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Il problema monolaterale di contatto dinamico con attrito di una trave su una fondazione alla Hetényi: un approccio agli elementi finiti

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Meccanica. — *The Dynamical Unilateral Contact Problem with Friction of a Beam with a Hetényi's Foundation: a Finite Element Approach.* Nota (*) di LUIGI ASCIONE e GIANCARLO BILOTTI, presentata dal Socio E. GIANGRECO.

ABSTRACT. — The hypotesis of unilateral contact between foundation structure and soil assumes an important meaning in many technical problems, when the area of contact between structure and foundation becomes small, owing either to the soil-structure relative stiffness or to the load condition and consequently the assumption of perfect bonding fails.

Much research has been carried out in this area, essentially in the field of statics. However there is very little research concerning the dynamical equilibrium problem with unilateral constraints to be found in literature.

In fact this problem presents far more analytical difficulties than statical ones. Neverteless there are some advantagies to be obtained by using simplified models of soil, which allow us to analyse the most significant mechanical features of the problem without too much analytical trouble.

In particular, in view of technical applications, Hetényi's model, which simulates the soil by means of bilateral springs connected by an elastic plate seems very useful.

In some previous work quoted in References the authors utilized this model to study the frictionless dynamical contact problem relative to beams and circular plates.

In the present paper, such analysis is extended to take into account the effects of frictional stresses at the interface between structure and soil.

The law of friction that we consider can be seen as a regularization of the classical Coulomb's law.

In particular we restrict our analysis to consider a beam case. The problem is formulated and analysed from a numerical point of view via finite elements. Two examples are examined. In the first one the load condition consists in a single force applied at the middle of the beam; in the second only the beam is loaded by vertical and horizontal forces in order to simulate the action on the foundation of a frame subjected to an earthquake.

The time integration of the dynamical equilibrium equations is carried out by means of a semi-implicit method.

The numerical results obtained show how the most important parameters involved in the analysis influence the solution.

KEY WORDS: Dynamics; Unilateral Contact; Friction; Soil-structure interaction; Finite elements,

RIASSUNTO. — *Il problema monolaterale di contatto dinamico con attrito di una trave su una fondazione alla Hetényi: un approccio agli elementi finiti.* L'ipotesi di contatto

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monolaterale tra strutture di fondazione e terreno assume un significato importante in tutti quei problemi tecnici, nei quali l'area di contatto tra struttura e fondazione diviene percentualmente piccola, sia per la rigidezza relativa dei corpi a contatto, sia per la condizione di carico, soprattutto in presenza di carichi ribaltanti come possono ad esempio essere le forze sismiche.

In questo contesto sono stati sviluppati negli ultimi anni diversi studi, che riguardano però principalmente il caso statico. Decisamente meno investigato risulta invece il problema dinamico sia sotto l'aspetto tecnico che numerico. Tale problema presenta infatti notevoli difficoltà di tipo matematico. Alcuni vantaggi possono essere conseguiti ricorrendo a modelli semplificati di suolo, che consentono di cogliere gli aspetti meccanici peculiari del problema superando in parte le suddette difficoltà.

Tra questi particolarmente adatto agli scopi tecnici risulta il modello di Hetényi, che simula il terreno mediante delle molle bilaterali, tipo Winkler, collegate da una piastra elastica in modo da tenere conto della effettiva capacità del terreno a trasmettere sforzi di taglio.

In alcuni lavori precedenti, citati in bibliografia, gli autori hanno già utilizzato tale modello per studiare il problema di contatto dinamico senza attrito relativo a travi o piastre circolari.

Nel presente lavoro tale analisi è estesa al caso in cui all'interfaccia tra suolo e struttura si producono sforzi tangenziali dovuti ad attrito. La legge di attrito esaminata rappresenta una regolarizzazione della classica legge di Coulomb e risulta particolarmente vantaggiosa in vista delle applicazioni numeriche.

Al momento attuale l'analisi è limitata al solo modello monodimensionale di trave. Il problema viene formulato e trattato dal punto di vista numerico mediante approssimazione agli elementi finiti. Si analizzano in particolare due esempi con diverse condizioni di carico. Nel primo caso il modello è sollecitato da un'unica forza verticale applicata in mezzeria e nel secondo da due forze verticali associate a sforzi di taglio orizzontali e coppie in modo da simulare le azioni esercitate sulle fondazioni da un telaio soggetto a sisma.

