Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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Quasilinear elliptic equations with discontinuous coefficients

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **82** (1988), n.1, p. 21–28.

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<http://www.bdim.eu/item?id=RLINA_1988_8_82_1_21_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1988.

Atti Acc. Lincei Rend. fis.
(8), LXXXII (1988), pp. 21-28

Calcolo delle variazioni. — *Quasilinear elliptic equations with discontinuous coefficients.* Nota (*) di LUCIO BOCCARDO e GIUSEPPE BUTTAZZO, presentata dal Socio E. DE GIORGI.

ABSTRACT. — We prove an existence result for equations of the form

$$\begin{cases} -D_i(a_{ij}(x,u)\,D_j u) = f & \text{in } \Omega \\ u \in H_0^1(\Omega) \end{cases}$$

where the coefficients $a_{ij}(x,s)$ satisfy the usual ellipticity conditions and hypotheses weaker than the continuity with respect to the variable s. Moreover, we give a counterexample which shows that the problem above may have no solution if the coefficients $a_{ij}(x,s)$ are supposed only Borel functions.

KEY WORDS: Quasilinear elliptic equations; Dirichlet problems; Semicontinuity; Calculus of variations.

RIASSUNTO. — *Equazioni ellittiche quasi lineari con coefficienti discontinui.* Si dimostra un teorema di esistenza per equazioni del tipo

$$\begin{cases} -D_i(a_{ij}(x,u)\,D_j u) = f & \text{in } \Omega \\ u \in H_0^1(\Omega) \end{cases}$$

dove i coefficienti $a_{ij}(x,s)$ verificano le usuali ipotesi di ellitticità ed ipotesi più deboli della continuità rispetto alla variabile s. Si mostra inoltre con un controesempio che il problema precedente può non avere nessuna soluzione se i coefficienti $a_{ij}(x,s)$ sono supposti soltanto boreliani.

1. INTRODUCTION

In this paper we consider quasilinear elliptic equations of the form

$$\begin{cases} -D_i(a_{ij}(x,u)\,D_j u) = f & \text{in } \Omega \\ u \in H_0^1(\Omega) \end{cases} \tag{1.1}$$

(the summation convention over repeated indices is adopted) where Ω is a

(*) Pervenuta all'Accademia il 14 luglio 1987.

bounded open subset of $\mathbf{R}^n$, $f \in H^{-1}(\Omega)$ is given, and the coefficients $a_{ij}(x, s)$ satisfy the standard ellipticity and boundedness condition

$$
(1.2) \qquad \begin{cases} \lambda \, |z|^2 \leq a_{ij}(x, s) \, z_i z_j \\ |a_{ij}(x, s)| \leq \Lambda \end{cases} \qquad (0 < \lambda \leq \Lambda)
$$

for almost all $x \in \Omega$, and all $s \in \mathbf{R}$, $z \in \mathbf{R}^n$.

Existence results for problem (1.1) are well-known in the literature (see for instance [6], [7]) when the coefficients $a_{ij}(x, s)$ are functions of Carathéodory type (i.e., measurable in x and continuous in s). However, equations of the form (1.1) with discontinuous (with respect to s) coefficients $a_{ij}(x, s)$ occur in many problems of physics. For example, if Ω is seen as a thermically conducting body, u its temperature, and f the density of heat source, the equation (1.1) governs the heat conduction in Ω, and the $a_{ij}(x, s)$ are the conductivity coefficients which may depend discontinuously on the temperature (for instance, in liquid-solid phase transition).

A simple case of discontinuous coefficients for which the existence of a solution of problem (1.1) holds, is when (see [3])

$$
a_{ij}(x, s) = \alpha_{ij}(x) \, b(s)
$$

where $\alpha_{ij}(x)$ and $b(s)$ are measurable functions satisfying (1.2). In fact, setting

$$
\mathrm{B}(s) = \int_0^s b(t) \, \mathrm{d}t ,
$$

and recalling the chain-rule for derivation (see [8], [10])

$$
\mathrm{D}(\mathrm{B}(u)) = \mathrm{B}'(u) \, \mathrm{D}u \qquad \text{for every } u \in \mathrm{H}^1(\Omega) ,
$$

it is enough to take $u = \mathrm{B}^{-1}(v)$, where v is the solution of the linear elliptic problem

$$
\begin{cases} -\mathrm{D}_i(\alpha_{ij}(x) \, \mathrm{D}_j v) = f & \text{in } \Omega \\ v \in \mathrm{H}_0^1(\Omega) . \end{cases}
$$

