Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

# Rendiconti

Sebastiano Giambò, Annunziata Palumbo

## Extended irreversible thermodynamics in hypoelasticity

*Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8*, Vol. **82** (1988), n.2, p. 253–258.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1988_8_82_2_253_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1988.

*Atti Acc. Lincei Rend. fis.*
*(8), LXXXII (1988), pp. 253-258*

**Meccanica dei continui.** — *Extended irreversible thermodynamics in hypoelasticity.* Nota di SEBASTIANO GIAMBÒ e ANNUNZIATA PALUMBO, presentata (*) dal Corrisp. T. MANACORDA.

ABSTRACT. - The constitutive equations of rate type for a class of thermo-hypo-elastic materials are derived within the framework of the extended irreversible thermodynamics.

KEY WORDS: Extended thermodynamics; Hypoelasticity.

RIASSUNTO. - *Termodinamica irreversibile estesa in ipoelasticità.* Si determinano, nell'ambito della termodinamica estesa, le equazioni costitutive per una classe di materiali ipoelastici.

## 1. INTRODUCTION

Recently, Olsen and Bernstein [1] have determined a class of hypoelastic materials, which obey the laws of thermodynamics, restricting the constitutive equation for heat flux to Fourier's law. However such an approach predicts the propagation of thermal signals with infinite speed.

The purpose of this paper is to propose a phenomenological theory for a class of thermo-hypo-elastic materials which removes this paradox and which is, at the same time, strictly consistent with the extended irreversible thermodynamics.

## 2. BALANCE EQUATIONS

As well known, the motion of the medium is described by the balance equations of mass, momentum and energy:

$$(1)\qquad \begin{cases} \dfrac{d\varrho}{dt} + \varrho\, d_{kk} = 0 \\[2ex] \varrho \dfrac{dv_i}{dt} = -\, t_{ij,j} \\[2ex] \varrho \dfrac{du}{dt} = -\, t_{ij} d_{ij} - q_{i,i} \end{cases}$$

where $\varrho$, $v_i$ and u are, respectively, the material density, the velocity and the specific

(*) Nella seduta del 20 novembre 1987.

internal energy of the medium. Moreover $t_{ij}$ is the stress tensor and $q_i$ is the heat flux while $d_{ij}$ denotes the rate of deformation tensor defined by:

$$d_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i}) \tag{2}$$

with the standard cartesian tensor notation.

On introduction of the deviatory stress and of the pressure by means of the following relations:

$$\begin{cases} s_{ij} = t_{ij} - p\delta_{ij} \\ p = \frac{1}{3} t_{kk}, \end{cases}$$

the system of balance equations (1) must be supplemented by constitutive laws for $q_i$, $s_{ij}$ and $p$.

## 3. The generalized Gibbs equation

According to the ideas of extended irreversible thermodynamics [2] – [11], we assume that the Gibbs function G depends on the absolute temperature T, on the stress tensor, as well as on the dissipative flux $q_i$. Supposing that the medium is isotropic, we postulate that G is a function of $t_{ij}$ and $q_i$ through the invariants p, $s = (s_{ij}s_{ij})^{1/2}$ and $q = (q_iq_i)^{1/2}$, i.e. $G = G(T, p, s, q)$.

Then, the specific internal energy u is defined by the Legendre transformation:

$$u = G + TS - pV - s\beta - q\alpha \tag{4}$$

where by analogy with the classical thermodynamics, we defined the specific entropy and the specific volume by:

$$\begin{cases} -\frac{\partial G}{\partial T} = S(T, p, s, q) \\ \frac{\partial G}{\partial p} = V(T, p, s, q) \end{cases} \tag{5}$$

while $\beta$ and $\alpha$ are given by:

$$\begin{cases} \frac{\partial G}{\partial s} = \beta(T, p, s, q) \\ \frac{\partial G}{\partial q} = \alpha(T, p, s, q) \end{cases} \tag{6}$$

From (4)-(6), differentiation of u yields the generalized Gibbs relation:

(7) $$\frac{Du}{Dt} = T\frac{DS}{Dt} - p\frac{DV}{Dt} - s\frac{D\beta}{Dt} - q\frac{D\alpha}{Dt}$$

where $\frac{D}{Dt}$ represents the objective derivative [12]:

(8) $$\frac{D}{Dt}t_{ij\ldots} = \frac{d}{dt}t_{ij\ldots} + t_{ik\ldots}\omega_{kj} + t_{jk\ldots}\omega_{ki} + \ldots,$$

$\omega_{ij} = \frac{1}{2}(v_{i,j} - v_{j,i})$ being the rotation tensor. Of course the use of the operator $\frac{D}{Dt}$ is needed in order to obtain objective equations.

## 4. Evolution equations for dissipative fluxes

Inserting the mass and energy balances into the generalized Gibbs equation, we get:

(9) $$\varrho T\frac{DS}{Dt} - s\varrho\frac{D\beta}{Dt} - q\varrho\frac{D\alpha}{Dt} + s_{ij}\tilde{d}_{ij} + q_{i,i} = 0$$

where $\tilde{d}_{ij} \equiv d_{ij} - \frac{1}{3}d_{kk}\delta_{ij}$ is the deviator of the rate of deformation tensor.

