Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Giancarlo Bilotti, Leonardo Giliberti

A simple mechanical model to analyse the rocking and sliding response of rigid blocks to earthquakes

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **82** (1988), n.2, p. 241–252.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1988_8_82_2_241_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1988.

Atti Acc. Lincei Rend. fis.
(8), LXXXII (1988), pp. 241-252

Meccanica. — *A simple mechanical model to analyse the rocking and sliding response of rigid blocks to earthquakes.* Nota di GIANCARLO BILOTTI e LEONARDO GILIBERTI, presentata (*) dal Socio E. GIANGRECO.

ABSTRACT. - In order to study the effects of earthquakes on tombstones and monumental columns in recent years the dynamical analysis of rigid blocks subjected to ground accelerations has interested many researchers. Mainly, the rocking motion has been investigated and many numerical difficulties have been pointed out in such analysis [1-2-3-4].

Some computational advantages can be obtained by modelling the bonding between two blocks or between block and foundation by means of an elastic layer of Winkler's springs not reacting in traction [5].

Nevertheless, many experimental observations show that the sliding phenomena can be relevant so that an analisys that only takes into account the rocking can be inadequate for technical purposes.

Recently, some mechanical models with elasto-plastic horizontal springs have been proposed in [6-7-8] which take into account such displacements.

Essentially, the sliding phenomenum is related to a friction idea. In this sense the aim of the present paper is to analyse the dynamical problem of a rigid block unilaterally constrained with friction against a Winkler's foundation; the analysis is carried out as a plane problem in the context of the linearized theory. Regarding the friction forces, we assume that they are related to the horizontal velocity of the block base through an analytical relationship, which regularizes the classical Coulomb's law. In this way we obtain a very simple mechanical model, which allows us to develop a comprehensive analysis of the block motion, including both rocking and sliding without too much analytical troubles. In particular in this paper we will present the first numerical results which we obtained referring to a rigid block subjected to horizontal seismic loads. They show both the effectiveness of the mechanical model and the efficiency of the algorithm solution proposed, which utilizes a semi-implicit integration method of the dynamical equations.

KEY WORDS: Rigid Blocks; Rocking; Sliding; Earthquakes.

RIASSUNTO - *Un modello meccanico semplice per analizzare i fenomeni di "rocking" e di "sliding" di blocchi rigidi investiti da sisma.* Negli ultimi anni l'analisi dinamica di blocchi rigidi soggetti ad accelerazioni al piede ha attirato l'attenzione di numerosi ricercatori nell'intento di valutare gli effetti dei terremoti sulle colonne monumentali.

Principalmente è stato studiato il fenomeno di "rocking", che, peraltro, ha messo in risalto le numerose difficoltà numeriche connesse con tale analisi [1-2-3-4]. Alcuni vantaggi numerici possono essere conseguiti schematizzando il contatto tra due blocchi o tra blocco e fondazione con un letto di molle elastiche alla Winkler non reagenti a trazione [5].

D'altra parte molte osservazioni sperimentali mostrano l'importanza del fenomeno di "sliding", per cui una analisi che tenga conto del solo "rocking" può dimostrarsi inadeguata per gli scopi tecnici.

Recentemente sono stati proposti alcuni modelli meccanici con molle orizzontali elastoplastiche [6-7-8], che permettono di tenere conto degli slittamenti orizzontali.

(*) Nella seduta del 12 dicembre 1987.

Essenzialmente il fenomeno di "sliding" è però legato al concetto di attrito, per cui, pur inquadrandosi in questo stesso ambito, il presente lavoro si propone un'analisi dinamica di un blocco rigido vincolato monolateralmente con attrito su una fondazione alla Winkler. L'analisi viene sviluppata come problema piano nell'ambito di una teoria linearizzata.

L'espressione che lega le forze di attrito alla velocità orizzontale della base del blocco è ottenuta regolarizzando opportunamente la classica legge di Coulomb. In tal modo si riesce a costruire un modello meccanico abbastanza semplice in grado di sviluppare un'analisi agevole di entrambi i fenomeni di "rocking" e di "sliding".

In particolare nel lavoro vengono presentati i primi risultati numerici ottenuti che illustrano l'efficacia del modello meccanico e l'efficienza dell'algoritmo di soluzione, che utilizza un metodo di integrazione numerica di tipo semi-implicito delle equazioni del moto.

1. Introduction

In order to study the effects of earthquakes on tombstones and monumental columns in recent years the dynamical analysis of rigid blocks subjected to ground accelerations has interested many researchers. Mainly, the rocking motion has been investigated and many numerical difficulties have been pointed out in such analysis [1-2-3-4].

