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Geometria. — *Holomorphic semigroups of holomorphic isometries.* Nota (*) del Corrisp. EDOARDO VESENTINI.

ABSTRACT. — A previous paper was devoted to the construction of non-trivial holomorphic families of holomorphic isometries for the Carathéodory metric of a bounded domain in a complex Banach space, fixing a point in the domain. The present article shows that such a family cannot exist if it contains a strongly continuous one parameter semigroup.

KEY WORDS: Carathéodory and Kobayashi metrics; Holomorphic isometries.

RIASSUNTO. — *Semigruppi olomorfi di isometrie olomorfe.* In un lavoro precedente è stata costruita una famiglia olomorfa non banale di isometrie olomorfe, per la metrica di Carathéodory, di un dominio limitato di uno spazio di Banach complesso, aventi un punto fisso comune. In questa nota si prova che famiglie siffatte non esistono se si impone la condizione aggiuntiva che esse contengano un semigruppo fortemente continuo ad un parametro reale.

Let U and D be domains in $\mathbf{C}$ and in a complex Banach space $\mathscr{E}$ and let f be a holomorphic map of $U \times D$ into D. It was shown in [2, proposition V. 1.10, pp. 120–112] that, if $f(z, .)$ is a bi-holomorphic automorphism of D for some $z \in U$, then f is independent of z.

Assuming now D to be a hyperbolic domain, the question naturally arises whether the same conclusion holds in the case in which $f(z, .)$ is a holomorphic isometry for the Kobayashi metric in D. An example was constructed in [9] where D is the open unit ball of the C*–algebra $\mathscr{L}(\mathscr{X})$ of all bounded linear operators in an infinite dimensional Hilbert space $\mathscr{X}$ and $f(z, .)$ is a holomorphic isometry for the Kobayashi metric in D, depending effectively on $z \in U$. A special feature of this example is that $f(z, .)$ fixes the centre of D for all $z \in U$.

In the present note holomorphic semigroups of holomorphic isometries will be investigated. More specifically the case will be considered in which: the domain U contains the strictly positive axis $\mathbf{R}_+^*$; D is a bounded domain in $\mathscr{E}$; $f(z, .)$ is a holomorphic isometry for the Carathéodory distance in D (*) for all $z \in \mathbf{R}_+^*$; setting $f(0, .) = \mathrm{id}$, the map $\mathbf{R}_+ \ni z \mapsto f(z, .)$ is a strongly continuous semigroup; $f(z, .)$ fixes some point $x_0 \in D$ for all $z \in \mathbf{R}_+^*$.

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(*) According to a theorem of L. Lempert, if the bounded domain D is convex, the Carathéodory and Kobayashi distances coincide [1].

It will be shown that, under the above conditions, $f(z, .)$ is necessarily independent of $z \in U$. This conclusion depends on some subharmonicity properties of the Carathéodory distance which were established in [8] and on properties of strongly continuous semigroups of linear isometries that will be investigated in section 1.

1. Let $\mathscr{L}(\mathscr{E})$ be the complex Banach algebra of all bounded linear operators in the complex Banach space $\mathscr{E}$. For any closed linear operator S on $\mathscr{E}$, $\sigma(S)$, $\sigma_r(S)$ and $r(S)$ will indicate respectively the spectrum, the residual spectrum and the resolvent set of S.

If $S \in \mathscr{L}(\mathscr{E})$ is a linear isometry of $\mathscr{E}$, then $\sigma(S) \subset \bar{\Delta}$, the closure of the open unit disc Δ of $\mathbf{C}$, and S is invertible if, and only if, $\sigma(S) \subset \partial\Delta$, the unit circle.

LEMMA 1.1. *If* $S \in \mathscr{L}(\mathscr{E})$ *is a linear isometry and if* ζ *is a boundary point of* $\sigma(S)$, *then* $|\zeta| = 1$.

Proof. The operator $S - \zeta I$ (I the identity operator on $\mathscr{E}$) is either zero (in which case the lemma trivially holds) or a generalized left divisor of zero [4, § 4.11, p. 129], i.e. there exists a sequence $\{A_n\}$ of operators $A_n \in \in \mathscr{L}(\mathscr{E})$ such that $\|A_n\| = 1$, and

$$\lim_{n\to\infty} (S - \zeta I) A_n = 0 .$$

Hence

$$\lim_{n\to\infty} \|S A_n\| = |\zeta| .$$

On the other hand,

$$\begin{aligned} \|S A_n\| &= \sup \{\|S A_n x\| : x \in \mathscr{E}, \|x\| \leq 1\} = \\ &= \sup \{\|A_n x\| : x \in \mathscr{E}, \|x\| \leq 1\} = \|A_n\| = 1 . \end{aligned}$$

Q.E.D.

