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## An improved result on irregularities in distribution of sequences of integers

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**Teoria dei numeri.** — *An improved result on irregularities in distribution of sequences of integers.* Nota (*) di JOHN H. HODGES, presentata dal Socio G. ZAPPA.

ABSTRACT. — In 1972 the author used a result of K.F. Roth on irregularities in distribution of sequences of real numbers to prove an analogous result related to the distribution of sequences of integers in prescribed residue classes. Here, a 1972 result of W.M. Schmidt, which is an improvement of Roth's result, is used to obtain an improved result for sequences of integers.

KEY WORDS: Irregularities; Distribution; Sequences; Integers.

RIASSUNTO. — *Un risultato più preciso sulle irregolarità nella distribuzione di successione di interi.* Nel 1972 l'Autore si servì di un risultato di K.F. Roth sulle irregolarità nelle distribuzioni delle successioni di numeri reali per dimostrare un analogo risultato relativo alle distribuzioni delle successioni di interi in prescritte classi-resto. Qui, facendo uso di un risultato di W.M. Schmidt del 1972, che migliora quello di Roth, l'Autore ottiene un analogo miglioramento del proprio risultato sulle successioni di interi.

1. Introduction and preliminaries. Let $\{\alpha_i\}$ be an arbitrary infinite sequence of real numbers in the unit interval $U = [0, 1)$. For any $\alpha \in U$ and large positive integer $n$, let $Z(n, \alpha)$ denote the number of $i$ with $1 \leqq i \leqq n$ such that $0 \leqq \alpha_i \leqq \alpha$. Furthermore, let

$$D(n, \alpha) = |Z(n, \alpha) - n\alpha|. \tag{1.1}$$

In 1954, K.F. Roth [3] proved that if N is any large integer $> 1$, then there exist an integer $n$ and $\alpha \in U$ with $1 \leqq n \leqq N$ such that

$$D(n, \alpha) > c_1 (\log N)^{1/2}, \tag{1.2}$$

where $c_1$ is a positive absolute constant. (Note that $\{\alpha_i\}$ is uniformly distributed modulo 1 iff $\lim_{n\to\infty} n^{-1} D(n, \alpha) = 0$ for all $\alpha \in U$). In 1972, Wolfgang M. Schmidt [5] improved this result by proving that there exist an integer $n$ and $\alpha \in U$ with $1 \leqq n \leqq N$ such that

$$D(n, \alpha) > 10^{-2} \log N. \tag{1.3}$$

(*) Pervenuta all'Accademia il 18 settembre 1987.

Earlier results of this sort are described in the papers by Roth and Schmidt cited above and a 1968 paper by Schmidt [4].

In [1], the author used Roth's inequality (1.2) to prove an analogue for irregularities in distribution of sequences of integers. In the present note, we apply Schmidt's result (1.3) to obtain an improved result for sequences of integers.

2. Distributions in intervals. Let $A = \{a_i\}$ be an arbitrary sequence of integers and $m, b$ be integers such that $m > 1$ and $0 \leqq b < m$. For any positive integer $n$, let $A(n, b, m)$ denote the number of $i$ such that $1 \leqq i \leqq n$ and $a_i \equiv b \pmod{m}$. (Note that A is uniformly distributed modulo $m$ as defined by Niven [2] iff $\lim_{n\to\infty} n^{-1} A(n, b, m) = 1/m$, for all $0 \leqq b < m$). Then define

$$S(n, b, m) = \left| \sum_{j=0}^{b} A(n, j, m) - n(b+1)/m \right| . \tag{2.1}$$

Thus, $S(n, b, m)$ is the magnitude of the total number of $a_i$ with $1 \leqq i \leqq n$ that lie in the $b+1$ residue classes (mod $m$) determined by $j = 0, 1, \ldots, b$, minus $n$ times the proportion of the residue classes (mod $m$) involved, i.e., minus the "expected number" of terms of A among the first $n$ terms that would lie in these $b+1$ residue classes if the terms of A were "evenly" distributed among all the $m$ different residue classes (mod $m$).

In [1], the author used Roth's result (1.2) to prove Theorem 1. If $A = \{a_i\}$ is any sequence of integers and N is any large integer $> 1$, then there exist integers $n, b, m$ with $1 \leqq n \leqq N$, $m > 1$ and $0 \leqq b < m$ such that

$$S(n, b, m) > c(\log N)^{1/2},$$

where $c$ is a positive absolute constant.

If $n$ is any fixed positive integer, we indicate by $S(n)$ the sup of $S(n, b, m)$ for all integers $b, m$ such that $0 \leqq b < m$. Then using Schmidt's result (1.3) we can prove the better result

THEOREM 2. If N is any large integer $> 1$, then there exists an integer $n$ with $1 \leqq n \leqq N$ such that

$$S(n) \geqq 10^{-2} \log N .$$

*Proof*: We need only modify in an appropriate way the proof of Theorem 1 given in [1]. With the given sequence A we associate the sequence $\{\alpha_i\}$ of real numbers, where for all $i$, $\alpha_i = a'_i/m$, $a'_i$ is the least positive residue (mod $m$) of $a_i$ and $m$ is an integer $> 1$ to be chosen later. If $Z(n, \alpha)$ is the counting function for the sequence $\{\alpha_i\}$ as defined in section 1, then by Schmidt's result (1.3) there exist $n$, $\alpha$ with $n$ an integer $1 \leqq n \leqq N$ and $\alpha$ real, $0 \leq \alpha < 1$ such that

$$D(n, \alpha) = | Z(n, \alpha) - n\alpha | > 10^{-2} \log N . \tag{2.2}$$

Define $b = [m\alpha]$ so that $0 \leqq b < m$ since $0 \leqq \alpha < 1$.

Since $a'_i = m\alpha_i \leqq [m\alpha] = b$ if and only if $a'_i/m = \alpha_i \leqq \alpha$, it follows that

$$Z(n, \alpha) = \sum_{j=0}^{b} A(n, j, m),$$

and substituting this expression into (2.2) gives

$$\left| \sum_{j=1}^{b} A(n, j, m) - n\alpha \right| = | \pm S(n, b, m) + \{n(b+1)/m - \\ - n\alpha\} | > 10^{-2} \log N. \tag{2.3}$$

Since $b \leqq m\alpha < b + 1$, it follows that $nb/m \leqq n\alpha < n(b+1)/m$ so that $0 < \{n(b+1)/m - n\alpha\} \leqq n/m$. If we choose $m$ to be any integer $> 1$ such that $m > c\, 10^2\, N/\log N$ for any fixed constant $c > 1$, then $n/m \leqq N/m < < c^{-1}\, 10^{-2} \log N$. Thus, in view of (2.3), for any such $c$ and subsequent values of $m, b, n$,

$$S(n, b, m) > c^{-1}(c-1)\, 10^{-2} \log N . \tag{2.4}$$

From (2.4) it follows that given N and $c > 1$, there exists an integer $n$ (depending on $m$ and so on $c$) with $1 \leqq n \leqq N$ such that

$$S(n) > c^{-1}(c-1)\, 10^{-2} \log N . \tag{2.5}$$

If we let $c \to \infty$ in (2.5), since $n$ can take on only a finite number of different values for fixed N, some fixed value of $n$ must occur with infinitely many different values of $c$ that increase without bound. For such a fixed value of $n$, and its associated values of $c$,

$$S(n) \geqq \lim_{c \to \infty} [c^{-1}(c-1)\, 10^{-2} \log N] = 10^{-2} \log N , \tag{2.6}$$

so that the theorem is proved.

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