Some computational advantages can be obtained by modelling the bonding between two blocks or between block and foundation by means of an elastic layer of Winkler's springs not reacting in traction [5].

Nevertheless, many experimental observations show that the sliding phenomena can be relevant so that an analysis which only takes into account the rocking can be inadequate for technical purposes.

Recently, some mechanical models with elasto-plastic horizontal springs have been proposed in [6-7-8] which take into account such displacements.

Essentially, the sliding phenomenum is related to a friction idea. In this sense the aim of the present paper is to analyse the dynamical problem of a rigid block unilaterally constrained with friction against a Winkler's foundation; the analysis is carried out as a plane problem in the context of the linearized theory. Regarding the friction forces, we assume that they are related to the horizontal velocity of the block base through an analytical relationship which regularizes the classical Coulomb's law. In this way we obtain a very simple mechanical model which allows us to develop a comprehensive analysis of the block motion, including both rocking and sliding without too much analytical troubles. In particular, in this paper we shall present the first numerical results obtained which refer to a rigid block subjected to horizontal and vertical ground accelerations. They show both the effectiveness of the mechanical model and the efficiency of solution algorithm proposed.

2. Formulation of the problem

In fig. 1 we consider the plane problem of a rigid body Ω unilaterally constrained against a foundation Φ.

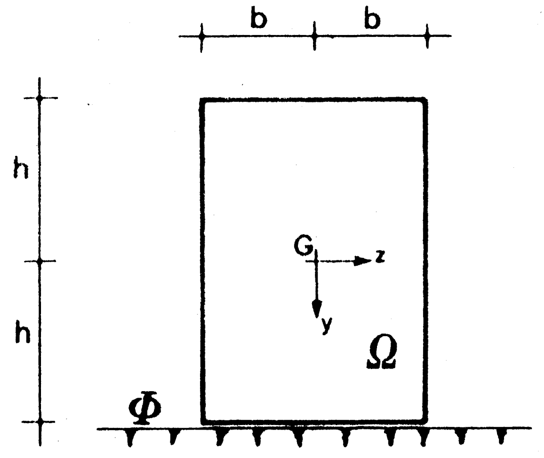

Fig. 1

Let

$\mathbf{u} = (v, w)$	be the displacement of the centre of mass,
ω	the body rigid rotation,
M	the body mass,
I_G	the mass moment of inertia of the body about G,
$\mathbf{f}^{(a)}$	the active force acting on the body,
$-\mathbf{f}^{(c)}$	the reactive force acting on the body surface in contact with the foundation,
$m^{(a)}$	the moment about G of active forces acting on the body,
$-m^{(c)}$	the moment about G of reactive forces,
N, T	the vertical and horizontal component of $\mathbf{f}^{(c)}$, respectively.

As is well known, the dynamical equilibrium equations of the body Ω can be written in the form [7]:

(2.1a) $$M\ddot{\mathbf{u}} = \mathbf{f}^{(a)} - \mathbf{f}^{(c)},$$

(2.1b) $$I_G\ddot{\omega} = m^{(a)} - m^{(c)},$$

where the point denotes the time derivative.

Regarding the reactive forces exerted by the foundation on the body, we shall distinguish between vertical and horizontal components. More precisely, we shall assume that in the vertical direction the foundation is realized by means of Winkler's springs not reacting in traction. Consequently, the vertical component N can be put in the form:

(2.2) $$N = \int_{-b}^{+b} K v_b^+ \, dz,$$

where K is the springs stiffness per unit of length, v_b the vertical component of the displacement exhibited by the generic point lying on the body base, the symbol $(\cdot)^+$ denotes the positive part of the real numer $(\cdot)$.

We recall [9] that, in the context of the linear theory, the displacement v_b is related to v and ω through the relation:

$$v_b = v - \omega z . \tag{2.3}$$

For what concerns the horizontal reactive forces, we shall assume that the component T of $\mathbf{f}^{(c)}$ is related to the velocity horinzontal component w of the body base by means of the following relation:

$$T = \gamma \dot{w}_b + (\epsilon - \gamma)(\dot{w}_b - \dot{w}_0)^+ , \tag{2.4}$$

where:

ϵ, γ are two assigned positive constants,
f is the coefficient of friction.

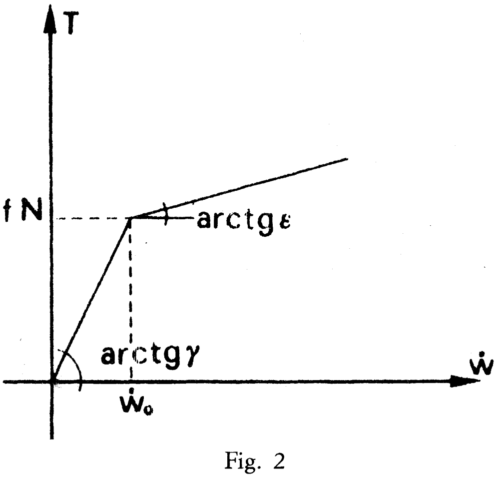

Fig. 2

Relation (2.4) is depicted in fig. 2.