L'integrazione rispetto al tempo delle equazioni del moto è effettuata mediante un metodo di tipo semi-implicito.

I risultati numerici ottenuti consentono di valutare l'influenza sulla soluzione dei principali parametri da cui essa dipende.

1. Introduction

The hypothesis of unilateral contact between foundation structure and soil assumes an important meaning in many technical problems, when the area of contact between structure and foundations becomes small, owing either to the soil-structure relative stiffness or to the load condition, and consequently the assumption of perfect bonding fails.

Much research has been carried out in this area, essentially in the field of statics [1-2]. However there is very little research concerning the dynamica equilibrium problem with unilateral constraints [3-4-5-6] to be found in literature.

In fact this problem presents far more analytical difficulties than statical ones Nevertheless there are some advantages to be obtained by using simplified models of soil, which allow us to analyse the most significant mechanical features of the problem without too much analytical trouble. A comprehensive review can be found in [7].

In particular, in view of technical applications, Hetényi's model (1946), which simulates the soil by means of bilateral springs connected by an elastic plate seems very useful.

In some previous works [8-9], we analysed the frictionless dynamical contact problem of beams and axisymmetric circular plates in unilateral contact with a Hetényi's foundation.

In the present paper, such analysis is extended to take into account the effects of frictional stresses, at the interface between structure and soil. The law of friction that we consider can be seen as a regularization of the classical Coulomb's law. Previous applications of the same law in the statical field can be found in [10-11].

In particular we will restrict our analysis to consider a beam case. The problem will be formulated and analysed from a numerical point of view.

The mathematical aspects will be the subject of a future paper.

For the present the numerical results, obtained by means of finite element, show how the most important parameters involved in the analysis influence the solution.

2. FORMULATION OF THE PROBLEM

With reference to fig. 1, let us consider the dynamical contact problem of the two linear elaetic beams τ_b and τ_s. The first one rests unilaterally on the second, which represents the Hetényi's foundation.

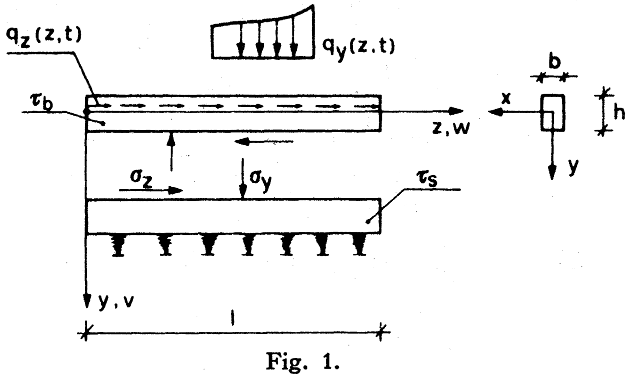


Fig. 1.

We assume that the beam τ_b undergoes both bending and stretching deformations, while the beam τ_s is supposed to be inextensible.

Let:

Ω be the open interval $]0,1[$, which schematizes the beam axis;

$\boldsymbol{u}_b = (v_b, w_b)$	the displacements of the beam τ_b;
v_s	the lateral deflection of the beam τ_s;
$\boldsymbol{\sigma} = (\sigma_y, \sigma_z)$	the interactions between τ_b and τ_s;
$\boldsymbol{q} = (q_y, q_z)$	the external loads acting on τ_b;
$\rho_i (> 0)$	the mass density of τ_i $(i = b, s)$;
$D_i (> 0)$	the flexural stiffness of τ_i;
$C_b (> 0)$	the stretching stiffness of τ_b;
$K (> 0)$	the stiffness of the spring elements;
$\dot{(\cdot)} = \frac{\partial}{\partial t}(\cdot)$	the time derivative $(t \in [0, T], T > 0)$
$H^n(\Omega)\ (n > 1)$	the Sobolev space of order n;
$L^2(\Omega)$	the space of the square summable functions on the interval Ω;
$\mathbf{V}_b = H^2(\Omega) \times H^1(\Omega)$	the space of the admissible displacements of τ_b;
$V_s = H^2(\Omega)$	the space of the admissible displacements of τ_s;
$< \cdot, \cdot >$	the duality pairing between dual spaces;
$(\cdot, \cdot)$	the inner product in $L^2(\Omega)$;
$\vert \cdot \vert$	the norm symbol in $L^2(\Omega)$;
$\Vert \cdot \Vert_V$	the norm symbol in the generic space V;
$\mathbf{A}_b : \mathbf{V}_b \rightarrow \mathbf{V}_b'$	the linear operator associated with the symmetric bilinear form $a_b(\boldsymbol{u}_b, \delta\boldsymbol{u}_b)$, which is defined on the space $\mathbf{V}_b$ as:

