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Analytic semigroups generated on a functional extrapolation space by variational elliptic equations

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Equazioni a derivate parziali. — *Analytic semigroups generated on a functional extrapolation space by variational elliptic equations.* Nota (*) di VINCENZO VESPRI, presentata dal Corrisp. E. VESENTINI.

ABSTRACT. — We prove that any elliptic operator of second order in variational form is the infinitesimal generator of an analytic semigroup in the functional space $C^{-1,\alpha}(\Omega)$ consinsting of all derivatives of hölder-continuous functions in Ω where Ω is a domain in R^n not necessarily bounded. We characterize, moreover the domain of the operator and the interpolation spaces between this and the space $C^{-1,\alpha}(\Omega)$. We prove also that the spaces $C^{-1,\alpha}(\Omega)$ can be considered as extrapolation spaces relative to suitable non–variational operators.

KEY WORDS: Analytic semigroups; Elliptic variational; Extrapolation spaces.

RIASSUNTO. — *Semigruppi analitici generali su uno spazio funzionale di estrapolazione da equazioni ellittiche variazionali.* Si prova che ogni operatore ellittico del II ordine di tipo variazionale è generatore infinitesimale di un semigruppo analitico nello spazio funzionale $C^{-1,\alpha}(\Omega)$ costituito dalle derivate di funzioni Hölderiane in Ω; Ω è un dominio non necessariamente limitato di R^n. Si caratterizza inoltre il dominio dell'operatore e gli spazi di interpolazione tra questo e lo spazio $C^{-1,\alpha}(\Omega)$. Si prova inoltre che gli spazi $C^{-1,\alpha}(\Omega)$ possono essere visti come spazi di extrapolazione relativi ad opportuni operatori non variazionali.

1. GENERATION OF ANALYTIC SEMIGROUPS

We consider an elliptic operator

$$\text{(1.1)} \qquad Eu = -\sum_{i,j} D_i(A_{ij} D_j u) - \sum_i D_i(A_i u) + \sum_i B_i D_i u + Cu$$

where A_{ij}, A_i, B_i, C are real valued functions defined on a possibly unbounded domain $\Omega \subseteq \mathbf{R}^n$ with $C^{3,\alpha}$ boundary. The growth of the coefficients of E will be described by means of a fixed C^∞ function $V(x) \geq 1$ such that:

$$\text{(1,2)} \qquad \sum_j |D_j V(x)| \leq k\, V(x) \qquad \forall x \in \overline{\Omega}$$

(*) Pervenuta all'Accademia il 5 agosto 1987.

where k is a positive constant. In fact we assume that there exists $\alpha \in]0, 1[$ such that

$$A_{ij}, A_i/(V)^{1/2} \in C_b^{0,\alpha}(\bar{\Omega}) \tag{1.3}$$

$$B_i/(V)^{1/2}, C/V \in C_b^0(\bar{\Omega}) \tag{1.4}$$

Moreover, we suppose that

$$C(x) = F(x)V(x) + D(x) \tag{1.5}$$

where $F, D \in C_b^{0,\alpha}(\bar{\Omega})$, $F(x) \geq F_0 > 0$. The ellipticity of E means that

$$\sum_{i,j} A_{ij}(x)\eta_i\eta_j \geq \nu |\eta|^2 \qquad \forall x \in \Omega, \qquad \forall \eta \in \mathbf{R}^n \tag{1.6}$$

where ν is a positive constant. Furthermore, we suppose that there exists $B \in [0, 1[$ such that

$$\sum_j |B_j(x)|^2 + \sum_i |A_i(x)|^2 \leq \nu F_0 B^2 V(x) \qquad \forall x \in \bar{\Omega} \tag{1.7}$$

Finally, we define the functional space $C^{-1,\alpha}(\Omega)$ as the space of the functions $f \in H^{-1}(\Omega)$ that admit a Hölder continuous representation, i.e., there exist $f_0, f_1, \ldots, f_n \in C_b^{0,\alpha}(\bar{\Omega})$, such that the equality $f = f_0 - \sum_i D_i f_i$ holds in the distributional sense. The norm $|f|_{-1,\alpha} = \sum_i |f_i|_{0,\alpha}$ makes $C^{-1,\alpha}$ a Banach space. Now we are able to state the main result of this paper.

