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Analisi matematica. — *On the inequalities associated with a model of Graffi for the motion of a mixture of two viscous, incompressible fluids.* Nota (*) di GIOVANNI PROUSE e ANNA ZARETTI, presentata dal Socio L. AMERIO.

ABSTRACT. — We demonstrate a theorem of existence and uniqueness on a large scale of the solution of a system of differential disequations associated to a Graffi model relative to the motion of two incompressible viscous fluids.

KEY WORDS: Partial differential equations and inequalities; Fluid dynamics; Mathematical models.

RIASSUNTO. — *Sulle disequazioni associate ad un modello di Graffi per il moto di una miscela di due fluidi viscosi incomprimibili.* Si dimostra un teorema di esistenza ed unicità in grande della soluzione di un sistema di disequazioni differenziali associato ad un modello di Graffi relativo al moto di una miscela di due fluidi viscosi incomprimibili.

The study of the motion of a mixture of two viscous, incompressible fluids in a closed basin Ω is of particular interest, for example, in the analysis of problems connected with pollution. The corresponding equations can be deduced, under more or less stringent assumptions, from the general equations of hydrodynamics (Navier–Stokes equations and mass conservation equation) and from experimental relationships regarding the behaviour of a mixture (Fick's law). In this law there appears the *molecular diffusion coefficient* λ and various mathematical models can be obtained depending on the assumptions made on λ.

If no assumption is made, the corresponding model has been studied by Beirao da Veiga [1] who has proved a *local* existence and uniqueness theorem for the solutions of the classical Cauchy–Dirichlet problem.

If λ is assumed to be "small" (and therefore all terms involving λ are neglected), the corresponding model has been studied by Antonov and Kazhikov [2], Ladyzenskaja and Solonnikov [3], Lions [4]; the results obtained are similar to the well-known ones for the classical Navier–Stokes equations, i.e. *global* existence of a *weak* solution (not necessarily unique), *local* existence and uniqueness of a *strong* solution.

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The model considered in the present note was proposed over 30 years ago by Graffi [5], who deduced it without introducing Fick's law; this model appears however to be "intermediate" between the two considered above, since it can be obtained by eliminating only some of the terms involving the molecular diffusion coefficient; such a partial elimination can be justified from a physical and mathematical point of view.

It is obvious that the results given in [1] hold also for the Graffi model, but the problem of global existence and uniqueness appears to be still open.

Denoting by $\vec{u}$ the mass velocity of the mixture, by p its pressure, by ρ its density, by $\vec{f}$ the external force and by μ the viscosity coefficient, the equations which constitute the Graffi model are

$$(1)\qquad \begin{aligned} &\rho\left(\frac{\partial \vec{u}}{\partial t} + (\vec{u}\,,\nabla)\,\vec{u} - \vec{f}\right) = -\nabla p + \mu\Delta\vec{u} \\ &\frac{\partial \rho}{\partial t} + \vec{u}\cdot\nabla\rho = \lambda\Delta\rho \\ &\operatorname{div}\vec{u} = 0\,, \end{aligned}$$

to which are associated the initial and boundary conditions

$$(2)\qquad \vec{u}\,(x\,,0) = \vec{\bar{u}}\,(x) \qquad \rho\,(x\,,0) = \bar{\rho}\,(x) \qquad (x \in \Omega)$$

$$(3)\qquad \vec{u}\,(x\,,t) = 0 \qquad \frac{\partial \rho\,(x\,,t)}{\partial \nu} = 0 \qquad (x \in \partial\Omega\,,\, t \in (0\,,\mathrm{T}))\,.$$

It must however be observed that the validity, from the physical point of view, of equations (1) is subject to certain limitations, which we call *consistency conditions*, which are assumed to hold for the deduction of (1) from the general hydrodynamical equations and from Fick's law; it is easy to see that these conditions can be expressed by the inequalities

$$(4)\qquad |\vec{u}| < \mathrm{M}_1 \quad , \quad |\nabla\rho| < \mathrm{M}_2 \quad , \quad |\Delta\rho| < \mathrm{M}_3 \quad , \quad \left|\frac{\partial \rho}{\partial t}\right| < \mathrm{M}_4\,,$$

where the M_i's are suitable constants.