Evidently eq. (2.4) represents a regularization of the classical Coulomb's law: it is easy to recognize that this one can be approximated by our law in the limit $\epsilon \to 0$ and $\gamma \to \infty$.

After these preliminaries, we can now formulate the dynamical problem under examination:

"Let $\mathbf{f}^{(a)}$, $m^{(a)}$, $\mathbf{u}_0$, ω_0, $\dot{\mathbf{u}}_0$, $\dot{\omega}_0$ be assigned quantities; find $\mathbf{u}(t)$, $\omega(t)$ defined on the interval $[0, t_1]$ which satysfies eqs. (2.1) and initial conditions:

$$\mathbf{u}(0) = \mathbf{u}_0 , \qquad \omega(0) = \omega_0 , \tag{2.5a}$$

$$\dot{\mathbf{u}}(0) = \dot{\mathbf{u}}_0 , \qquad \dot{\omega}(0) = \dot{\omega}_0 ." \tag{2.5b}$$

The mathematical features of this problem will be analysed in a future paper; in this work we are mainly interested in analysing its physical and numerical aspects.

3. SOLUTION ALGORITHM

In order to formulate the solution algorithm that we have utilized in solving problem (2.1), firstly we develop some topics regarding the evaluations of the reactive forces N, T, $m^{(c)}$.

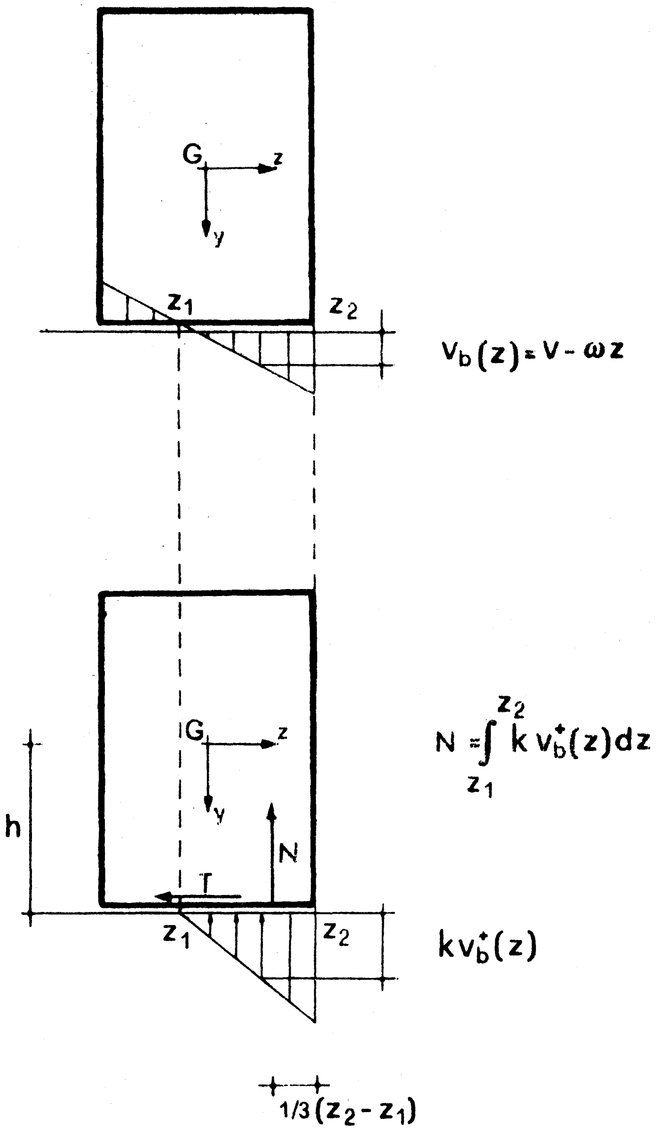

Fig. 3a

Fig. 3b

Relatively to the vertical reaction N, we observe that eq. (2.2) can also be put in the form (fig. 3a):

$$N = \int_{z_1}^{z_2} K(v - \omega z)\,dz = K\,|z|_{z_1}^{z_2}\,v - K\left|\frac{z}{2}\right|_{z_1}^{z_2}\omega, \qquad (3.1)$$

z_1 and z_2 being the coordinates of the end points of the contact region between the body Ω and the foundation Φ.