Unfortunately, this simple argument cannot be applied to general equations of the form (1.1); thus, our approach is based on two steps: the first one consists in (Section 2) proving that, under some mild assumptions on $a_{ij}(x, s)$, the operator

$$
u \to \mathrm{D}_i(a_{ij}(x, u) \, \mathrm{D}_j u)
$$

is weakly continuous between $H^1(\Omega)$ and $H^{-1}(\Omega)$, and the second consists in (Section 3) proving that this weak continuity implies the surjectivity.

In the last section, we give a simple one-dimensional example to show that the sole hypothesis (1.2) is not sufficient to get the existence result for problem (1.1).

2. WEAK CONTINUITY OF QUASILINEAR OPERATORS

In this section we consider operators $A : H^1(\Omega) \to L^2(\Omega)$ of the form

$$Au = a(x, u) D_j u \tag{2.1}$$

where Ω is a bounded open subset of $\mathbf{R}^n$, $j \in \{1, \ldots, n\}$ is an integer, and $a : \Omega \times \mathbf{R} \to \mathbf{R}$ is a function. We denote by $\mathscr{B}_k$ and $\mathscr{L}_k$ the Borel and Lebesgue σ-algebras in $\mathbf{R}_k$ respectively; if $E \in \mathscr{L}_k$ we denote by $|E|$ the Lebesgue measure of E. Our main result is the following.

THEOREM 2.1. *Assume that:*

(2.2) *the function $a(x, s)$ is bounded and $\mathscr{L}_k \otimes \mathscr{L}_1$-measurable;*

(2.3) *for every $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset \Omega$ such that $|\Omega - K_\varepsilon| < \varepsilon$, and for every $R > 0$ the family of functions $\{a(\cdot, s)\}_{|s| \leq R}$ is equicontinuous on K_ε.*

Then, the operator A defined in (2.1) is sequentially weakly continuous between $H^1(\Omega)$ and $L^2(\Omega)$.

Proof. Arguing as in [2] we may assume that $a(x, s)$ is a Borel function, so that the operator A is well-defined between $H^1(\Omega)$ and $L^2(\Omega)$. It remains to show that for every $v \in L^2(\Omega)$

$$\int_\Omega Au_h\, v\, dx \to \int_\Omega Au\, v\, dx \qquad \text{whenever} \quad u_h \to u \text{ in } H^1(\Omega).$$

Since $a(x, s)$ is bounded, we may restrict ourselves to the case $v \in \mathscr{D}(\Omega)$; moreover, changing $a(x, s)$ into $-a(x, s)$, it is enough to prove that for every $v \in \mathscr{D}(\Omega)$ the functional

$$F(u, \Omega) = \int_\Omega v(x)\, a(x, u)\, D_j u\, dx$$

is sequentially weakly lower semicontinuous on $H^1(\Omega)$. Fix $v \in \mathscr{D}(\Omega)$ and set for every $m > 0$

$$f_m(x, s, z) = [v(x)\, a(x, s)\, z_j] \vee (-m)$$

$$F_m(u, \Omega) = \int_\Omega f_m(x, u, Du)\, dx .$$

By Theorem 4.15 of [1] the functionals F_m are sequentially weakly lower semicontinuous on $H^1(\Omega)$; moreover, if $u_h \to u$ weakly in $H^1(\Omega)$ we have denoting by c an arbitrary constant

$$F_m(u_h, \Omega) \leq F(u_h, A_{m,h}) \leq F(u_h, \Omega) + c \int_{\Omega - A_{m,h}} |v|\, |Du_h|\, dx \leq \tag{2.4}$$

$$\leq F(u_h, \Omega) + c\, |\Omega - A_{m,h}|^{1/2} \left[\int_{\Omega - A_{m,h}} |Du_h|^2\, dx \right]^{1/2} \leq F(u_h, \Omega) + c$$