$$a_b(\boldsymbol{u}_b, \delta\boldsymbol{u}_b) = D_b \int_0^e \frac{d^2 v_b}{dz^2}\frac{d^2 \delta v_b}{dz^2}\, dz + C_b \int_0^e \frac{dw_b}{dz}\frac{d\delta w_b}{dz}\, dz \tag{2.1}$$

$$\forall\ \boldsymbol{u}_b, \delta\boldsymbol{u}_b \in \mathbf{V}_b,$$

$A_s : V_s \rightarrow V_s$ — the linear operator associated with the symmetric bilinear form $a_s(v_s, \delta v_s)$ defined on the space V_s as:

$$a_s(v_s, \delta v_s) = D_s \int_0^e \frac{d^2 v_s}{dz^2}\frac{d^2 \delta v_s}{dz^2}\, dz, \tag{2.2}$$

$$\forall\ v_s, \delta v_s \in V_s.$$

As is will known [2], the mathematical properties of the bilinear forms $a_b(\cdot, \cdot)$ and $a_s(\cdot, \cdot)$ are the following:

$$\forall\ \mu > 0 \quad , \quad \exists\ \delta_b, \delta_s > 0:$$

$$a_b(\boldsymbol{u}_b, \boldsymbol{u}_b) + \mu \vert \boldsymbol{u}_b \vert^2 \geq \delta_b \Vert \boldsymbol{u}_b \Vert^2_{V_b} \quad \forall\ \boldsymbol{u}_b \in \mathbf{V}_b, \tag{2.3a}$$

(2.3 *b*) $$a_s(v_s, v_s) + \mu \, | v_s |^2 \geq \delta_s \| v_s \|^2_{V_s} \; \forall \, v_s \in V_s .$$

For that which concerns the friction law, we assume that, a shear stress is acting at the interface between structure and foundation.

Its expression is the following:

(2.4) $$\sigma_z = \varepsilon_1 \left(\dot{w}_b + \dot{\varphi} \frac{h}{2} \right) + (\varepsilon_2 - \varepsilon_1) \left(\left| \dot{w}_b + \dot{\varphi} \frac{h}{2} \right| - \dot{w}_y \right)^+ \frac{\dot{w}_b + \dot{\varphi} \frac{h}{2}}{\left| \dot{w}_b + \dot{\varphi} \frac{h}{2} \right|} ,$$

where:

ε_1, ε_2	are two assigned positive constants,
$\varphi = - \frac{dv_b}{dz}$	is the bending slope of τ_b,
h	is the beam depth,
$\dot{w}_y = f\sigma_y / \varepsilon_1$,	
$f (> 0)$	is the friction coefficient,
$\| \cdot \|$	is the symbol of absolute value,
$(\cdot)^+$	denotes the positive part of the number $(\cdot)$.

Relation (2.4) is depicted in fig. 2; we observe that for $\varepsilon_2 \to 0$ and $\varepsilon_1 \to \infty$ it approximates the classical law of dynamical friction due to Coulomb.

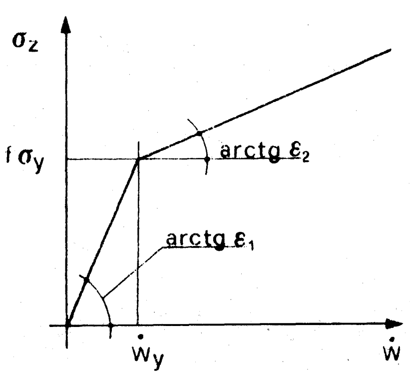


Fig. 2.