COROLLARY 1.2. *If* $S \in \mathscr{L}(\mathscr{E})$ *is a linear isometry, either* $\sigma(S) = \bar{\Delta}$ *or* $\sigma(S) \subset \partial\Delta$ *and* S *is an isometric linear automorphism of* $\mathscr{E}$.

LEMME 1.3. *If* $S \in \mathscr{L}(\mathscr{E})$ *is a non-surjective linear isometry, then* $\Delta \subset \subset \sigma_r(S)$.

Proof. Clearly every $\zeta \in \Delta$ is not an eigenvalue of S. Assume that $\zeta \in \Delta$ is a point of the continuous spectrum of S. Then, given any $y \in \mathscr{E}$, there is a sequence $\{x_n\}$ of vectors $x_n \in \mathscr{E}$ such that

$$\lim_{n\to\infty} (\zeta I - S) x_n = y .$$

Since

$$\| (\zeta\, \mathrm{I} - \mathrm{S}) (x_m - x_n) \| \geq | \, \| \mathrm{S} (x_m - x_n) \| - | \zeta | \, \| x_m - x_n \| \, | = \\ = (1 - | \zeta |) \, \| x_m - x_n \| ,$$

the fact that $\{(\zeta \mathrm{I} - \mathrm{S})\, x_n\}$ is a Cauchy sequence implies that $\{x_n\}$ is a Cauchy sequence and therefore converges to some point $x \in \mathscr{E}$. But then $(\zeta \mathrm{I} - \mathrm{S})\, x = \lim_{n \to \infty} (\zeta \mathrm{I} - \mathrm{S})\, x_n = y$, and so $(\zeta \mathrm{I} - \mathrm{S})\, \mathscr{E} = \mathscr{E}$. By the injectivity of $\zeta \mathrm{I} - \mathrm{S}$, that implies that $\zeta \in r(\mathrm{S})$, contradicting Corollary 1.2.

Q.E.D.

Let $\mathrm{T} : \mathbf{R}_+ \to \mathscr{L}(\mathscr{E})$ be a strongly continuous semigroup of linear isometries of $\mathscr{E}$. Let X be the infinitesimal generator of T. Since $\mathrm{T}(t)$ is a contraction, the spectrum $\sigma(\mathrm{X})$ of X is contained in the closure of the left half–plane $\Pi_l = \{\zeta \in \mathbf{C} : \mathrm{Re}\, \zeta < 0\}$. The question will now be investigated whether the semigroup T is eventually differentiable, i.e. whether there exists some $t_0 \geq 0$ such that the function $t \mapsto \mathrm{T}(t)\, x$ is of class C^1 on $(t_0, +\infty)$ for every $x \in \mathscr{E}$.

PROPOSITION 1.4. *If the linear isometry* $\mathrm{T}(t)$ *is not surjective for some* $t > 0$, *then* $\mathrm{T}(t)$ *is not surjective for every* $t > 0$ *and the semigroup* T *is not eventually differentiable.*

Proof. *a*) If $\mathrm{T}(t)$ is not surjective for some $t > 0$, then by Corollary 1.2 $\sigma(\mathrm{T}(t)) = \bar{\Delta}$, and, by Lemma 1.3 every $\zeta = \rho\, e^{i\theta}$, with $0 < \rho < 1$, $\theta \in \mathbf{R}$, belongs to the residual spectrum $\sigma_r(\mathrm{T}(t))$. Thus [5; 6, Theorem 2.5, p. 47] there exists $n \in \mathbf{Z}$ and $\lambda = \lambda' + i\lambda'' \in \sigma_r(\mathrm{X})$ $(\lambda', \lambda'' \in \mathbf{R})$ such that $\lambda' = \frac{1}{t} \log \rho$ and $\lambda'' = \frac{\theta + 2n\pi}{t}$. That implies that the intersection of $\sigma_r(\mathrm{X})$ with any vertical line contained in Π_l is non-empty. On the other hand, if $\mathrm{T}(s)$ is surjective for some $s > 0$, then $\sigma(\mathrm{T}(s)) \subset \partial\Delta$, and the inclusion [5; 6, p. 45]

$$\sigma(\mathrm{T}(s)) \supset e^{s\, \sigma(\mathrm{X})}$$

implies that $\sigma(\mathrm{X}) \subset i\mathbf{R}$. That proves that, if $\mathrm{T}(t)$ is not surjective for some $t > 0$, then $\mathrm{T}(t)$ is not surjective for every $t > 0$.

b) Assume from now on that the linear isometry $\mathrm{T}(t)$ is not surjective for every $\mathrm{t} > 0$.