Now, we can express the entropy balance equation (9) in the standard form:

(10) $$\varrho\frac{DS}{Dt} + \operatorname{div} J = \mathfrak{s},$$

in which the entropy flux and entropy production are given respectively by:

(11) $$J = \gamma(T, p, s)q$$

(12) $$\mathfrak{s} = T^{-1}\left[(T\gamma - 1)q_{i,i} - s_{ij}\tilde{d}_{ij} + s\varrho\frac{D\beta}{Dt} + q\varrho\frac{D\alpha}{Dt} + Tq_i\gamma_{,i}\right].$$

As a first approach, we suppose the absence of coupling between the heat flux and the stress tensor; therefore the positive character of $\mathfrak{s}$ leads to the following restrictions:

(13) $$\begin{cases} \gamma = T^{-1} \\ s_{ij}\tilde{d}_{ij} = s\varrho\dfrac{D\beta}{Dt} \\ qT\varrho\dfrac{D\alpha}{Dt} - q_iT_{,i} \geqq 0 \end{cases}$$

Now, we note that only the dissipative flux $q_i$ has the property of vanishing at the equilibrium, while this requirement is not fulfilled by $t_{ij}$ whose equilibrium value is not zero (cf. [13], [14], [15]).

Then, we expand the quantities defined by (5) and (6) around their local equilibrium values:

$$(14)\qquad \begin{cases} S = S_{eq}(T, p, s) + \left(\dfrac{\partial S}{\partial q}\right)_{eq} q + 0(2) \\ V = V_{eq}(T, p, s) + \left(\dfrac{\partial V}{\partial q}\right)_{eq} q + 0(2) \\ \beta = \beta_{eq}(T, p, s) + \left(\dfrac{\partial \beta}{\partial q}\right)_{eq} q + 0(2) \\ \alpha = \left(\dfrac{\partial \alpha}{\partial q}\right)_{eq} q + 0(2) = \tilde{\alpha}(T, p, s)\, q + 0(2) \end{cases}$$

From the equality of the second derivatives of G with respect to T and q, p and q, s and q, it follows that:

$$\left(\frac{\partial S}{\partial q}\right)_{eq} q = -\left(q\frac{\partial \tilde{\alpha}}{\partial T}\right)_{eq} = 0$$

$$\left(\frac{\partial V}{\partial q}\right)_{eq} q = \left(q\frac{\partial \tilde{\alpha}}{\partial p}\right)_{eq} = 0$$

$$\left(\frac{\partial \beta}{\partial q}\right)_{eq} q = \left(q\frac{\partial \tilde{\alpha}}{\partial s}\right)_{eq} = 0,$$

therefore up to second order S, V and $\beta$ reduce to their local equilibrium values and $\tilde{\alpha}$ is equal to a constant $\alpha_0$.

Consequently, the inequality $(13)_3$ leads to the following evolution equation for the heat flux:

$$(15)\qquad q_i = \chi\left(\varrho \alpha_0 T \frac{Dq_i}{Dt} - T_{,i}\right) \quad (\chi > 0).$$

The next step is to determine thermo-hypo-elastic constitutive equations which satisfy $(13)_2$.

First of all we note that:

$$(16)\qquad \frac{\partial G}{\partial s_{ij}} = \frac{\beta}{s} s_{ij},$$

Then eq. $(13)_2$ takes the form (cf. [1])

$$(17)\qquad s_{ij}\tilde{d}_{ij} = \varrho s_{ij} \frac{D}{Dt}\left(\frac{\partial G}{\partial s_{ij}}\right).$$

Making use of the simplifying assumption $\dfrac{\partial^2 G}{\partial p \partial s} = 0$ [1], which implies:

$$
(18) \qquad \begin{aligned} \varrho &= \varrho(T, p) \\ \beta &= \beta(T, s) \end{aligned}
$$

and differentiating the relation (16) with respect to time, eq. (17) becomes:

$$
(19) \qquad s_{ij}\tilde{d}_{ij} = \left[\frac{\varrho}{s}\left(\beta_s - \frac{\beta}{s}\right) s_{ij}\frac{Ds}{Dt} + \varrho\frac{\beta}{s}\frac{Ds_{ij}}{Dt} + \varrho\frac{\beta_T}{s}s_{ij}\frac{DT}{Dt}\right] s_{ij},
$$

where the subscript denotes partial derivative with respect to that variable.

