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Teoria dei gruppi. — *On automorphisms fixing subnormal subgroups of soluble groups.* Nota di SILVANA FRANCIOSI e FRANCESCO DE GIOVANNI, presentata (*) dal Socio G. ZAPPA.

ABSTRACT. – The group $Aut_{sn}G$ of all automorphisms leaving invariant every subnormal subgroup of the group G is studied. In particular it is proved that $Aut_{sn}G$ is metabelian if G is soluble, and that $Aut_{sn}G$ is either finite or abelian if G is polycyclic.

KEY WORDS: Automorphisms; Subnormal subgroups; Soluble groups.

RIASSUNTO. – *Sugli automorfismi che fissano i sottogruppi subnormali dei gruppi risolubili.* Si prende in esame il gruppo $Aut_{sn}G$ degli automorfismi che fissano tutti i sottogruppi subnormali del gruppo G. In particolare si prova che se G è un gruppo risolubile il gruppo $Aut_{sn}G$ è metabeliano, mentre se G è policiclico il gruppo $Aut_{sn}G$ risulta abeliano oppure finito.

INTRODUCTION

An automorphism of a group G is called a *power automorphism* if it maps every subgroup of G onto itself; the set PAutG of all power automorphism of G is a normal subgroup of the automorphism group AutG of G. The structure of PAutG was described by Cooper in [1] (see also [6]).

More generally, it is of interest to investigate the structure of the group of all automorphisms of a group G which leave some specified subgroups of G invariant. In this direction the group Aut_nG of all automorphisms of the group G which fix every normal subgroup of G was studied in [2].

Our object here is to obtain information about the group $Aut_{sn}G$ of all automorphisms of G which leave every subnormal subgroup of G invariant. Clearly $Aut_{sn}G$ is a normal subgroup of AutG and, if G is soluble, also $Aut_{sn}G$ is soluble, since its commutator subgroup stabilizes the derived series of G.

Our main results are the following:

THEOREM A. – *Let G be a group. If G is either hyperabelian or hypoabelian, then the group $Aut_{sn}G$ is metabelian.*

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THEOREM B. – *Let G be a polycyclic group. Then the group* $Aut_{sn}G$ *is either finite or abelian.*

Here recall that a group G is called *hyperabelian* if it has an ascending normal series with abelian factors, and G is *hypoabelian* if it has a descending (normal) series with abelian factors.

Theorems A and B show how the behaviour of $Aut_{sn}G$ for a soluble group G is similar to that of a soluble T-group, i.e. a soluble group in which normality is a transitive relation. Such groups were described by Gaschütz [3] and Robinson [6]. However, in spite of these similarities, the structure of $Aut_{sn}G$ can be quite different from that of soluble T-groups, as it is shown by examples in section 3.

In the investigation of $Aut_{sn}G$ an important role is played by the *Wielandt subgroup* $\omega(G)$ of G, i.e. the intersection of the normalizers of all subnormal subgroups of G. In particular it will be shown that $Aut_{sn}G$ acts trivially on the factor group $G/\omega(G)$.

Our notation is standard and can be found in [8]. In particular:

InnG is the group of all inner automorphisms of the group G,

$K \ltimes H$ is the semidirect product of H and K, where H is the normal factor.

A power automorphism τ of a group G is called *homogeneous* if elements of the same order of G are mapped by τ to the same power. Recall that every power automorphism of an abelian group is homogeneous (see [1]).

PROOF OF THE THEOREMS

The following two lemmas are needed in the proofs of the Theorems.

LEMMA 2.1. – *For each group G, the group* $Aut_{sn}G$ *acts trivially on the factor group* $G/\omega(G)$.

Proof. The group $(\mathrm{Inn}G)\cap(Aut_{sn}G)$ is the set of all inner automorphisms of G fixing every subnormal subgroup, and hence it corresponds to $\omega(G)/Z(G)$ in the natural isomorphism between InnG and $G/Z(G)$. Clearly we have that $[\mathrm{Inn}G, Aut_{sn}G] \leq (\mathrm{Inn}G)\cap(Aut_{sn}G)$, so that $[G, Aut_{sn}G] \leq \omega(G)$.

LEMMA 2.2. – *Let G be a group and let* Θ *be a soluble subgroup of* $Aut_{sn}G$ *which is normalized by InnG. Then* Θ *is metabelian.*

Proof. The set H of all elements of G which induce on G an automorphism belonging to Θ is a normal subgroup of G, which is contained in the Wielandt subgroup $\omega(G)$ of G. In particular H is a soluble T-group, and hence it is metabelian (see [6]). Since $[\mathrm{Inn}G,\Theta] \leq (\mathrm{Inn}G)\cap\Theta$, it follows that $K = [G,\Theta]$ is contained in H. Therefore the commutator subgroup $L = K'$ of K is an abelian normal subgroup of G.