Likewise, the expression (2.4) of T can be rewritten as:

$$T = \gamma(\dot{w} + \dot{\omega} h) + \alpha(\epsilon - \gamma)(\dot{w} + \dot{\omega} h) - \alpha\beta(\epsilon - \gamma)\dot{w}_0, \tag{3.2}$$

where we have put:

$$\alpha = \begin{cases} 0 & \text{if } |\dot{w} + \dot{\omega} h| - \dot{w}_0 \le 0 \\ 1 & \text{if } |\dot{w} + \dot{\omega} h| - \dot{w}_0 > 0 \end{cases} \tag{3.3}$$

and

$$\beta = \begin{cases} -1 & \text{if } \dot{w} + \dot{\omega} h < 0 \\ +1 & \text{if } \dot{w} + \dot{\omega} h > 0. \end{cases} \tag{3.4}$$

Finally, for that which concerns $m^{(c)}$, it is easy to verify (fig. 3b) that:

$$m^{(c)} = -K \int_{-b}^{+b} (v - \omega z)^+ z\, dz + Th =$$

$$= -K \int_{z_1}^{z_2} (v - \omega z) z\, dz + [\gamma(\dot{w} + \dot{\omega} h) + \alpha(\epsilon - \gamma)(\dot{w} + \dot{\omega} h) - \alpha\beta(\epsilon - \gamma)\dot{w}_0] h = \tag{3.5}$$

$$= -K \left|\frac{z^2}{2}\right|_{z_1}^{z_2} v + K \left|\frac{z^3}{3}\right|_{z_1}^{z_2} \omega + [\gamma + \alpha(\epsilon - \gamma)] h \dot{w} + [\gamma + \alpha(\epsilon - \gamma)] h^2 \dot{\omega} - \alpha\beta(\epsilon - \gamma)\dot{w}_0 h.$$

Putting now:

$$\chi = \gamma + \alpha(\epsilon - \gamma), \tag{3.6a}$$

$$\lambda = \alpha\beta(\epsilon - \gamma), \tag{3.6b}$$

$$K_1 = K|z|_{z_1}^{z_2}, \tag{3.6c}$$

$$K_2 = K \left|\frac{z^2}{2}\right|_{z_1}^{z_2}, \tag{3.6d}$$

$$K_3 = K \left|\frac{z^3}{3}\right|_{z_1}^{z_2}, \tag{3.6e}$$

the system of differential equations (2.1) can be rewritten in a scalar form as follows:

$$M\ddot{v} = f_y^{(a)} - K_1 v + K_2 \omega, \tag{3.7a}$$

$$M\ddot{w} = f_z^{(a)} - \chi\dot{w} - \chi h \dot{\omega} + \lambda \dot{w}_0, \tag{3.7b}$$

$$I_G \ddot{\omega} = m^{(a)} + K_2 v - \lambda h \dot{w} - K_3 \omega - \chi h^2 \dot{\omega} + \lambda h \dot{w}_0, \tag{3.7c}$$

where $f_y^{(a)}$ and $f_z^{(a)}$ are the components of $f^{(a)}$ along the y and z axes, respectively.

3.1 *Time integration*

The time integration of eqs. (3.7) is carried out by means of a semi-implicit method [10], already utilized in [11] in studying some elastodynamic problems with unilateral constraints.

Velocities and accelerations are approximated by central finite differences and the solution at time $t + \Delta t$ is calculated by the dynamical equilibrium equations (3.7) relative to time t, assuming for the reaction vertical component N its value at time $t + \vartheta\Delta t$, where ϑ is fixed number between 0 and 1.

In more explicit terms, the step-by-step solution algorithm is the following:

$$\mathbf{U}_{t+\Delta t} = \hat{\mathbf{M}}(\mathbf{U}_\vartheta)^{-1}\hat{\mathbf{q}}(\mathbf{U}_\vartheta), \tag{3.8}$$

where we put:

$$\mathbf{U}_t = \begin{bmatrix} v(t) \\ w(t) \\ \omega(t) \end{bmatrix}, \tag{3.9}$$

$$\mathbf{U}_\vartheta = \mathbf{U}_t + \vartheta(\mathbf{U}_{t+\Delta t} - \mathbf{U}_t), \tag{3.10}$$

$$\hat{\mathbf{M}}(\mathbf{U}_\vartheta) = \begin{bmatrix} \left(\dfrac{M}{\Delta t^2} + \vartheta K_1\right), \ 0, \ -\vartheta K_2 \\ 0, \ \left(\dfrac{M}{\Delta t^2} + \dfrac{\chi}{2\Delta t}\right), \ \dfrac{\chi h}{2\Delta t} \\ -\vartheta K_2, \ \dfrac{\chi h}{2\Delta t}, \ \left(\dfrac{I_G}{\Delta t^2} + \dfrac{\chi h^2}{2\Delta t} + \vartheta K_3\right) \end{bmatrix}, \tag{3.11}$$

