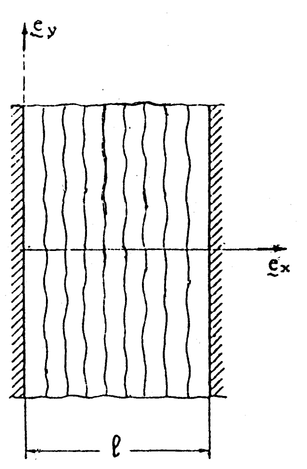
We consider an incompressible viscoelastic fluid whose extra stress tensor field $\mathbf{T}(\mathbf{x}, t)$ is related to the velocity field $\mathbf{v}(\mathbf{x}, t)$ by

$$\mathbf{T}(\mathbf{x}, t) = \int_0^{+\infty} G(\tau)\mathbf{A}(\mathbf{x}, t-\tau)\, d\tau, \qquad \mathbf{A} = 2\,\mathrm{sym}\,\nabla\mathbf{v}.$$

The relaxation function $G\colon R^+ \to R^+$ is here assumed only to be integrable and with finite first moment:

$$\Gamma = \int_0^{+\infty} G(\tau)\,\tau\, d\tau < +\infty .$$

Suppose that the fluid flows in the region between two parallel infinite walls, which in a Cartesian frame lie at $x = 0$ and $x = \ell$ (see Fig.).



Moreover, let the velocity field have the special form

(1.1) $$\mathbf{v}(\mathbf{x},t) = u(x,t)\,\mathbf{e}_y, \qquad x = \mathbf{x}\cdot\mathbf{e}_x,$$

and let the condition of adhesion to the walls

(1.2) $$u(0,t) = u(\ell,t) = 0$$

be satisfied. Such a fluid motion is a channel flow: it is a special *lineal flow*, in the terminology of [2] (see, in particular, Sects. 106 and 111). As any lineal flow, it is isochoric:

(1.3) $$\operatorname{div}\mathbf{v} = 0.$$

If there is no external force, the equation of motion is

(1.4) $$\varrho\frac{\partial \mathbf{v}}{\partial t} = -\nabla p + \int_0^{+\infty} G(\tau)\,\Delta\mathbf{v}(\mathbf{x},t-\tau)\,d\tau,$$

where ϱ is the mass density and $p(\mathbf{x},t)$ is an arbitrary hydrostatic pressure. When $\nabla p \equiv 0$, by (1) and (3), equation (4) becomes

(1.5) $$\varrho\, u_t(x,t) = \int_0^{+\infty} G(\tau)\,u_{xx}(x,t-\tau)\,d\tau.$$

We will confine ourselves to non-zero regular solutions of (5) and (2) which are bounded for all $t\in(-\infty,+\infty)$, if there be any. Precisely, let Σ be the strip

$$\Sigma = \{(x,t)\mid x\in[0,\ell];\ t\in(-\infty,+\infty)\},$$

we will consider solutions belonging to the class of functions

$$U = \{u\in C^2(\Sigma)\mid 0 \leqq \sup|u(x,t)| < +\infty\}.$$

A qualitative property of such solutions is expressed by the following

THEOREM – *Let* $u\in U$ *be a solution of* (5) *and* (2). *If*

(1.6) $$\ell > \pi\sqrt{\frac{2\Gamma}{\varrho}},$$

then the following two statements cannot both be true.

(a) *There is a time, say* t_1, *when the flow stops:*

(1.7) $$u(x,t_1) = 0, \qquad \textit{for all } x\in[0,\ell].$$

(b) *There is a time, say* t_2, *when the flow is unidirectional and not zero at any interior point:*

(1.8) $$\operatorname{sgn}(u(x',t_2)) = \operatorname{sgn}(u(x'',t_2)) \neq 0,$$

for all pairs $x', x'' \in (0,\ell)$.

REMARK 1 – The result just stated has a transparent mechanical interpretation: when the channel is sufficiently wide, if the fluid is at rest at the time t_1, even if $t_1 = -\infty$ or $t_1 = +\infty$, then the lineal flow (1) is never driven in only one way throughout the channel.