Every element τ of Θ acts on L as a homogeneous power automorphism, and hence $[x,\tau]$ acts trivially on L for each element x of G. Thus $L \leq Z(K)$ and K is nilpotent. It follows that Θ induces on K a group of power automorphisms. Then Θ' stabilizes the series $G \geq K \geq 1$, and Θ is metabelian (see [4]).

Proof of theorem A

Suppose first that G is hyperabelian, and write $\Gamma = Aut_{sn}G$. The set H of all elements of G fixed by $\Gamma^{(2)}$ is a normal subgroup of G. Assume that H is properly contained in G, and let K/H be a non-trivial abelian normal subgroup of G/H. Then Γ induces a group of power automorphisms on K/H, so that Γ' acts trivially on K/H and $\Gamma^{(2)}$ stabilizes the series $K > H \geq 1$. Therefore $\Gamma^{(3)}$ acts trivially on K. The group Γ^* of all automorphisms induced by Γ on K is a soluble subgroup of $Aut_{sn}K$, and by Lemma 2.2 it follows that Γ^* is metabelian. Hence $\Gamma^{(2)}$ acts trivially on K, which is impossible. Therefore H = G and Γ is metabelian.

Suppose now that G is hypoabelian, and write again $\Gamma = Aut_{sn}G$. Let N be the smallest normal subgroup of G such that $\Gamma^{(2)}$ acts trivially on G/N. Since Γ induces a group of power automorphisms on N/N′, then Γ' acts trivially on N/N′ and $\Gamma^{(2)}$ stabilizes the series $G \geq N \geq N'$. Therefore $\Gamma^{(3)}$ acts trivially on G/N′. The group Γ^* of all automorphisms induced by Γ on G/N′ is a soluble subgroup of $Aut_{sn}(G/N')$, and by Lemma 2.2 it follows that Γ^* is metabelian. Hence $\Gamma^{(2)}$ acts trivially on G/N′ and N = N′. This shows that N = 1 and Γ is metabelian.

REMARK. – (a) Theorem A becomes false for SI-groups in general, as can be seen by the consideration of a non-soluble locally nilpotent T-group (see [5] for examples of such groups).
(b) It is well-known that in any T-group G the derived series stops with $G^{(2)}$. This is not true for $Aut_{sn}G$ when G is an arbitrary group. In fact, if G is one of the orthogonal groups $O_8^+(3), O_8^+(5), O_8^+(7)$ and $\Gamma = Aut_{sn}G$, then $\Gamma^{(2)} \neq \Gamma^{(3)}$.

Proof of theorem B

The Wielandt subgroup A of G is a finitely generated soluble T-group, and hence it is either finite or abelian (see [6]). Write $\Gamma = Aut_{sn}G$ and let Θ be the centralizer of A in Γ. Clearly Γ is polycyclic, as a soluble group of automorphisms of the polycyclic group G (see [8] Part 1, p. 82).

Suppose first that A is finite, and let e be the exponent of A. By Lemma 2.1 Θ acts trivially on G/A and A, so that $\tau^e = 1$ for each $\tau \in \Theta$. Therefore Θ is finite. This shows that Γ is finite in this case.

Suppose now that A is infinite, and hence abelian. If $\Gamma = \Theta$, then Γ is obviously abelian. Assume that Θ is properly contained in Γ. The factor group Γ/Θ has order 2, since it is isomorphic with a group of power automorphisms of the non-periodic abelian group A. If μ is any element of $\Gamma \backslash \Theta$, then μ acts as the inversion on A and hence has order 2; it follows that $\Gamma = \langle\mu\rangle \ltimes \Theta$.

For each $\tau \in \Theta$ and $x \in G$, we have that $x^{\mu} = xa$, $x^{\tau} = xb$, where a and b are elements of A, and hence $x^{\mu\tau\mu} = (xa)^{\tau\mu} = (x^{\tau}a)^{\mu} = (xba)^{\mu} = xb^{-1} = x^{\tau^{-1}}$. Then $\Gamma' = [\Theta,\mu] = \Theta^2$ and Γ/Γ' is finite.

Let p be an odd prime number, and assume that the factor group G/A^p is infinte. Then the Fitting subgroup F/A^p of G/A^p is a nilpotent non-periodic subgroup. Since μ acts as a power automorphism on F/A^p and as the inversion on A/A^p, it follows that F/A^p is abelian (see [1], Corollary 4.2.3) and μ acts as the inversion on it. This is impossible, since μ acts trivially on F/A. This contradiction shows that G/A^p is finite; in particular G/A is a finite group. Since A normalizes every subnormal subgroup of G, we obtain that the subnormal subgroups of G have bounded defect. Then it is well-known that the intersection of an arbitrary collection of subnormal subgroups of G is subnormal, and hence G is finite-by-nilpotent by a result of Robinson [7]. If N is a finite normal subgroup of G with nilpotent factor group G/N, then Γ induces a group of power automorphisms on G/N, so that Γ' acts trivially on $G/(A \cap N)$. Since Γ' acts trivially on A, it follows that Γ' has finite exponent, and hence it is finite. Therefore Γ is finite. The proof of Theorem B is complete.