THEOREM 1.1. *Consider the Dirichlet problem*

$$\begin{cases} (E + \lambda)u = f \in C_b^{-1,\alpha}(\Omega) \\ u(x) = 0 \qquad \forall x \in \partial\Omega \end{cases} \tag{1.8}$$

where $\lambda \in \mathbf{C}$. *Assume that conditions* (1.2)-(1.7) *hold. Then there exists* $\omega > 0$ *such that for every* $\lambda : \operatorname{Re}\lambda > \omega$ *problem* (1.8) *has a unique solution* $u \in C^{1,\alpha}(\bar{\Omega}) \cap \cap C_0^0(\bar{\Omega})$. *Moreover the following estimate holds:*

$$|\lambda|\,|u|_{-1,\alpha} + |\lambda|^{1/2}\{|u|_{0,\alpha} + |V^{1/2}u|_{-1,\alpha}\} + |u|_{1,\alpha} + \tag{1.9}$$
$$+ |V^{1/2}u|_{0,\alpha} + |Vu|_{-1,\alpha} \leq c\,|f|_{-1,\alpha}$$

where c *is independent of* λ, f *and* u.

Sketch of the proof. By using a suitable partition of unity we reduce problem (1.8) into many problems in bounded domains. In order to apply the results of [5], we change by a homothetical transformation the above problems into ones with uniformly bounded coefficients. At this point (1.9) follows using Theorem 5.2 of [5], applying the reverse transformation and collecting all estimates (see also [1]).

Remark 1.2. From estimate (1.9) it follows that the operator (1.1) under Dirichlet boundary conditions is the infinitesimal generator of an analytic semigroup in $C^{-1,\alpha}$.

2. INTERPOLATION AND EXTRAPOLATION SPACES

For the applications to linear and nonlinear parabolic equations it is interesting to characterize the interpolation and extrapolation spaces between the domain of the operator (1.1) and the space $C^{-1,\alpha}(\Omega)$ (for the related definitions and more details see respectively [3] and [2]). Before stating the main result of this section let us define $C^{0,\alpha}(\Omega, V)$ as the space of the bounded α–Hölder functions u such that $Vu \in C^{0,\alpha}(\bar{\Omega})$. Analogously $C^{1,\alpha}(\Omega, V)$ $(C^{-1,\alpha}(\Omega, V))$ is the space of the functions $u \in C^{1,\alpha}(\bar{\Omega})$ $(u \in C^{-1,\alpha}(\Omega)$, resp.) such that $Vu \in C^{1,\alpha}(\bar{\Omega})$ $(C^{-1,\alpha}(\Omega)$ resp.).

THEOREM 2.1. *Consider the operator* E *defined in* (1.1) *with domain* $C^{1,\alpha}(\bar{\Omega}) \cap C_0^{0,\alpha}(\Omega, V^{1/2}) \cap C^{-1,\alpha}(\Omega, V)$. *Assume that conditions* (1.2)-(1.7) *hold. Then* $D_E(\vartheta, \infty)$ *is isomorphic to the set of functions* $u \in C^{-1,\alpha}(\Omega, V^{\vartheta})$ *such that*

$$
\begin{aligned}
&u \in C^{-1,\alpha+2\vartheta}(\Omega) && \text{for } \vartheta \in]0, (1-\alpha)/2[\\
&u \in C_0^{0,2\vartheta+\alpha-1}(\bar{\Omega}) && \text{for } \vartheta \in](1-\alpha)/2, 1-\alpha/2[\\
&u \in C^{1,2\vartheta+\alpha-2}(\bar{\Omega}) \cap C_0^0(\bar{\Omega}) && \text{for } \vartheta \in]1-\alpha/2, 1[\,.
\end{aligned}
$$

Proof. In this situation, by a partition of the unity the global problem is reduced to local problems (see [4] chapter 3). Assume, for the sake of simplicity $\Omega = \mathbf{R}^n$ and let x_0 be a point of $\mathbf{R}^n$. We consider the interpolation spaces $(C^{1,\alpha}(B(x_0, 1)) \cap C_0^{0,\alpha}(B(x_0, 1), V^{1/2}) \cap C^{-1,\alpha}(B(x_0, 1), V), C^{-1,\alpha}(B(x_0, 1))_{\vartheta,\infty}$ where $0 < \vartheta < 1$ and $B(x_0, 1) = \{x \in \mathbf{R}^n : |x - x_0| \leq 1\}$. By a homothetical transformation of rate $V^{1/2}(x_0)$ and a translation our problem reduces to characterizing the interpolation spaces $(C^{1,\alpha}(B(0, V^{1/2}), V^{\alpha+1/2}) \cap C_0^0(\overline{B(0, V^{1/2})}), C^{-1,\alpha}(B(0, V^{1/2}), V^{-1/2})_{\vartheta,\infty}$. By the results of [5] the above interpolation spaces are isomorphic to $C^{1+\alpha-2\vartheta}(B(0, V^{1/2}), V^{1/2+\alpha-\vartheta})$ $(\cap C_0^0(\overline{B(0, V^{1/2})}))$ for $\vartheta < (1-\alpha)/2))$. We have used the notation $C^{\beta}(\Omega) = C^{[\beta],[\beta-\beta]}(\bar{\Omega})$, $[\beta]$ is the integral part of $\beta \in \mathbf{R} \setminus \mathbf{Z}$. Therefore by the reverse homothetical transformation, the interpolation spaces in $B(x_0, 1)$ are isomorphic to $C^{1+\alpha-2\vartheta}(B(x_0, 1)) \cap C^{-1,\alpha}(B(x_0, 1), V^{1-\vartheta}(x_0))$ $(\cap C_0^0(\overline{B(x_0, 1)}))$ for $\vartheta < (1-\alpha)/2)$.