$$\hat{\mathbf{q}}(\mathbf{U}_\vartheta) = \begin{bmatrix} f_y^{(a)} - \dfrac{M}{\Delta t^2} v(t-\Delta t) + \left[\dfrac{2M}{\Delta t^2} + K_1(\vartheta - 1)\right] v(t) + K_2(1-\vartheta)\,\omega(t) \\ f_z^{(a)} - \left(\dfrac{M}{\Delta t^2} - \dfrac{\chi}{2\Delta t}\right) w(t-\Delta t) + \dfrac{2M}{\Delta t} w(t) + \dfrac{\chi h}{2\Delta t}\,\omega(t+\Delta t) + \lambda \dot{w}_0 \\ m^{(a)} - \left(\dfrac{I_G}{\Delta t^2} - \dfrac{\chi h^2}{2\Delta t}\right)\omega(t-\Delta t) - \left[K_3(1-\vartheta) - \dfrac{2I_G}{\Delta t}\right]\omega(t) + \\ + \dfrac{\chi h}{2\Delta t} w(t-\Delta t) + K_2(1-\vartheta) v(t) + \lambda h \dot{w}_0 \end{bmatrix}. \tag{3.12}$$

Eq. (3.8) is valid if $t > 0$; in particular at time $t = 0$ we put:

$$\mathbf{U}_{\Delta t} = \mathbf{U}_0 + \dot{\mathbf{U}}_0 \Delta t. \tag{3.13}$$

In order to calculate the non-linear terms in eqs. (3.11) and (3.12), the following iterative procedure is employed:

$$\mathbf{U}^{(K)}_{t+\Delta t} = \hat{\mathbf{M}}(\mathbf{U}^{(K-1)}_{\vartheta})^{-1}\hat{\mathbf{q}}(\mathbf{U}^{(K-1)}_{\vartheta}), \tag{3.14}$$

i.e. at k-nth step the matrices $\hat{\mathbf{M}}$ and $\hat{\mathbf{q}}$ are evaluated by using the values of $\mathbf{U}_{\vartheta}$ founded at (k-1)-nth step; in particular at this step we calculate the parameters (3.6) which appear in eqs. (3.11) and (3.12).

For k = 1, we put:

$$\mathbf{U}^{(0)}_{\vartheta} = \mathbf{U}_t, \tag{3.15}$$

$$\dot{\mathbf{U}}^{(0)}_{\vartheta} = \frac{1}{2\Delta t}(\mathbf{U}_t - \mathbf{U}_{t-\Delta t}). \tag{3.16}$$

4. Numerical results and conclusions

The solution algorithm previously described is here utilized to get some numerical results relative to the motion of a rigid block subjected to horizontal seismic loads.

The block dimensions are b = 0.5 m and h = 1.0 m, while the soil parameters are the following: $K = 10^9$ Kg m^{-2}, $\epsilon = 10^{-9}$ Kg sec m^{-1}, $\gamma = 10^9$ Kg sec m^{-1}.

Such values allow us to simulate a rigid support with Coulomb's friction forces.

More precisely, the load condition consists of a vertical force corresponding to the self-weight ($f^{(a)}_y$ = 4000 Kg) and in a horizontal one (fig. 4) which corresponds to the seismic action ($f^{(a)}_z$ = 1500 Kg).

The time-step used in the numerical integration of eqs. (3.7) is $\Delta t = 1 \times 10^{-4}$ sec, while the parameter ϑ is taken to be 0.5.

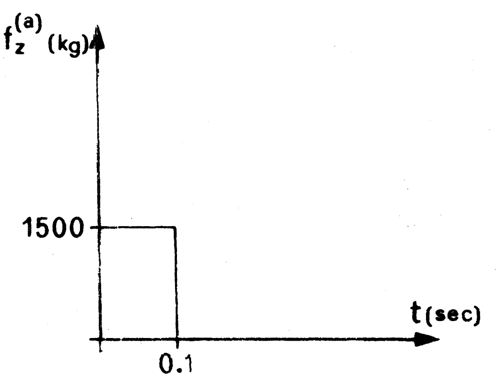

Fig. 4

Figs. 5-6-7 show the numerical results that we have obtained by choosing the following values of the friction's coefficient: $f = 0.6, 0.3, 0.2$. The initial conditions taken into account are $w(0) = \omega(0) = 0$, $v(0) = f_y^{(a)}/2bK$.