After these preliminaries, denoting by a prime the derivative respect to z, the problem under examination can be formulated as follows:

Problem P· Let $\boldsymbol{q}(z, t)$ be an element of $L^\infty(0, T; \mathbf{V}_b)$; find $u_b \in L^\infty(0, T; \mathbf{V}_b)$, $v_s \in L^\infty(0, T; V_s)$ and $\sigma_y \in L^\infty(0, T; V_s')$ satisfying to the conditions:

$$(2.5\,a) \qquad < \rho_b \ddot{\boldsymbol{u}}_b + \mathbf{A}_b \boldsymbol{u}_b - \boldsymbol{q} + \boldsymbol{\sigma} , \delta \boldsymbol{u} > + < \frac{h}{2} \sigma_z , \delta v_b' > = 0 ,$$

$$(2.5\,b) \qquad < \rho_s \ddot{v}_s + \mathrm{A}_s v_s + k v_s - \sigma_y , \delta v_s > = 0 ,$$

$$(2.5\,c) \qquad v_s - v_b \geq 0 ,$$

$$(2.5\,d) \qquad \sigma_y \geq 0 ,$$

$$(2.5\,e) \qquad < \sigma_y , v_s - v_b > = 0$$

$$\forall\ \delta \boldsymbol{u}_b \in \mathbf{V}_b \text{ and } \forall\ v_s \in \mathrm{V}_s .$$

Relations (2.5 *a*) and (2.5 *b*) represent the virtual work equations relative to the beam τ_b and to the foundation τ_s respectively. We recall that the shear component σ_z of the interaction $\boldsymbol{\sigma}$ is expressed in terms of the unknowns by means of eq. (2.4). The set of equations (2.4) and (2.5 *c*)-(2.5 *e*) completely characterize the contact condition between the bodies τ_b and τ_s. In particular eq. (2.5 *c*) imposes that the deflection of a generic point of τ_b must be greater than the corresponding point of the foundation τ_s; eq. (2.5 *d*) constraints the possible vertical reations between the beam and the foundation to positive (compression) values (fig. 1); finally, eq. (2.5 *e*) states that the vertical component of the interaction σ_y is different from zero only where the relative displacement $v_s - v_b$ is zero, i.e. where the beam is in contact with the foundation.

2.1. *An auxiliary problem.*

In studying the contact problem formulated in the previous section, it is helpful [9] to refer to an auxiliary problem, which is simpler than the original one. This new problem can be formulated by modifying the contact conditions as fig. 3 shows.

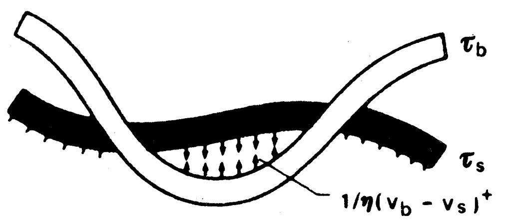


Fig. 3.

We admit that the beam τ_b can present lateral deflections greater than the corresponding deflections of the foundation. Nevertheless, where the condition $v_b - v_s > 0$ is verified, some springs of stiffness $1/\eta$ are active and they oppose the interprenetation of the two bodies. It is easy to conjecture that when $\eta \to 0$, i.e. when the spring stiffness increases indefinitely, the auxiliary problem approximates the previous one.

The mathematical aspect related to the auxiliary problem will be the subject of a future paper. For the present we only formulate this problem, which is later utilized to get numerical results.

In view of that, let $\boldsymbol{q}$ be an element of the space $L^\infty(0, T; \mathbf{V}'_b)$; the auxiliary problem is the following:

Problem A. Find $\boldsymbol{u}_b \in L^\infty(0, T; \mathbf{V}_b)$ and $v_s \in L^\infty(0, T; V_s)$ such that:

(2.6.)
$$
\begin{aligned}
&<\rho_b \ddot{\boldsymbol{u}}_b, \delta \boldsymbol{u}_b> + <\mathbf{A}_b \boldsymbol{u}_b, \delta \boldsymbol{u}_b> + <\rho_s \ddot{v}_s, \delta v_s> + \\
&+ <k\, v_s, \delta v_s> + <\frac{1}{\eta}(v_b - v_s)^+, \delta v_b - \delta v_s> + \\
&- <\boldsymbol{q}, \delta \boldsymbol{u}_b> + <\sigma_z, \delta w_b> - <\frac{h}{2}\sigma_z, \delta v'_b> = 0, \\
&\forall\, \delta \boldsymbol{u}_b \in \mathbf{V}_b \text{ and } \forall\, v_s \in V_s.
\end{aligned}
$$

In eq. (2.6) the expression of σ_z is given by eq. (2.4).