The study of equations (1) with the consistency conditions (4) leads to the introduction of the following system of differential inequalities

$$(5)\qquad \begin{aligned} &\int_0^t\!\int_\Omega \left[\rho\left(\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\nabla)\,\vec{u} - \vec{f}\right) - \mu\Delta\vec{u}\right](\vec{u} - \vec{\varphi})\,d\Omega\,d\eta \leq 0 \\ &\int_0^t\!\int_\Omega \left(\frac{\partial \rho}{\partial t} + \vec{u}\cdot\nabla\rho - \lambda\Delta\rho\right)\left(\frac{\partial \rho}{\partial t} - \frac{\partial \psi}{\partial t}\right) d\Omega\,d\eta \leq 0 \\ &\operatorname{div}\vec{u} = 0 \end{aligned}$$

$\forall\vec{\varphi}\,,\psi$ belonging to appropriate convex sets.

More precisely, setting $N^1 = \{\vec{v} \in H_2(\Omega): \operatorname{div} \vec{v} = 0\}$, $K_1 = \{\vec{v} \in L^2(\Omega): |\vec{v}| \leq M_1\}$, $K_2 = \{g \in L^2(\Omega): |\nabla g| \leq M_2, |\Delta g| \leq M_3\}$, $K_3 = \{h \in L^2(\Omega): |h| \leq M_4\}$, we shall say that $\{\vec{u}, \rho\}$ is a *solution of* (5) *in* $[0, T]$ *satisfying* (2), (3) if:

i) $\vec{u}(t) \in L^\infty(0, T; N^1 \cap K_1) \cap L^2(0, T; H^2)$, $\vec{u}'(t) \in L^\infty(0, T; N^0) \cap \cap L^2(0, T; N^1)$, $\rho(t) \in L^\infty(0, T; K_2)$, $\rho'(t) \in L^\infty(0, T; K_3)$, $\vec{u}(0) = \bar{\vec{u}}$, $\rho(0) = \bar{\rho}$;

ii) u, ρ *satisfy* (5) *a.e. in* $(0, T) \,\forall \vec{\varphi}(t) \in L^2(0, T; N^1 \cap K^1)$, $\psi(t) \in \in L^\infty(0, T; K_2)$ with $\psi'(t) \in L^\infty(0, T; H^1 \cap K_3)$.

The following *global existence and uniqueness* theorem holds: *Assume that* $\bar{\vec{u}} \in N^1 \cap H^2(\Omega) \cap K_1$, $\bar{\rho} \in H^1(\Omega) \cap K_2$, $\vec{f}(t) \in H^1(0, T; L^2(\Omega))$ *and that* Ω *is an open, bounded, convex set* $\subset R^3$, *with* $\partial\Omega$ *constituted by a finite number of surfaces of class* C^2. *There exists then one, and only one, pair* $\{\vec{u}, \rho\}$ *satisfying* i), ii).

The solution therefore takes its values, a.e. on $[0, T]$, in the same functional spaces as the initial data.

The complete proof of this theorem is given in a paper to appear in the Rendiconti dell'Accademia Nazionale dei XV.

In order to give a physical interpretation of the theorem stated above we observe that, as is well-known, if $\{\vec{u}, \rho\}$ is a solution of (5) ans satisfies (4) in $[0, t')$ $(0 < t' \leq T)$, then $\{\vec{u}, \rho\}$ satisfies (1) in the sense of distributions on $\Omega \times (0, t')$. Hence, *if the solutions of* (5) *satisfy the consistency conditions in* $Q' = \Omega \times (0, t')$, *they are also solutions in* Q' *of Graffi's model.*

The *global* result obtained for (5) may therefore correspond to a *local* one for (1). This latter however differs from the local results which can be obtained directly for (1) since in these last the local time interval does not have a precise physical meaning, since it depends on a priori estimates involving embedding constants, etc.; in our case, on the contrary, the interval $[0, t')$ represents the *largest* time interval in which the solution satisfies the consistency conditions, i.e. has physical meaning. If t' were $= 0$, we should, in particular, conclude that the Graffi model is not acceptable (with the given data).

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