The statement easily follows by using a suitable net of $\mathbf{R}^n$ and combining together the corresponding relations. ■

Let us consider the following operator:

$$\begin{cases} \tilde{E} : \{u \in C^{1,\alpha}(\bar{\Omega}) \cap C_0^0(\bar{\Omega}) \quad \text{such that} \quad Eu \in C_0^{0,\beta}(\bar{\Omega})\} \to C_0^{0,\beta}(\bar{\Omega}) \\ \tilde{E}u = Eu . \end{cases}$$

We want to characterize the interpolation spaces $D_{\tilde{E}}(\vartheta, \infty)$ for small ϑ.

In this situation the result also follows by a suitable localization and by Theorem 6.2 for [5].

THEOREM 2.1. *Under the assumptions* (1.2)-(1.7)

$$D_{\tilde{E}}(\vartheta, \infty) = \begin{cases} C_0^{0,\beta+2\vartheta}(\bar{\Omega}) \cap C_0^{0,\vartheta}(\Omega, V^\beta) & 0 < \vartheta < (1-\beta)/2 \\ C^{1,2\vartheta+\beta-1}(\bar{\Omega}) \cap C_0^{0,\beta}(\Omega, V^\vartheta) & (1-\beta)/2 < \vartheta < (1-\beta+\alpha/2) \end{cases}$$ ■

Now, we characterize the space $C^{-1,\alpha}$ as an extrapolation space.

Let us consider this non-variational elliptic operator:

$$Au = -\sum_{i,j} A_{ij} D_{ij} u + \sum_i B_i D_i u + Cu$$

Assume that

(1.9) $$A_{ij} \in C_b^1(\bar{\Omega}), \; B_i/(V^{1/2}) \qquad \text{and} \qquad C/V \in C_b^{0,\alpha}(\bar{\Omega})$$

It is known ([1]) that under assumptions (1.2), (1.5)-(1.6), (1.9) the operator $A : C^{2,\alpha}(\bar{\Omega}) \cap C^{1,\alpha}(\bar{\Omega}, V^{1/2}) \cap C_0^{0,\alpha}(\bar{\Omega}, V) \to C_0^{0,\alpha}(\bar{\Omega})$ is the infinitesimal generator of an analytic semigroup. Let us define $\tilde{C}_\alpha^{0,\beta}(\Omega, V)$, $1 > \alpha > \beta > 0$, as the space of the functions f locally β-Hölder continuous which admit a representation $f = f_1 + f_2$ where $f_1 \in C_0^{0,\beta}(\bar{\Omega})$ and $f_2 \in C_0^{0,\alpha}(\Omega, V^{(\beta-\alpha)/2})$. Analogously $\tilde{C}^{-1,\beta}(\Omega, V)$ is the space of the functions g admitting a representation $g = g_1 + g_2$ where $g_1 \in C^{-1,\beta}(\Omega)$ and $g_2 \in C_0^{0,\alpha}(\Omega, V^{(\beta-1-\alpha)/2})$.

By using the characterization of the extrapolation spaces given in [5] and a suitable localization, it is possible to prove:

THEOREM 3.1. *Under the assumptions* (1.2), (1.5)-(1.6), (1.9)

$$D_A(\vartheta - 1, \infty) = \begin{cases} \tilde{C}^{0,2\vartheta+\alpha-2}(\Omega, V) & 1 - \alpha/2 < \vartheta < 1 \\ \tilde{C}^{-1,2\vartheta+\alpha-1}(\Omega, V) & (1-\alpha)/2 < \vartheta < 1 - \alpha/2 . \end{cases}$$

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