PROOF – The idea of the proof is to show that, whenever (6) holds, by assuming both (7) and (8), we arrive at a contradiction.

Before that we need some preliminaries. Let us define the function

$$v(t) = \frac{1}{\ell}\int_0^\ell \sin\left(\frac{\pi x}{\ell}\right) u(x,t)\,dx. \tag{1.9}$$

It follows from (5), (2) and (9) that v(t) solves the equation

$$\dot v(t) = -\mu \int_0^\infty G(\tau) v(t-\tau)\,d\tau \quad \text{with } \mu = \frac{1}{\varrho}\left(\frac{\pi}{\ell}\right)^2. \tag{1.10}$$

From now on a superposed dot indicates differentiation with respect to t.

Let w(t) denote any solution of the adjoint equation

$$\dot w(t) = \mu \int_0^{+\infty} G(\tau) w(t+\tau)\,d\tau. \tag{1.11}$$

Then, by combining (10) and (11), we have

$$w(t)\left(\dot v(t) + \mu \int_0^{+\infty} G(\tau) v(t-\tau)\,d\tau\right) + v(t)\left(\dot w(t) - \mu \int_0^{+\infty} G(\tau) w(t+\tau)\,d\tau\right) = 0. \tag{1.12}$$

Equation (12) can easily be rewritten as

$$\frac{d}{dt}\left[v(t)w(t) - \mu \int_0^{+\infty} G(\tau)\left(\int_t^{t+\tau} w(s) v(s-\tau)\,ds\right) d\tau\right] = 0,$$

and, after a change of variable in the inner integral, as

$$\frac{d}{dt}\left[v(t)w(t) - \mu \int_0^{+\infty} G(\tau)\left(\int_0^{t} w(t+\sigma) v(t+\sigma-\tau)\,d\sigma\right) d\tau\right] = 0.$$

Thus there is a constant a, depending on v and w, such that

$$v(t)w(t) - \mu \int_0^{+\infty} G(\tau)\left(\int_0^{\tau} w(t+\sigma) v(t+\sigma-\tau)\,d\sigma\right) d\tau = a, \tag{1.13}$$

for every $t \in (-\infty, +\infty)$. It is easy to check that if v(t) solves (10), then $v(-t+T)$ solves (11) for any T. By setting $w(t) = v(-t+T)$ in (13), we get

$$v(t)v(-t+T) = \mu \int_0^{+\infty} G(\tau)\left(\int_0^{\tau} v(-t-\sigma+T) v(t+\sigma-\tau)\,d\sigma\right) d\tau + a. \tag{1.14}$$

If $u \in U$ is a non-zero solution of (5) and (2), it follows from (8) and (9) that

(1.15) $$0 < \sup_{(-\infty,+\infty)} |v(t)| < +\infty .$$

Set $M = \sup_{(-\infty,+\infty)} |v(t)|$. By the first of (15), for any fixed $\epsilon \in (0,1)$ there is a $\bar{t} \in (-\infty, +\infty)$ such that

(1.16) $$|v(\bar{t})| \geqq M(1-\epsilon) > 0 .$$

Set $T = 2\bar{t}$ and $t = \bar{t}$ in (14), so that

(1.17) $$M^2[(1-\epsilon)^2 - \mu\Gamma] \leqq |a| .$$

On the other hand, by applying the second of (15) to (14), we get

(1.18) $$|a| \leqq M|v(t)| + M^2\mu\Gamma ,$$

for every $t \in (-\infty, +\infty)$. At $t = t_1$ (7), (9), and (18) imply

(1.19) $$|a| \leqq M^2\mu\Gamma .$$

Now inequality (17) is consistent with (19) only if

$$(1-\epsilon)^2 \leqq 2\mu\Gamma ,$$

or, since ϵ was arbitrary, only if

$$1 \leqq 2\mu\Gamma ;$$

by the definition of μ in (10) this contradicts (6). Our proof is complete.

REMARK 2 – Note that, provided $\Gamma < +\infty$, the theorem holds also if

$$\lim_{\tau\to 0^+} G(t) = +\infty ,$$

as may be the case for some fluids (see *e.g.* [3] , Sect. 6).