$$|\Omega - A_{m,h}|^{1/2}$$

where $A_{m,h} = \{x \in \Omega : v(x)\, a(x, u_h(x))\, D_j u_h(x) \geq -m\}$. We have

$$|\Omega - A_{m,h}| \leq |\{c\, |v(x)|\, |Du_h(x)| > m\}| \leq \frac{c}{m} \int_\Omega |v|\, |Du_h|\, dx \leq \frac{c}{m} ,$$

so that, by (2.4)

$$F(u, \Omega) \leq F_m(u, \Omega) \leq \liminf_{h \to \infty} F_m(u_h, \Omega) \leq \liminf_{h \to \infty} F(u_h, \Omega) + c\, m^{-1/2} ,$$

and this achieves the proof. ■

Remark 2.2. Note that hypothesis (2.3) is satisfied for instance in the following cases:

(i) $a(x, s)$ is measurable in x and continuous in s;

(ii) $a(x, s) = \alpha(x)\, b(s)$ with α and b measurable functions.

Remark 2.3. By Theorem 2.1 every operator of the form

$$Au = -D_i(a_{ij}(x, u)\, D_j u)$$

is sequentially weakly continuous between $H^1(\Omega)$ and $H^{-1}(\Omega)$ provided that the coefficients $a_{ij}(x, s)$ satisfy hypotheses (2.2) and (2.3).

Remark 2.4. If $a(x, s)$ is only measurable in s and continuous in x, the operator A in (2.1) may be not sequentially weakly continuous. For a counter-example we refer to [1], Section 5, Example 6.

3. A SURJECTIVITY RESULT

In this section X denotes a Hilbert space and $T : X \to X$ is a mapping. Our surjectivity result is the following.

THEOREM 3.1. *Assume that*

(i) T *is sequentially weakly continuous (i.e.,* $x_h \rightharpoonup x \Rightarrow T(x_h) \rightharpoonup T(x)$*)*;

(ii) *there exist* $a > 0$ *and* $b \geq 0$ *such that*

$$\langle T(x), x \rangle \geq a \| x \|^2 - b \qquad \textit{for every } x \in X;$$

(iii) *there exists* $c > 0$ *such that*

$$\| T(x) \| \leq c (1 + \| x \|) \qquad \textit{for every } x \in X.$$

Then T *is surjective.*

Proof. We use an idea of Stampacchia (see [9]). Let $y \in X$; we want to solve the equation $T(x) = y$ or, equivalently, the equation

$$x + t(y - T(x)) = x \tag{3.1}$$

for some $t > 0$. Denote by $S : X \to X$ the mapping

$$S(x) = x + t(y - T(x)).$$

Then, we are looking for a fixed point of S. We have for every $x \in X$

$$\| S(x) \|^2 = \| x \|^2 + t^2 \| y \|^2 + t^2 \| T(x) \|^2 + 2t \langle x, y \rangle - 2t \langle x, T(x) \rangle -$$
$$- 2t \langle y, T(x) \rangle \leq \| x \|^2 (1 + t^2 c^2 - 2at) + K(t)(1 + \|x\|)$$

where $K(t)$ is a suitable constant depending on t. Taking $t = a/c^2$ we get

$$\| S(x) \|^2 \leq \| x \|^2 \left(1 - \frac{a^2}{c^2}\right) + K(a/c^2)(1 + \| x \|)$$

so that

$$\| S(x) \| \leq c_1 \| x \| + c_2 \tag{3.2}$$

for suitable constants c_1 and c_2 with $c_1 < 1$. By (3.2), there exists $R > 0$ such that

$$\| x \| \leq R \Rightarrow \| S(x) \| \leq R .$$

Set $B_R = \{x \in X : \| x \| \leq R\}$; by hypothesis (i) the mapping $S : B_R \to B_R$ is weakly continuous, so that by the Schauder-Tychonoff fixed point theorem (see [4], page 74) it admits a fixed point $x_0 \in B_R$ which is a solution of equation (3.1). ■

Remark 3.2. A result similar to Theorem 3.1 holds for mappings $T : X \to X'$ where X' is the dual space of X. In fact, if $J : X \to X'$ denotes the Riesz isomorphism, it is enough to apply Theorem 3.1 to the mapping $J \circ T$.