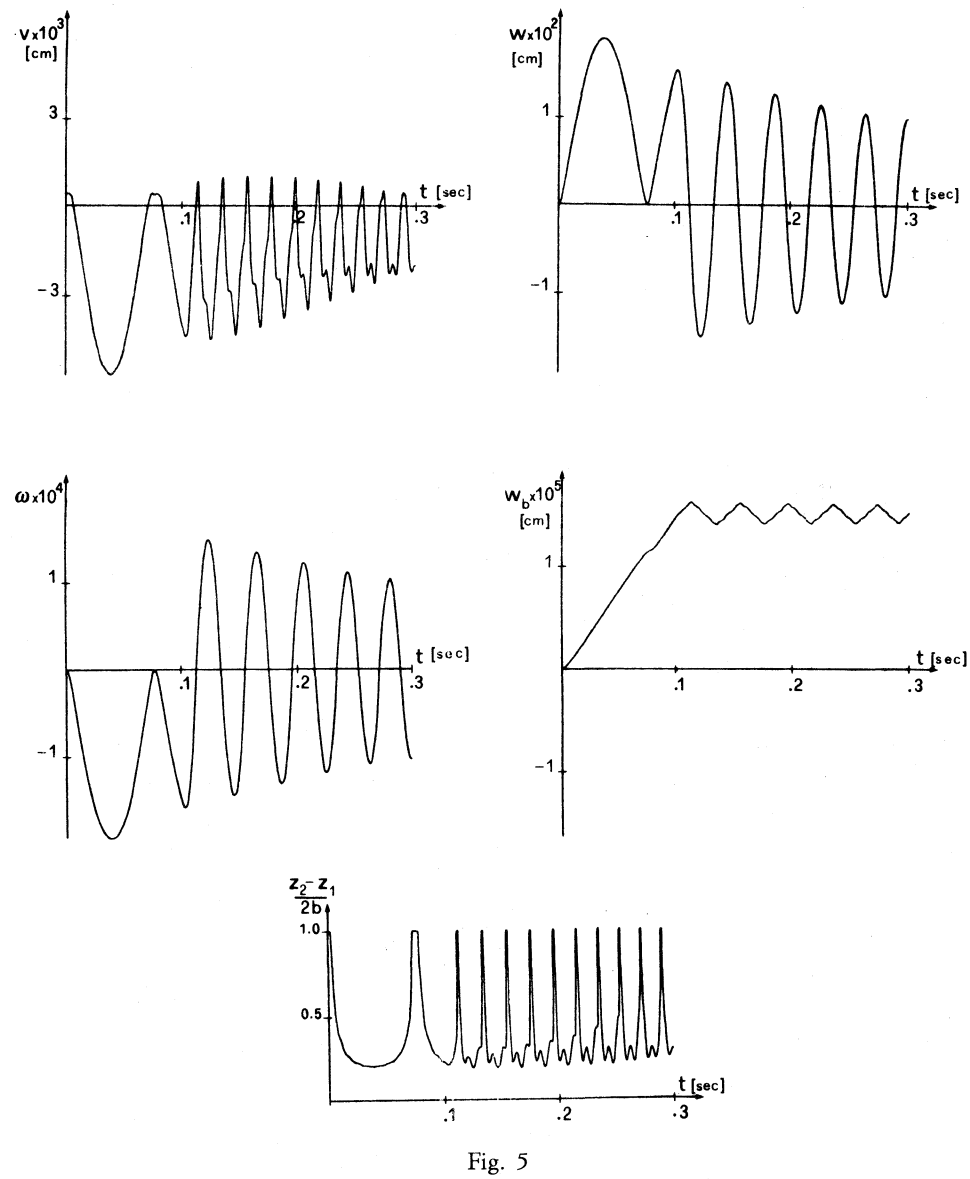

Fig. 5

The plots refer to the vertical and horizontal displacement of the block centre, to the rigid rotation and to the horizontal displacement exhibited by the base of the block respectively. Further in the same figures we show the percentage of the contact area versus the time t.