3. APPLICATIONS

In this section we present some numerical results relative to the two examples shown in fig. 4.

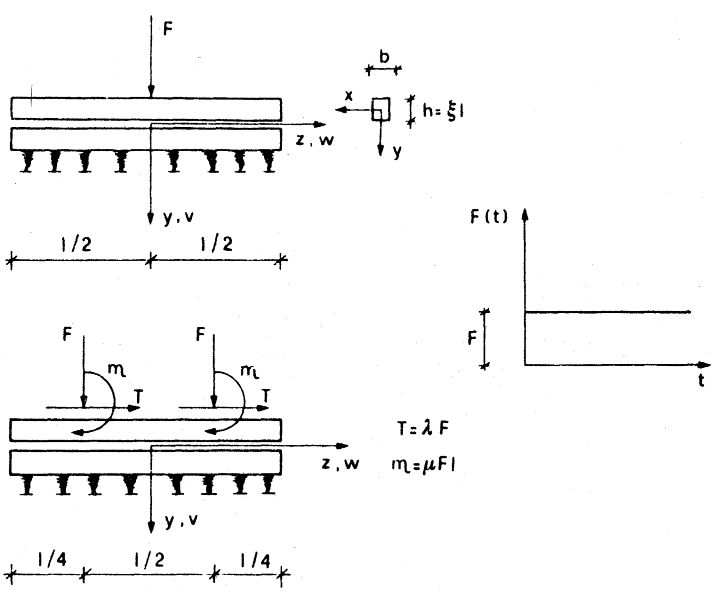


Fig. 4.

The finite element discretization is applied to the variational formulation (2.6) of Problem A.

In order to deal with dimensionless quantities, we put:

(3.1 *a*) $$\zeta = z/l ,$$

(3.1 *b*) $$V_i = v_i/l , \; (i = b , s)$$

(3.1 *c*) $$W_b = w_b/l ,$$

(3.1 *d*) $$\tau = t/T_0 ,$$

where:

(3.1 *e*) $$T_0 = l^2 (\rho_b/D_b)^{1/2} .$$

By substituting eq. (3.1) into eq. (2.6) it is easy to verify that the solution of the dynamical problem A depends on the following dimensionless parameters:

(3.2 *a*) $$\xi = h/l ,$$

(3.2 *b*) $$\bar{\varepsilon}_i = \varepsilon_i \, l^2/\sqrt{\rho_b D_b} , \quad (i = 1 , 2)$$

(3.2 *c*) $$\bar{\eta} = \eta D_b/l^4 ,$$

(3.2 *d*) $$\alpha = D_s/D_b ,$$

(3.2 *e*) $$\beta = kl^4/D_b ,$$

(3.2 *f*) $$\gamma = C_b l^2/D_b ,$$

(3.2 *g*) $$\bar{\rho} = \rho_s/\rho_b ,$$

(3.2 *h*) $$f .$$

3.1. *Solution algorithm*.

Following the semidiscrete approximation method [12], we construct approximations of u_b, v_s of the form:

(3.3 *a*) $$u_b(z , t) = \sum_{i=1}^{2N} u_{bi}(t) f_i(z) ,$$

(3.3 *b*) $$v_s(z , t) = \sum_{i=1}^{N} v_{si}(t) g_i(z) ,$$

where f_i are cubic Hermitian polynomials and g_i are linear polynomials; N is the number of nodes in the finite element mesh.

By standard calculations, the auxiliary problem can be set in the discretized form:

$$\mathbf{M}\,\ddot{\boldsymbol{s}} + \mathbf{K}(\boldsymbol{s})\,\boldsymbol{s} + \mathbf{C}(\dot{\boldsymbol{s}})\,\dot{\boldsymbol{s}} = \boldsymbol{q}\,, \tag{3.4}$$

where $\boldsymbol{s}$ is the vector of nodal displacements, $\mathbf{M}$ is the mass matrix, $\boldsymbol{q}$ is the load vector, $\mathbf{K}(\boldsymbol{s})$ is the stiffness matrix and $\mathbf{C}(\dot{\boldsymbol{s}})$ is a damping matrix deriving from the discretization of the last two terms in eq. (2.6).