4. The existence result

Let Ω be a bounded open subset of $\mathbf{R}^n$ and let $f \in H^{-1}(\Omega)$; consider the problem

$$\begin{cases} -D_i(a_{ij}(x, u) D_j u) = f & \text{in } \Omega \\ u \in H_0^1(\Omega) . \end{cases} \tag{4.1}$$

On the coefficients $a_{ij}(x, s)$ we assume that:

(4.2) every $a_{ij}(x, s)$ is measurable in (x, s) and satisfies property (2.3);

(4.3) the ellipticity and boundedness condition (1.2) is satisfied.

By using Theorem 2.1, Theorem 3.1 and Remark 3.2, we obtain immediately the following existence result.

Theorem 4.1. *Assume* (4.2) *and* (4.3). *Then, for every* $f \in H^{-1}(\Omega)$ *problem* (4.1) *admits at least a solution.*

Remarks 4.2. Problems with lower order terms and non-zero boundary condition, like

$$\begin{cases} -D_i(a_{ij}(x, u) D_j u) + D_i(a_i(x, u)) + b_i(x, u) D_i u + a(x, u) = 0 & \text{in } \Omega \\ u - \phi \in H_0^1(\Omega) \end{cases}$$

(with $\phi \in H^1(\Omega)$), can be treated in a similar way provided the coefficients $b_i(x, s)$ are measurable in (x, s) and satisfy property (2.3), $a_i(x, s)$ and $a(x, s)$ are Carathéodory functions, and the usual bounds on b_i, a_i, a are satisfied (see for instance [6]).

Remark 4.3. In [5] it is proved that the quasilinear structure $-\mathrm{D}_i(a_{ij}(x, u)\mathrm{D}_j u)$ is a necessary condition for the sequential weak continuity of Leray-Lions operators.

When condition (2.3) is not satisfied, in general the existence for problem (4.1) may fail, as the following example shows.

Example 4.4. Let $n=1$, $\Omega =]0, 1[$, and

$$a(x, s) = \begin{cases} 1+x & \text{if } s = x \\ 1 & \text{if } s \neq x . \end{cases}$$

The function $a(x, s)$ is a Borel function which does not satisfy property (2.3). Consider the problem

$$\begin{cases} (a(x, u)\, u')' = 0 \\ u(0) = 0 \quad u(1) = 1 \end{cases} \tag{4.4}$$

and assume by contradiction that a solution u exists. Setting

$$\mathrm{A} = \{x \in \Omega : u(x) = x\},$$

by (4.4) we obtain

$$(1+x)\, 1_{\mathrm{A}}(x) \times 1_{\Omega-\mathrm{A}}(x)\, u'(x) = c \qquad \text{a.e. in } \Omega \tag{4.5}$$

where c is a suitable constant and 1_{A}, $1_{\Omega-\mathrm{A}}$ are the characteristic functions of A, $\Omega - \mathrm{A}$ respectively. By (4.5) we have

$$1 + x = c \qquad \text{a.e. in A},$$

so that A is negligible, and so

$$u'(x) = c \qquad \text{a.e. in } \Omega .$$

The boundary conditions in (4.4) then imply that $u(x) = x$, which contradicts the fact that A is negligible.

If instead of equations we deal with elliptic systems, the existence result of Theorem 4.1 may fail, even if the coefficients do not depend on the x variable. In fact, the following example holds.

Example 4.5. Let $n = 1$, $\mathrm{N} = 2$, $\Omega =]0, 1[$. Consider the problem

$$\begin{cases} (a(u, v)\, v')' = 0 \\ (b(u, v)\, u')' = 0 \end{cases}$$

with the boundary conditions

$$u(0) = 0\,, \quad u(1) = 1\,, \quad v(0) = 0\,, \quad v(1) = 1\,.$$

Take

$$b(u, v) \equiv 1$$

$$a(u, v) = \begin{cases} 1 + |u| & \text{if } v = u \\ 1 & \text{if } v \neq u\,. \end{cases}$$

Then we have $u(x) = x$, and v must satisfy the equation

$$(a(x, v)\, v')' = 0 \qquad \text{with} \quad v(0) = 0\,,\ v(1) = 1\,,$$

which, by Example 4.4 has no solution.

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