The solution algorithm proposed appears very efficient and in all cases examin-

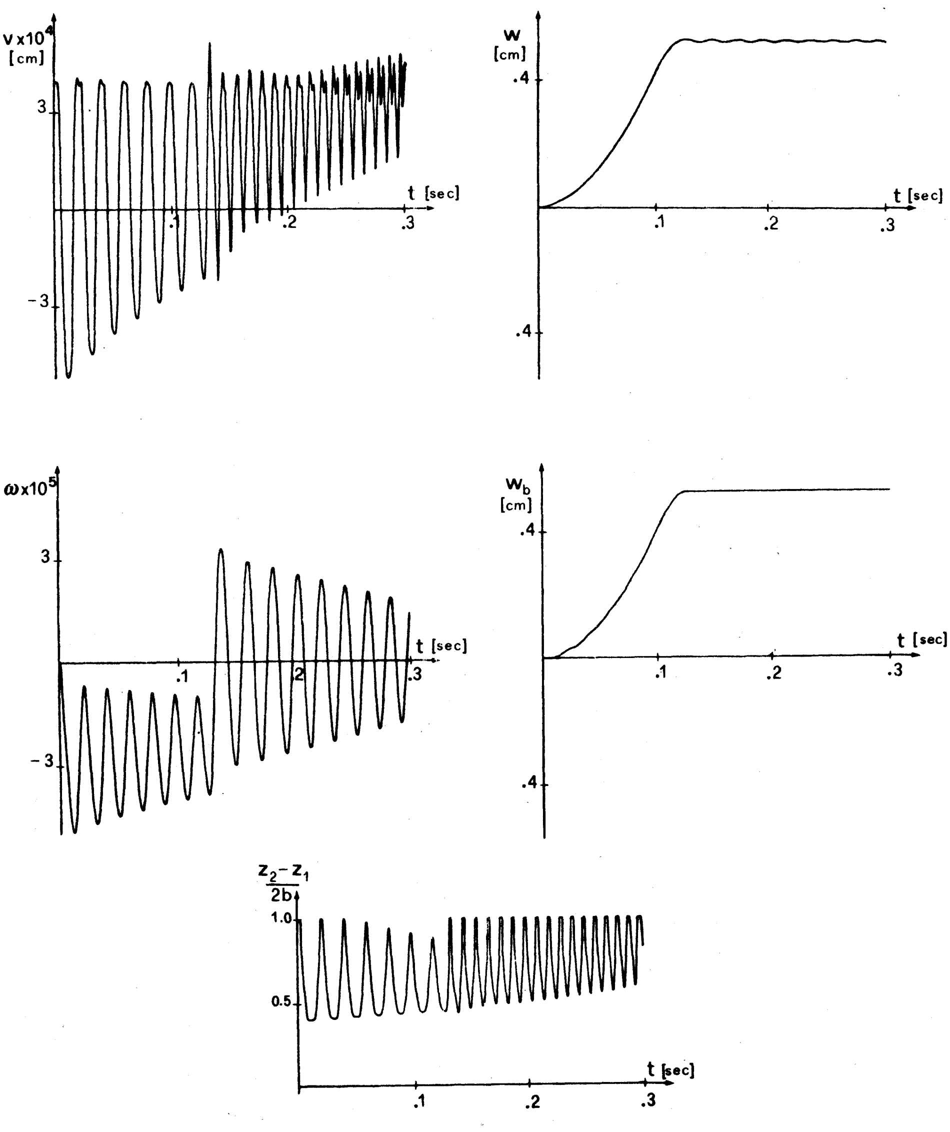

Fig. 6

ed the convergence is reached in few iterations (2-4 step). In the same cases the use of Newmark's scheme gave problems of numerical stability.

Figs. 5-6-7 clearly show the influence of the friction forces and make clear the importance of the block sliding as far as the friction's coefficient diminishes.

Due to the lack of other numerical results in the literature, no assessment of accuracy can be made.