Some counterexamples

The first example shows that $\mathrm{Aut}_{sn}G$ is not always a T-group.

Example 1. – Let p be an odd prime, and let $G = \langle x, y, | x^{p^2} = y^2 = (xy)^2 = 1\rangle$. Then every subnormal subgroup of G is characteristic, and hence $\Gamma = \mathrm{Aut}_{sn}G = \mathrm{Aut}G$. Since every automorphism of $\langle x\rangle$ is induced by an automorphism of G, it follows that p divides the order of Γ/Γ'. Furthermore Γ' acts trivially on $G/\langle x\rangle$ and $\langle x\rangle$, so that it is a p-group. Thus it is well-known that Γ is not a T-group (see [6]).

If G is a soluble p-group of finite exponent, then G is a Baer group (see [8] Part 2, p. 49), so that $\mathrm{Aut}_{sn}G = \mathrm{PAut}G$ is abelian, and it is a p-group of finite exponent, provided that G is not abelian (see [1], Corollary 5.1.2). The situation is different for p-groups of infinte exponent.

Example 2. – Let G be the wreath product of a p^{∞}-group and a group of order p, and let B be the base group of G; then $G = \langle x\rangle \ltimes B$ is the semidirect product of B and a group $\langle x\rangle$ of order p. It is easy to see that every subnormal subgroup of G either is contained in B or contains [B,x]. Let $\mu \neq 1$ be a p-adic integer such that $\mu \equiv 1(p)$, and let n be the least positive integer such that $\mu \not\equiv 1(p^n)$). The positions

$$x^\delta = x, \quad b^\delta = b^\mu \quad \text{for all} \quad b \in B$$

define an automorphism δ of G acting as μ on B and G/[B,x]. Therefore δ belongs to $\mathrm{Aut}_{sn}G$. If τ is an inner automorphism of G which is induced by an element u of B and has order p^n, then obviously $\tau \in \mathrm{Aut}_{sn}G$ and $\delta\tau \neq \tau\delta$ since $u^{-1}u \notin Z(G)$. This shows that $(\mathrm{Inn}G) \cap (Z(\mathrm{Aut}_{sn}G))$ is finite, and therefore $\mathrm{Aut}_{sn}G$ is not nilpotent. Of course in this case $\mathrm{Aut}_{sn}G$ is not periodic.

It is well-known that a torsion-free soluble T-group is abelian ([6]); the following example shows that $\mathrm{Aut}_{sn}G$ can be non-nilpotent when G is a torsion-free soluble group.

Example 3. – Let H be the additive group of rational numbers (multiplicatively written), and let a be the automorphism of H defined by the position $h^a = h^n$ for all $h \in H$, where n is an integer such that $|n| > 1$. Put $G = \langle a \rangle \ltimes H$. For each element $x \in G/H$ we have that $[H,x] = H$. This shows that every subnormal subgroup of G either is contained in H or contains H. Therefore the automorphism δ of G defined by the positions

$$a^\delta = a, \quad h^\delta = h^{-1} \quad \text{for all} \quad h \in H$$

belongs to $\mathrm{Aut}_{sn}G$. If τ is the inner automorphism of G induced by an element $k \neq 1$ of H, we have that $\delta\tau \neq \tau\delta$ since $Z(G) = 1$. Therefore the group of inner automorphisms of G induced by elements of H has trivial intersection with the centre of $\mathrm{Aut}_{sn}G$, and hence $\mathrm{Aut}_{sn}G$ is not nilpotent. Note also that the group $\mathrm{Aut}_{sn}G$ is not torsion-free.

The last example shows that $\mathrm{Aut}_{sn}G$ is not always locally supersoluble.

Example 4. – Let H be the additive group of rational numbers (multiplicatively written), and let a be the inversion of H. Put $G = \langle a \rangle \ltimes H$. Clearly every proper subnormal subgroup of G is contained in H. Let u be a non-trivial element of H and let r be a rational number greater than 1; then the positions

$$a^\sigma = au, \quad h^\sigma = h^r \quad \text{for all} \quad h \in H$$

define an automorphism σ of G which belongs to $\mathrm{Aut}_{sn}G$. If τ is the inner automorphism of G induced by a and $\delta = \tau\sigma\tau$, it is not difficult to see that the group $\Lambda = \langle \sigma, \delta \rangle$ is a torsion-free normal subgroup of $\Theta = \langle \sigma, \tau \rangle$. Let $\langle \mu \rangle$ be a Θ-invariant cyclic subgroup of Λ; then there exist an element v of H and a positive rational number s such that $a^\mu = av$ and $h^\mu = h^s$ for all $h \in H$. Since the conjugates of μ by τ and σ are equal either to μ or to μ^{-1}, it follows that μ is the identity. Therefore Λ does not contain a cyclic non-trivial Θ-invariant subgroup. This shows that the group $\mathrm{Aut}_{sn}G$ is not locally supersoluble.

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