We observe that both $\mathbf{K}$ and $\mathbf{C}$ are non-linear functions of $\boldsymbol{s}$ and $\dot{\boldsymbol{s}}$ respectively, because of the presence of the terms $1/\eta\,(v_b - v_s)^+$ and $(|\,\dot{w}_b + \dot{\varphi}h/z\,| + - \dot{w}_y)^+$ in their expression.

The solution of eq. (3.4) is obtained trough a semi-implicit integration method [13].

More precisely, the velocities and the accelerations are approximated by central finite differences and the solution at time $t + \Delta t$ is calculated by the dynamical equilibrium equations (3.3) relative to time t, assuming for the elastic forces $\mathbf{K}(\boldsymbol{s})\,\boldsymbol{s}$ their value at time $t + \theta t$, where θ is a fixed number between 0 and 1.

In more explicit terms, the step-by-step solution algorithm is the following:

$$\boldsymbol{s}_{t+\Delta t} = \hat{\mathbf{M}}^{-1}(\boldsymbol{s}_\theta\,, \dot{\boldsymbol{s}}_t)\,\hat{\boldsymbol{q}}\,(\boldsymbol{s}_\theta\,, \dot{\boldsymbol{s}}_t)\,, \tag{3.5}$$

where:

$$\boldsymbol{s}_\theta = \boldsymbol{s}_t + \theta\,(\boldsymbol{s}_{t+\Delta t} - \boldsymbol{s}_t)\,, \tag{3.6 a}$$

$$\dot{\boldsymbol{s}}_t = \frac{1}{2\,\Delta t}\,(\boldsymbol{s}_{t+\Delta t} - \boldsymbol{s}_{t-\Delta t})\,, \tag{3.6 b}$$

$$\hat{\mathbf{M}}\,(\boldsymbol{s}_\theta\,, \dot{\boldsymbol{s}}_t) = \frac{1}{\Delta t^2}\,\mathbf{M} + \theta\,\mathbf{K}(\boldsymbol{s}_\theta) + \frac{1}{2\,\Delta t}\,\mathbf{C}\,(\dot{\boldsymbol{s}}_t)\,, \tag{3.6 c}$$

$$\hat{\boldsymbol{q}}\,(\dot{\boldsymbol{s}}_\theta\,, \dot{\boldsymbol{s}}_t) = \boldsymbol{q}_t + \left[\frac{2}{\Delta t^2}\,\mathbf{M} + (\theta - 1)\,\mathbf{K}\,(s_\theta)\right]\boldsymbol{s}_t + $$

$$- \left[\frac{1}{\Delta t^2}\,\mathbf{M} - \frac{1}{2\,\Delta t}\,\mathbf{C}\,(\dot{\boldsymbol{s}}_t)\right]\boldsymbol{s}_{t-\Delta t}\,. \tag{3.6 d}$$

Eq. (3.5) is valid if $t > 0$; in particular at time $t = 0$ we put:

$$s_{\Delta t} = s_0 + \dot{s}_0\,\Delta t\,. \tag{3.7}$$

In order to calculate the non–linear terms in eq. (3.6 c) and (3.6 d), the following iterative procedure is employed:

$$\boldsymbol{s}_{t+\Delta t}^{(k)} = \hat{\mathbf{M}}^{-1} (\boldsymbol{s}_\theta^{(k-1)}, \dot{\boldsymbol{s}}_t^{(k-1)})\, \hat{\boldsymbol{q}}\, (\boldsymbol{s}_\theta^{(k-1)}, \dot{\boldsymbol{s}}_t^{(k-1)}), \tag{3.8}$$

being:

$$\boldsymbol{s}_\theta^{(k-1)} = \boldsymbol{s}_t + \theta\, (\boldsymbol{s}_{t+\Delta t}^{(k-1)} - \boldsymbol{s}_t) \tag{3.9 a}$$