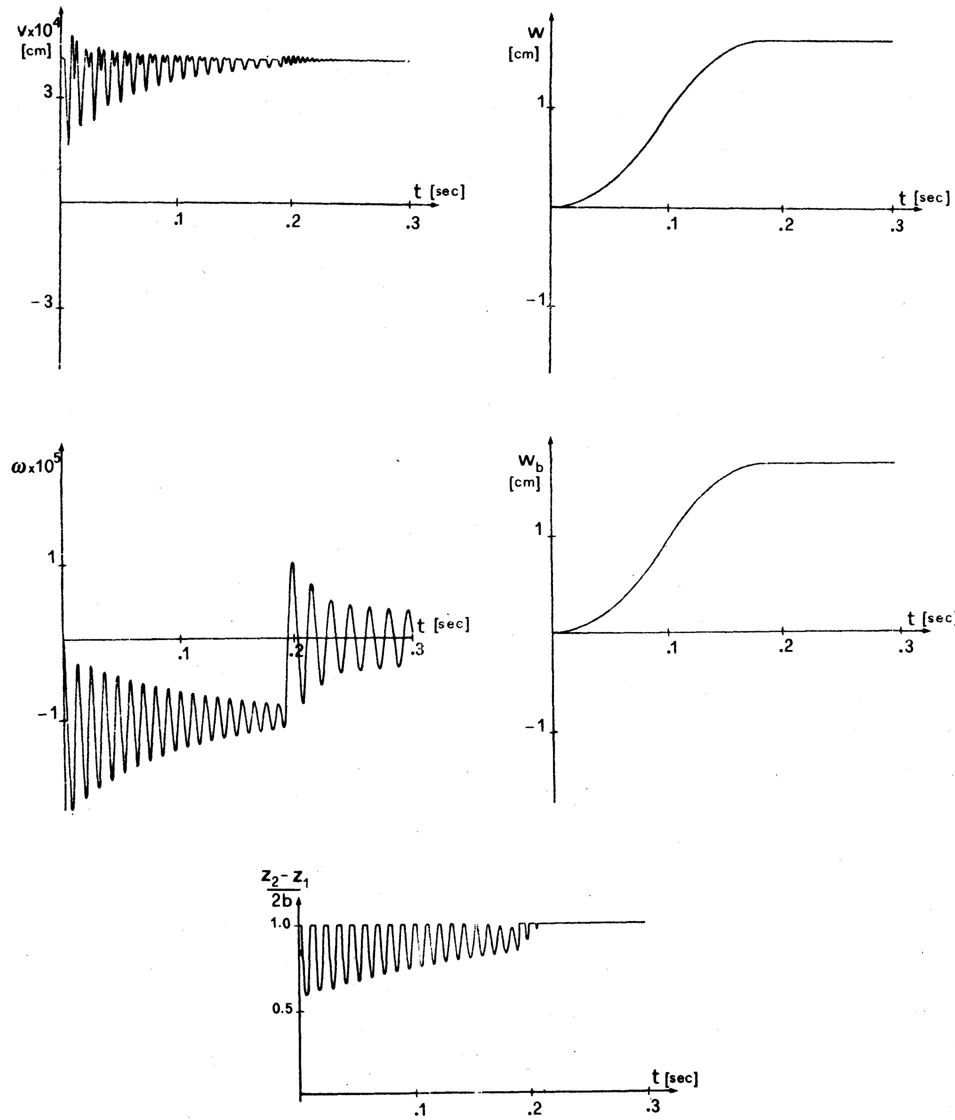

Fig. 7

In the near future from one side we will analyse the case of more blocks and from the other one we will examine thoroughly the mathematical topics related to the differential problem (2.1).

5. References

[1] Housuer G.W.: « The Behaviour of Inverted Pendulum Structures During Earthquakes », Bull. Seismo. Soc. Am. 53, 404-417, 1963.

[2] Aslam M., Godden W.G. and Scalise D.T.: « Earthquake Rocking Response of Rigid Bodies », J. Struct. Div., ASCE, 106, 377-392, 1980.

[3] Muto K. *et al.*, « 3-D Non-Linear Dynamic Analysis Stacked Blocks », II Conf. on Dynamic Response of Structures, ASCE, pp. 917-930, 1981.

[4] Giannini R.: « Analisi Dinamica dei Sistemi di Blocchi Sovrapposti », II Convegno L'Ingegneria Sismica in Italia, pp. 6.47-6.42, Rapallo 1984.

[5] Blasi C. and Spinelli P.: « Un Metodo di Calcolo Dinamico per Sistemi Formati da Blocchi Rigidi Sovrapposti », Ingegneria Sismica, Anno III, N. 1 gen. apr., 1986.

[6] Angotti F. and Toni P.: « Oscillazioni non Lineari di Corpi Rigidi su Semispazi Elastici Monolaterali », Congresso Nazionale AIMETA, pp. 134-143, Genova 7-8-9 ottobre 1982.

[7] Andreaus U., D'Asdia P. and Di Paolo A.: « Sul Comportamento dei Giunti tra Conci Lapidei: un Modello di Giunto Attritivo », III Convegno Nazionale A.I.I.S., pp. 665-690, Roma, ottobre 1987.

[8] Blasi C. and Spinelli P.: « Sul Consolidamento Antisismico di Colonne Lapidee: un Esempio », III Convegno Nazionale A.I.I.S., pp. 95-106, Roma, ottobre 1987.

[9] Gurtin M.: An Introduction to Continuum Mechanics, Academic Press, 1981.

[10] Glowinski R., Lions J.L. and Trémolieres R.: Analyse Numérique des Inequation Variationnelles, Dunod-Paris, 1976.

[11] Ascione L. and Bilotti G.: « The Dynamical Unilateral Contact Problem with Friction of a Beam with Hetényi's Foundation: a Finite Element Approach », «Atti dell'Accademia Nazionale dei Lincei», Rend. fis. (8), LXXXII (1988), 61-73.