Since, by Lemma 1.3, $\Delta \subset \sigma_r(\mathrm{T}(t))$ for all $t > 0$, there is no eigenvalue of X in Π_l [5; 6 Theorem 2.4, p. 46]. Hence for any $\lambda = \lambda' + i\lambda''$ with $\lambda' < 0$, $\lambda'' \in \mathbf{R}$, $e^{\lambda t} \in \sigma_r(\mathrm{T}(t))$ [5;6, Theorem 2.5, p. 47].

It will be shown now that, given $\lambda' < 0$ and $\mathrm{K} > 0$ there is some $\lambda = \lambda' + i\lambda''$ with $\lambda'' \in \mathbf{R}$ such that $|\lambda''| > \mathrm{K}$. If that is not the case, there exists $M > 0$ such that, if $\lambda' + i\lambda'' \in \sigma_r(X)$, then $|\lambda''| \leq M$. For any $t > 0$, the image of the segment $\{\lambda' + il : -M \leq l \leq M\}$ by the map $\zeta \mapsto e^{\zeta t}$ is the arc $\{e^{\lambda' t} e^{il t} : -M \leq l \leq M\}$ whose length tends to zero as t tends to zero. Since, on the other hand, $\Delta \subset \sigma_r(T(t))$, the image of $\sigma_r(X) \cap \cap \{\lambda' + i\mathbf{R}\}$ by the map $\zeta \mapsto e^{\zeta t}$ must cover the entire circle with centre 0 and radius $e^{\lambda' t}$ [5; 6, Theorem 2.5, p. 47]. Contradiction.

c) If T is eventually differentiable, by a theorem of A. Pazy [6, Theorem 4.7, p. 54] there exist $a \in \mathbf{R}$, $b > 0$ such that

$$\{\zeta \in C : \operatorname{Re} \zeta \geq a - b \log |\operatorname{Im} \zeta|\} \subset r(X).$$

But that contradicts the conclusion of *b)*.

Q.E.D.

Remark. Part *b)* of the proof might be obtained also as a consequence of the fact that the spectral mapping theorem holds for eventually differentiable strongly continuous semigroups [3, p. 87].

COROLLARY 1.5. *If* T *is an eventually differentiable strongly continuous semigroup of linear isometries, then* $T(t)$ *is surjective for every* $t \geq 0$.

2. Let D be a bounded domain in $\mathcal{E}$. The Carathéodory pseudodistance c_D is a distance defining on D the relative topology [2, Theorem IV.2.2, p. 92]. For every $x \in D$ the Carathéodory differential metric $\gamma_D(x; .)$ is a norm which is equivalent to the norm of $\mathcal{E}$ [2, Lemma V.2.1, pp. 121-122].

Let $g : \mathbf{R}_+ \times D \to D$ be such that:

i) *the map* $t \mapsto g(t, .)$ *is a homomorphism of the additive semigroup* $\mathbf{R}_+$ *into the semigroup of all holomorphic maps of* D *into* D, *and* $\lim_{t \downarrow 0} g(t, x) = x$ *for every* $x \in D$;

ii) *there exists* $x_0 \in D$ *such that* $g(t, x_0) = x_0$ *for all* $t > 0$.

Given $t \geq 0$ and $x \in D$, $dg(t, x)$ stands for the differential of $g(t, .)$ with respect to the second variable, evaluated at the point x.

Condition ii) implies that $T : t \mapsto dg(t, x_0)$ $(t \geq 0)$ is a semigroup of bounded linear operators in $\mathcal{E}$.

LEMMA 2.1. *The semigroup* T *is strongly continuous.*

Proof. Let $B(x_0, r)$ and $B_c(x_0, r)$ be the open balls with centre x_0 and radius $r > 0$ for the norm-distance and the Carathéodory distance c_D. Since these two distances are equivalent on any closed ball (for the norm-distance) completely interior to D [2, Theorem IV.2.2, p. 92], for any $r > 0$ there exist $r_0 > 0$ and $r_1 > 0$ such that $B(x_0, r_0) \subset B_c(x_0, r) \subset B(x_0, r_1)$. The Cauchy formula yields, for any $v \in \mathscr{E}$, with $\| v \| < r_0$,

$$\left\| dg(t, x_0)\, v - v \right\| = \frac{1}{2\pi} \left\| \int_0^{2\pi} e^{-i\theta} \left(g(t, x_0 + e^{i\theta} v) - (x_0 + e^{i\theta} v) \right) d\theta \right\|$$

$$\leq \frac{1}{2\pi} \int_0^{2\pi} \| g(t, x_0 + e^{i\theta} v) - (x_0 + e^{i\theta} v) \| \, d\theta ,$$

whence,

$$\lim_{t \downarrow 0} \| dg(t, x_0)\, v - v \| = 0 .$$

Q.E.D

The differential metric γ_d is the derivative of c_d; more exactly, according to a theorem of H.-J. Reiffen [4; 7],

$$\lim_{s \to 0} \frac{1}{|s|} c_D(x + sv, x) = \gamma_D(x; v)$$

locally uniformly in $x \in D$, $v \in \mathscr{E}$.