$$\dot{\boldsymbol{s}}_t^{(k-1)} = \frac{1}{2\,\Delta t}\, (\boldsymbol{s}_{t+\Delta t}^{(k-1)} - \boldsymbol{s}_{t-\Delta t}) \tag{3.9 b}$$

$$\boldsymbol{s}_{t+\Delta t}^{(0)} = \boldsymbol{s}_t\,. \tag{3.9 c}$$

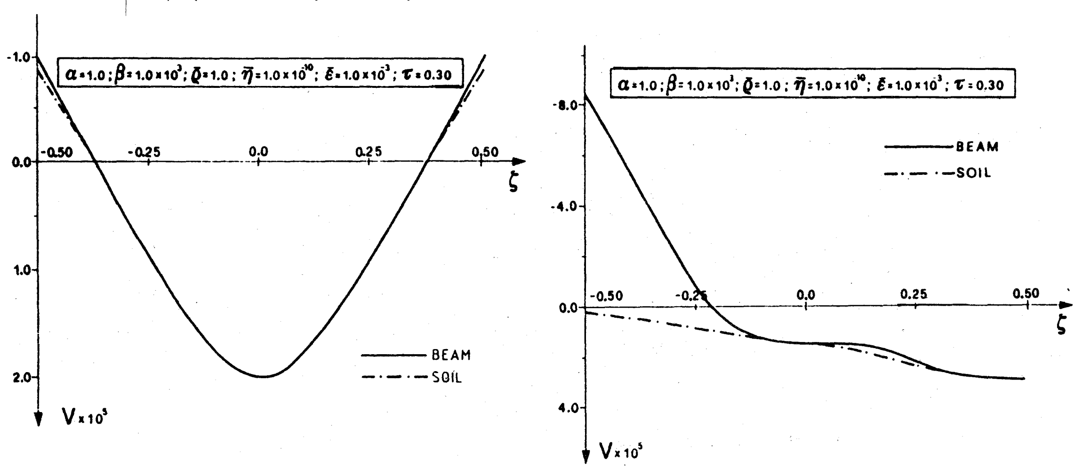
3.2. *Numerical results and conclusions.*

The following numerical results have been obtained by means of a uniform finite element mesh of 40 elements. The initial conditions for displacements and velocities were assumed to be zero and the parameter θ in the semi-implicit scheme was taken to be 0.5.

Further on we assumed the following values of the dimensionless parameters (3.2):

$$\begin{aligned} &\bar{\varepsilon}_1 = 1.0 \times 10^3 \;, && \bar{\varepsilon}_2 = 1.0 \times 10^{-3} \;, && \bar{\eta} = 1.0 \times 10^{-10}, \\ &\alpha = 1.0 \;, && \bar{\rho} = 1.0 \;, && \xi = 0.1 \;, \\ &\gamma = 1.2 \times 10^3 \;, && f = 0.8 \;, && Fl^2/D_b = 1.0 \times 10^{-3}. \end{aligned} \tag{3.10}$$

With reference to these values, figs 5 a-5 b show the plots of the lateral deflections along the beam axis for an assigned time $\tau = 0.30$. The first one refers to the load condition a); while the second refers one to the load condition b) ($\lambda = 0.1$, $\mu = 2.0$).



a) *b*)

Fig. 5.

The same figures show how the load condition can modify the contact region between structure and foundation. In fact, for the same values of the stiffness parameters α and β, we observe that the contact is almost complete under the load condition *a*), while the unbonding area becomes relevant in presence of moments and horizontal loads, as appears under seismic loadings.

The dependence on the time for the displacements is analyzed in figs. 6 *a*-6 *b*.

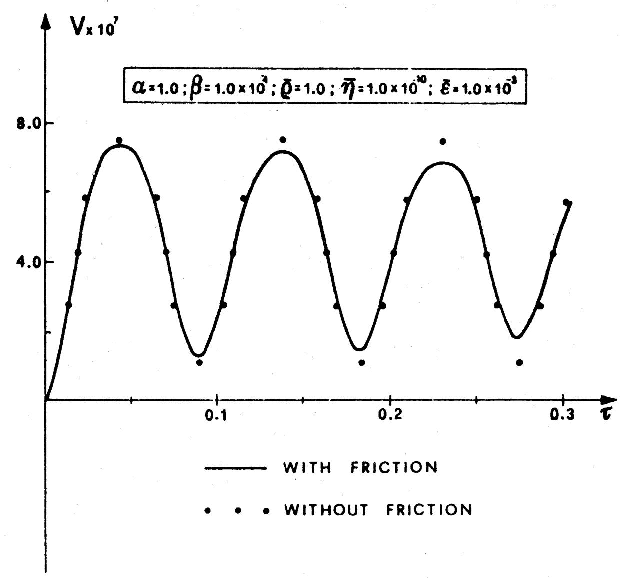


Fig. 6 *a*.

