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On a definition for formations

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Algebra. — *On a definition for formations.* Nota (*) di MARCO BARLOTTI, presentata dal Socio G. ZAPPA.

ABSTRACT. — By constructing appropriate faithful simple modules for the group GL (2 , 3) , the author shows that certain « local » definitions for formations are not equivalent.

KEY WORDS: Group; Formation; GL (2 , 3); SL (2 , 3); Faithful simple modle.

RIASSUNTO. — *Su un criterio di definizione « locale » per le formazioni.* Con la costruzione di moduli semplici fedeli per GL (2 , 3) si mostra che due criteri per la definizione « locale » di formazioni dati in [1] non sono equivalenti.

1.

All the groups considered in this paper are assumed to be finite and soluble.

For every positive integer n , let $\mathscr{F}(n)$ be a non-empty formation and let $\mathscr{X}(n)$ be a (possibly empty) isomorphism-closed class of groups. Let $\mathscr{F}_1$ denote the class of all the groups G such that, for every prime p and for any chief factor H/K of order divisible by p , the automorphism group induced on H/K by the $\mathscr{F}(p)$-residual of G belongs to $\mathscr{X}(p)$; and let $\mathscr{F}_2$ denote the class of all the groups G such that, for every positive integer n and for any chief factor H/K of order n, the automorphism group induced on H/K by the $\mathscr{F}(n)$-residual of G belongs to $\mathscr{X}(n)$. The resulting classes $\mathscr{F}_1, \mathscr{F}_2$ depend of course upon our choice of the $\mathscr{F}(n)$'s and the $\mathscr{X}(n)$'s; we call a class of groups $\delta^{(1)}$-definable ($\delta^{(2)}$-definable) if it can appear as $\mathscr{F}_1$ (as ($\mathscr{F}_2$) in the above construction with an appropriate choice of the $\mathscr{F}(n)$'s and of the $\mathscr{X}(n)$'s .

In [1] we proved that a $\delta^{(1)}$-definable or $\delta^{(2)}$-definable class is a (Σ-closed) formation and remarked that every $\delta^{(1)}$-definable formation is also $\delta^{(2)}$-definable. We now want to show that there exist $\delta^{(2)}$-definable formations which are not $\delta^{(1)}$-definable, i.e. that we do get more formations by allowing the $\mathscr{F}(n)$'s and the $\mathscr{X}(n)$'s to range over all the prime-powers.

In section 3 we exhibit faithful simple modules V_2 and V_4 of dimension 2 and 4 respectively for the group $G_0 = SL(2, 3)$ over the field with 11 elements. Denote by G_2 and G_4 the semi-direct products $[V_2]\, G_0$ and $[V_4]\, G_0$;

(*) Pervenuta all'Accademia il 18 settembre 1987.

let $\mathscr{F}_2$ be the $\delta^{(2)}$-definable formation obtained by choosing $\mathscr{F}(n) = \{1\}$ for every n, $\mathscr{X}(11^2) = \{G_0\}$, $\mathscr{X}(11^4) = \varnothing$ and $\mathscr{X}(n) = \mathscr{S}$ (the class of all groups) for any other positive integer n: it is clear that $G_2 \in \mathscr{F}_2$ and $G_4 \notin \mathscr{F}_2$. Now let $\mathscr{F}_1$ be any $\delta^{(1)}$-definable formation obtained by a choice of $\mathscr{F}^*(p)$'s and $\mathscr{X}^*(p)$'s such that $G_2 \in \mathscr{F}_1$. Then $\mathscr{X}^*(11)$ is not empty: in fact the group of automorphisms induced on V_2 by the $\mathscr{F}^*(11)$-residual R of G_2 (a normal subgroup of G_0) must belong to $\mathscr{X}^*(11)$; now, whatever $\mathscr{F}^*(11)$ may be, the $\mathscr{F}^*(11)$-residual of G_4 induces on V_4 the same automorphism group as R does on V_2, whence $G_4 \in \mathscr{F}_1$ and $\mathscr{F}_1 \neq \mathscr{F}_2$.

Sections 2 and 3 are devoted to some results on the representations of GL (2, 3) and SL (2, 3) which justify the above example. The attention on the group GL (2, 3) was motivated by the last example of chapter 16 in [3].

The author wants to thank professor T.O. Hawkes for a friendly discussion of the subject.

2.

We fix some notation for this section.

Let p be an odd prime such that $p \equiv 2 \pmod 3$, and let **F** be the field with p elements. Let G denote either the group GL (2, 3) or its normal subgroup SL (2, 3); in both cases, G has a unique normal subgroup of order 8 which we shall denote by Q (Q is isomorphic to the quaternion group, and the quotient group G/Q is isomorphic in the former case to the non-abelian group of order 6 and in the latter case to the cyclic group of order 3).

All the tensor products will be over **F**.

In this section we prove the following

THEOREM. *Suppose that there exists a faithful simple* **F**G*-module* V *of dimension* 2 *over* **F**; *let* U *be a simple* **F**G*-module such that Ker* (G *on* U) $=$ Q (*such a module exists and has dimension* 2 *over* **F** *by* [5], II.3.10 *if* G $=$ SL (2, 3) *and by* [2] 2.2 *if* G $=$ GL (2, 3)).

Then the tensor product $V \otimes U$ *is a faithful simple* **F**G*-module* (*and has dimension* 4 *over* **F**).