Equation (19) holds if we assume the constitutive equation:

$$
(20) \qquad \tilde{d}_{ij} = \frac{\varrho}{s}\left(\beta_s - \frac{\beta}{s}\right) s_{ij}\frac{Ds}{Dt} + \varrho\frac{\beta}{s}\frac{Ds_{ij}}{Dt} + \varrho\frac{\beta_T}{s}s_{ij}\frac{DT}{Dt}.
$$

Now, eliminating $\dfrac{Ds}{Dt}$ from (20) by means of (19), we deduce the following constitutive equation for the deviatory stress:

$$
(21) \qquad \frac{Ds_{ij}}{Dt} = \frac{1}{\varrho}\left[\frac{1}{s^2}\left(\frac{1}{\beta_s} - \frac{s}{\beta}\right) s_{ij}s_{kl}d_{kl} + \frac{s}{\beta}\left(d_{ij} - \frac{1}{3}d_{kk}\delta_{ij}\right)\right] - \frac{\beta_T}{\beta_s}\frac{s_{ij}}{s}\frac{DT}{Dt}.
$$

In order to derive the rate equation for the pressure, we write the continuity equation $(1)_1$ in a different way taking account of $(5)_2$ and $(18)_1$. Whereupon we get:

$$
(22) \qquad \frac{Dp}{Dt} = -\frac{1}{\varrho_p}\left(\varrho d_{kk} + \varrho_T\frac{DT}{Dt}\right).
$$

Equations (21) and (22) can be combined to yeld the thermo-hypo-elastic equations:

$$
(23) \qquad \begin{aligned} \frac{Dt_{ij}}{Dt} = \frac{1}{\varrho}\left[\frac{1}{s^2}\left(\frac{1}{\beta_s} - \frac{s}{\beta}\right) s_{ij}s_{kl}d_{kl} + \frac{s}{\beta}d_{ij} - \left(\frac{\varrho^2}{\varrho_p} - \frac{1}{3}\frac{s}{\beta}\right) d_{kk}\delta_{ij}\right] + \\ - \left(\frac{\varrho_T}{\varrho_p}\delta_{ij} - \frac{\beta_T}{\beta_s}\frac{s_{ij}}{s}\right)\frac{DT}{Dt} \end{aligned}
$$

Finally, in order to obtain a complete set of dynamic governing equations for the medium, it is necessary to transform the derivative $\dfrac{Du}{Dt}$ in $(1)_3$. Setting $c_{p,s} = T\dfrac{\partial S}{\partial T}$, the generalized Gibbs equation (6) can be written as:

$$
(24) \qquad \begin{aligned} \frac{Du}{Dt} = \left(c_{p,s} + T\frac{\beta_T^2}{\beta_s}\right)\frac{DT}{Dt} + \varrho_T\frac{T}{\varrho^2}\frac{Dp}{Dt} - \frac{1}{\varrho}\left(1 + \frac{T}{s}\frac{\beta_T}{\beta_s}\right) s_{ij}d_{ij} + \\ - \frac{p}{\varrho}d_{kk} - \alpha_0 q_i\frac{Dq_i}{Dt}, \end{aligned}
$$

therefore the first law of thermodynamics $(1)_3$ takes the following form:

$$\varrho\left(c_{p,s}+T\frac{\beta_T^2}{\beta_s}-\frac{T}{\varrho^2}\frac{\varrho_T^2}{\varrho_p}\right)\frac{DT}{Dt}-T\frac{\varrho_T}{\varrho_p}d_{kk}-\frac{T}{s}\frac{\beta_T}{\beta_s}s_{ij}d_{ij}-\frac{q_i}{\chi T}(q_i+\chi T_{,i})+q_{i,i}=0. \tag{25}$$

Hence our system is completely characterized by the following thirteen unknown variables

$$T,\ p,\ v_i,\ s_{ij},\ q_i$$

which are completely determined by the equations (25), (22), $(1)_2$, (21) and (15).

ACKNOWLEDGEMENT

This work was supported by G.N.F.M. - C.N.R. and M.P.I. of Italy.

## REFERENCES

[1] OLSEN E.T. and BERNESTEIN B., *Arch Rat. Mech. Anal., 86, 291* (1984).
[2] MULLER I., *J. Phys., 198, 329* (1967).
[3] LEBON G., *Int. J. Engng. Sci., 18, 727* (1980).
[4] LEBON G., *Bull. Acad. Roy. Sc. Belgique, 64, 45* (1978).
[5] LEBON G., JOU D. and CASAS-VAZQUEZ J., *J. Phys. A, 13, 273* (1980).
[6] LEBON G. and RUBI J.M., *J. Non Equil. Thermodyn., 5, 285* (1980).
[7] LEBON G. and JOU D., *J. Chem. Phys., 77, 970* (1982).
[8] MULLER I., *Arch. Rat. Mech. Anal., 41, 319* (1971).
[9] SHIHLIU I. and MULLER I., *Arch. Rat. Mech. Anal., 83, 285* (1983).
[10] GRECO A. and MULLER I., *Arch. Rat. Mech. Anal., 85, 279* (1984).
[11] SHIH LIU I., *Arch. Rat. Mech. Anal., 88, 1* (1985).
[12] JAUMANN G., *Akad. Wiss. Wien Sitzungsber IIA, 120, 385* (1911).
[13] TRUSDELL C., *Arch. Rat. Mech. Anal., 4, 83* (1985).
[14] NARIBOLI G.A., *J. Math. Anal. Appl., 8, 57* (1964).
[15] MAUGIN G.A., *Commun. Math. Phys., 53, 233* (1977).