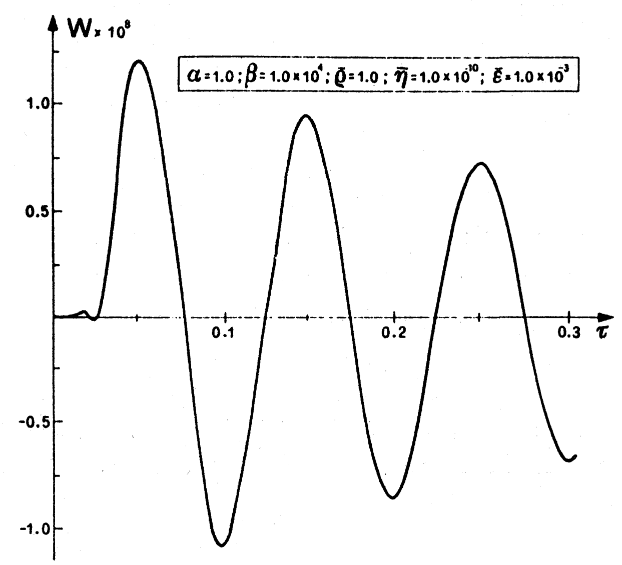


Fig. 6 *b*.

In particular, these figures show the plots of the vertical and horizontal displacements exhibited by the beam middle and end cross section, respectively (load condition *a*).

It is evident the damping effect due to the presence of the frictional forces.

Finally figs. 7 *a*-7 *b* make a comparison between the beam lateral deflections (load condition *a*) in presence of friction or not.

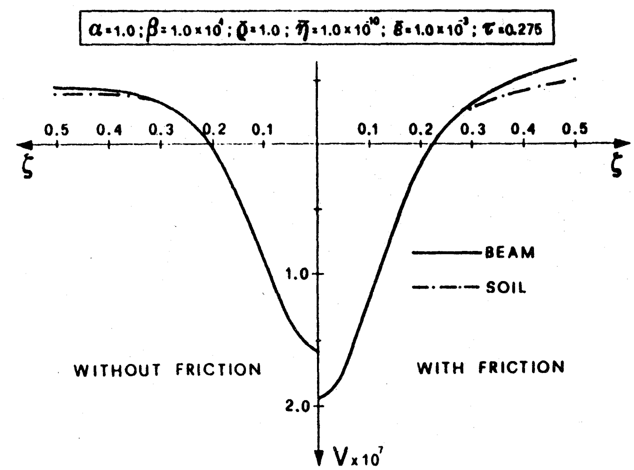


Fig. 7 *a*.

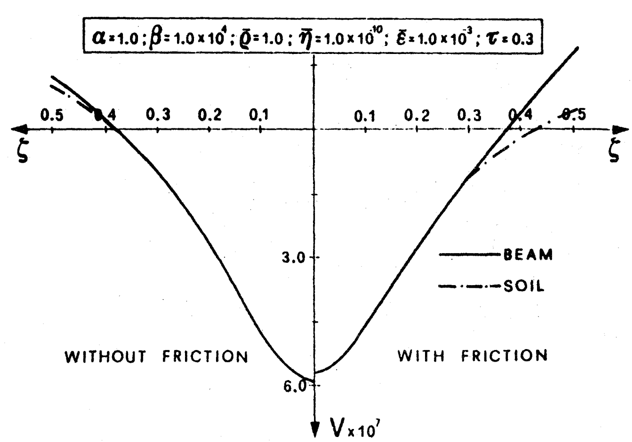


Fig. 7 *b*.

We observe that the friction can modify either the contact area or the value of the maximum lateral deflection; in our case we found differences bigger than 20%.

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