Proof. Let $W = V \otimes U$.

We start with some considerations on the restrictions V_Q and W_Q.

Suppose that V_Q were the direct sum of two one-dimensional **F**Q-modules: since such modules cannot be faithful for Q (which is not abelian), they would both be centralised by the unique minimal normal subgroup of Q; hence V_Q wouldn't be a faithful **F**Q-module, which is a contradiction since V is by hypothesis faithful for G. So

(2.1) V_Q is a faithful simple **F**Q-module.

Moreover, since Q acts trivially on U and acts faithfully on V, we have that

(2.2) W_Q is a faithful **F**Q-module.

Now let $\{v_1, v_2\}$ be a basis for V and $\{u_1, u_2\}$ be a basis for U over **F**; since $\{v_1 \otimes u_1, v_2 \otimes u_1, v_1 \otimes u_2, v_2 \otimes u_2\}$ is a basis for W over **F**, we have that $W = (V \otimes u_1) \oplus (V \otimes u_2)$ as a **F**-vector space. Since Q acts trivially on U, each $V \otimes u_i$ is a **F**Q-submodule of W_Q isomorphic to V_Q; thus

(2.3) W_Q is a direct sum of two faithful simple FQ-modules isomorphic to V_Q.

Now we consider the **F**G-module W. Since Q contains the unique minimal normal subgroup of G,

(2.4) any **F**G-submodule of W which is faithful for **Q** is a faithful **F**G-module.

Thus, by (2.2), W is faithful for G. Suppose by way of contradiction that W is not a simple **F**G-module: since any decomposition of W in a direct sum of **F**G-submodules yields a decomposition of W_Q in a direct sum of **F**Q-submodules, by (2.3) and the theorem of Jordan-Hölder we have, using again (2.4), that

(2.5) if W is not a simple **F**G-module, then

$$W = V_1 \oplus V_2$$

and each V_i $(i = 1, 2)$ is a faithful simple **F**G-module of dimension 2.

Denote by T a (fixed) Sylow 3-subgroup of G. Each V_i in (2.5) is faithful for T, hence (by [5], II.3.10) is a simple **F**T-module. Thus

(2.6) if W is not a simple **F**G-module, then W_T is a direct sum of two faithful simple **F**T-modules each of dimension 2 over **F**.

Since, by their definition, both V_T and U_T are faithful for T, they must be isomorphic to the unique faithful simple **F**T-module M (which has dimension 2 by [5] II.3.10), hence

(2.7) $W_T \simeq M \otimes M$.

To complete our proof (through comparison of (2.6) and (2.7)), we show that $M \otimes M$ has one-dimensional **F**T-submodules.

Let m be any non-zero element of M, and let t be a generator of T; then $\{m, mt\}$ is a basis for M over **F** and $\{m \otimes m, mt \otimes m, m \otimes mt, mt \otimes mt\}$ is a basis for $M \otimes M$ over **F**. Direct computation, and the fact that in **F** -1 is the unique cubic root of -1 (see [4], 3.7.1.), shows that $mt^2 = -m - mt$. Thus we have

$$(m \otimes mt - mt \otimes m)\, t = m \otimes mt - mt \otimes m,$$

$$(m \otimes m + m \otimes mt + mt \otimes mt)\, t = m \otimes m + m \otimes mt + mt \otimes mt$$

and the proof is complete.

3.

Let a, b be elements of GL(2, 3) of order 6 and 8 respectively, such that GL(2, 3) $= \langle a, b \rangle$; then SL(2, 3) $= \langle a, b^2 \rangle$. To exhibit a representation for GL(2, 3) of degree n over the field $\mathbf{F}$ we give the images of a and b as $n \times n$ matrices over $\mathbf{F}$.

The smallest prime $p \equiv 2 \pmod{3}$ yielding a faithful irreducible representation of GL(2, 3) of degree 2 over GF(p) is $p = 11$; we get in fact two non-equivalent representations f and g, given by

$$f(a) = g(a) = \mathrm{A} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \quad f(b) = \mathrm{B} = \begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix} \text{ and } g(b) = \mathrm{B}_1 = \begin{pmatrix} 0 & 1 \\ 1 & -3 \end{pmatrix}.$$

These yield just one faithful irreducible representation of SL(2, 3), because B^2 and B_1^2 are conjugate via

$$\begin{pmatrix} 3 & 3 \\ -3 & -5 \end{pmatrix}$$

which centralises A.

Applying the theorem proved in section 2, we obtain a faithful irreducible representation φ for GL(2, 3) (and one for SL(2, 3)) by considering e.g.

$$\varphi(a) = \mathrm{A}' = \begin{pmatrix} 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 \\ 1 & -1 & -1 & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix} \quad \text{and} \quad \varphi(b) = \mathrm{B}' = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 3 \\ 1 & 0 & 3 & 0 \end{pmatrix}$$

Note that B′ and

$$\mathrm{B}_1' = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -3 \\ 1 & 0 & -3 & 0 \end{pmatrix} \quad \text{are conjugate via} \quad \begin{pmatrix} 1 & -2 & 1 & -2 \\ 2 & -1 & 2 & -1 \\ -1 & 2 & 2 & -4 \\ -2 & 1 & 4 & -2 \end{pmatrix}$$

which centralizes A′.

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