2. NON-ZERO BOUNDED CHANNEL FLOWS

First consider the specific relaxation function G:

$$G(\tau) = \gamma e^{-\alpha\tau}, \quad \tau \geqq 0 ,$$

where γ and α are positive constants. For this G the reduced equation (1.10) for v is just a second order ordinary differential equation with constant coefficients whose characteristic values have negative real parts. The only globally bounded solution of (1.10) is zero. Since this argument also holds for every Fourier coefficient of u, the only $u \in U$ which solves (1.5) and (1.2) is $u \equiv 0$.

Now fix $T > 0$, and for $n = 0, 1, 2, \ldots$ set $\gamma_n = G(nT)$, with G any integrable relaxation function. The piecewise linear interpolant G_T is given by

$$G_T(\tau) = \gamma_n + (\tau - nT)\left(\frac{\gamma_{n+1} - \gamma_n}{T}\right), \quad \tau \in [nT, (n+1)T], \quad n=0, 1, 2, \ldots$$

We seek a solution of (1.5) and (1.2) of the form

$$u(x,t) = u_0 \sin\left(\frac{N\pi x}{\ell}\right)\sin\left(\frac{2\pi t}{T}\right).$$

We will have such a solution whenever

$$\int_0^{+\infty} G_T(\tau)\cos\left(\frac{2\pi\tau}{T}\right)d\tau = 0$$

and

$$\int_0^{+\infty} G_T(\tau)\sin\left(\frac{2\pi\tau}{T}\right)d\tau = \varrho\frac{2\pi}{T}\left(\frac{\ell}{N\pi}\right)^2.$$

A straightforward computation shows that the first condition is always satisfied and the second is equivalent to

$$\frac{T}{2\pi}G(0) = \varrho\frac{2\pi}{T}\left(\frac{\ell}{N\pi}\right)^2.$$

If it happens that for some positive integer N

$$2\sqrt{\frac{\varrho}{G(0)}}\,\frac{\ell}{T} = N,$$

then

$$u(x,t) = \frac{u_0}{2}\left\{\sin\left[\frac{2\pi}{T}\left(\sqrt{\frac{\varrho}{G(0)}}\,x+t\right)\right]+\sin\left(\frac{2\pi}{T}\left(\sqrt{\frac{\varrho}{G(0)}}\,x-t\right)\right]\right\}$$

is a bounded solution of (1.5) and (1.2) when G is replaced by G_T.

REMARK 3 - If G is integrable, then surely $\sum_{n=0}^{\infty}\gamma_n$ converges. If G is decreasing and has finite first moment Γ, then

$$\Gamma_T = \int_0^{+\infty} G_T(\tau)\,\tau\,d\tau = \frac{T^2}{6}G(0) + T^2\sum_{n=1}^{\infty} n\gamma_n.$$

Observe that in this case $T^2\sum_{n=1}^{\infty} n\,\gamma_n \leqq \Gamma + T^2\sum_{n=1}^{\infty}\gamma_n < \infty$. If (1.6) holds for Gr, then $N > \frac{2\sqrt{3}\pi}{3} > 1$, which is certainly consistent with our result in Sect. 1.

REMARK 4 - The case when G is positive, decreasing, and convex, with finite support (finite memory) is well understood. Hale [1] (Sects. 13, 14) shows precisely under what additional conditions a continuous initial state can approach a non-zero bounded (periodic) solution.

REMARK 5 - The classical incompressible Newtonian fluid corresponds to the case in which the measure $G(\tau)\,d\tau$ is replaced by an atomic measure with a single atom of magnitude η at zero. Thus equation (1.5) is replaced by

$$\varrho\, u_t(x,t) = \eta u_{xx}(x,t)\,, \qquad \eta > 0\,.$$

As is shown by again considering the Fourier coefficients of u, the latter equation and (1.2) have no non-zero solution belonging to U.

ACKNOWLEDGEMENT

One of the authors (M.J.L.) would like to thank the *Consiglio Nazionale delle Ricerche* for its support during the preparation of this note.

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