LEMMA 2.2. *If* i) *and* ii) *hold and if* $g(t, .)$ *is a holomorphic isometry for* c_D *for all* $t \geq 0$, *then* $T : t \mapsto dg(t, x_0)$ *is a strongly continuous semigroup of linear isometries for the norm* $\gamma_D(x_0; .)$.

Let U be a domain in **C** and let f be a holomorphic map of $U \times D$ into D.

LEMMA 2.3. *If, for every pair of points* x, y *in* D, *there is* $z \in U$ *such that*

$$c_D(f(z, x), f(z, y)) = c_D(x, y), \tag{1}$$

then, for every $z \in U$, $f(z, .)$ *is an isometry for* c_D.

Proof. For $z \in U$, $f(z, .)$ is a holomorphic map of D into D, and therefore, given x, y in D,

$$c_D(f(z, x), f(z, y)) \leq c_D(x, y).$$

The function $z \mapsto \log c_D(f(z, x), f(z, y))$ is subharmonic on U [8, Theorem I, p. 216], and therefore also the function $z \mapsto c_D(f(z, x), f(z, y))$ is subharmonic on U. If, given x, y on D, there is $z_0 \in U$ such that

$$c_D(f(z_0, x), f(z_0, y)) = c_D(x, y),$$

by the maximum principle $c_D(f(z,x), f(z,y))$ is independent of z, and thus (1) holds for all $z \in U$. Since this happens for all x, y in D, the conclusion follows.

Q.E.D.

For $x \in D$, let $\Gamma(x) = \{z \in U : f(z,x) = x\}$. Since the function $z \mapsto \log c_D(f(z,x), x)$ is subharmonic on U, by a classical theorem of H. Cartan, either the set $\Gamma(x)$ is a G_δ with outer capacity zero or $c_D(f(z,x), x) = 0$ for all $z \in U$. That proves

LEMMA 2.4. *If $\Gamma(x)$ has positive outer capacity, then $f(z,x) = x$ for all $z \in U$.*

From now on U will be a domain in **C** containing the strictly positive real axis $\mathbf{R}_+^*$. Let g be the restriction of f to $\mathbf{R}_+^* \times D$. By Lemma 2.4, if g satisfies condition ii), or, more in general, if $g(t, x_0) = x_0$ when t varies on a subset of $\mathbf{R}_+^*$ having positive outer capacity, then $f(z, x_0) = x_0$ for all $z \in U$.

Define $f(0,.)$ by setting $f(0,x) = x$ for all $x \in D$. Accordingly let $g(0,.) = f(0,.)$. For any $z \in U$, $df(z,x) \in \mathscr{L}(\mathscr{E})$ will denote the differential of $f(z,.)$ with respect to the second variable, evaluated at $x \in D$.

THEOREM I. *Let the domain U contain $\mathbf{R}_+^*$. If there exists $x_0 \in D$ such that $f(t, x_0) = x_0$ for all $t > 0$, and if the map $T : t \mapsto df(t, x_0)$ $(t \geq 0)$ is a strongly continuous semigroup of linear isometries for the norm $\gamma_D(x_0; .)$, then $f(z,x) = x$ for all $x \in D$.*

Proof. Let C be the open unit ball for the norm $\gamma_D(x_0; .)$. For every $z \in U$, $df(z, x_0)$ defines a holomorphic map of C into C.

The function $t \mapsto dg(t, x_0)$ is the restriction to $\mathbf{R}_+^*$ of the holomorphic map $z \mapsto df(z, x_0)$ of U into $\mathscr{L}(\mathscr{E})$. Hence the semigroup T is differentiable. Corollary 1.5. implies then that $df(t, x_0)$ is a surjective linear isometry for $\gamma_D(x_0; .)$, and thus defines a holomorphic automorphism of C. By Proposition V.1.10 (pp. 120-121) of [2], $df(z, x_0) = I$ for all $z \in U$. By H. Cartan's uniqueness theorem $f(z,.)$ is the identity on D for all $z \in U$.

Q.E.D.

Lemmas 2.2, 2.3 and 2.4 imply then

THEOREM II. *Let U contain $\mathbf{R}_+^*$ and let f be a holomorphic map of U $\times$ D into D such that, setting $f(0,x) = x$ for all $x \in D$, the restriction of f to $\mathbf{R}_+ \times D$ defines a semigroup of holomorphic isometries for c_D for which $\lim_{t \downarrow 0} f(t,x) = x$ for all $x \in D$. If there exists $x_0 \in D$ such that $f(t, x_0) = x_0$ when t varies on a subset of $\mathbf{R}_+^*$ having positive outer capacity, then $f(z,x) = x$ for all $z \in U$, $x \in D$.*

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