Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Aldo Bressan

On control theory and its applications to certain problems for Lagrangian systems. On hyperimpulsive motions for these. II. Some purely mathematical considerations for hyper-impulsive motions. Applications to Lagrangian systems

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **82** (1988), n.1, p. 107–118.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1988_8_82_1_107_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1988.

Atti Acc. Lincei Rend. fis.
(8), LXXXII (1988), pp. 107-118

Fisica matematica. — *On control theory and its applications to certain problems for Lagrangian systems. On hyperimpulsive motions for these.* II. *Some purely mathematical considerations for hyper-impulsive motions. Applications to Lagrangian systems.* Nota(*) del Corrisp. ALDO BRESSAN.

ABSTRACT. — See Summary in Note I. First, on the basis of some results in [2] or [5]–such as Lemmas 8.1 and 10.1–the general (mathematical) theorems on controllizability proved in Note I are quickly applied to (mechanic) Lagrangian systems. Second, in case Σ, χ, and M satisfy conditions (11.7) when $\mathscr{Q}$ is a polynomial in $\dot{\gamma}$, conditions (C)–i.e. (11.8) and (11.7) with $\mathscr{Q} \equiv 0$–are proved to be necessary for treating satisfactorily Σ's hyper-impulsive motions (in which positions can suffer first order discontinuities).

This is done from a general point of view, by referring to a mathematical system $(\mathscr{M})\ \dot{z} = \mathrm{F}(t, \gamma, z, \dot{\gamma})$ where $z \in \mathbf{R}^m$, $\gamma \in \mathbf{R}^{\mathrm{M}}$, and $\mathrm{F}(\ldots)$ is a polynomial in $\dot{\gamma}$. The afore-mentioned treatment is considered satisfactory when, at a typical instant t, (i) the anterior values z^- and γ^- of z and γ, together with γ^+, determine z^+ in a certain physically natural way, based on certain sequences $\{\gamma_n(\cdot)\}$ and $\{z_n(\cdot)\}$ of regular functions that approximate the 1st order discontinuities (γ^-, γ^+) and (z^-, z^+) of $\gamma(\cdot)$ and $z(\cdot)$ respectively, (ii) for z^- and γ^- fixed, z^+ is a continuous function of γ^+, and (iii) if γ^+ tends to γ^-, then $z(\cdot)$ tends to a continuous function and, for certain simple choices of $\{(\gamma_n(\cdot)\}$ the functions $z_n(\cdot)$ behave in a certain natural way.

For $\mathrm{M} > 1$, conditions (i) to (iii) hold only in very exceptional cases. Then their 1-dimensional versions (*a*) to (*c*) are considered, according to which (i) to (iii) hold, so to say, along a trajectory $\tilde{\gamma}$ ($\in \mathrm{C}^3$) of γ's discontinuity (γ^-, γ^+), chosen arbitrarily; this means that $\tilde{\gamma}$ belongs to the trajectory of all regular functions $\gamma_n(\cdot)$ ($n \in \mathbf{N}_*$), i.e. $\gamma_n(t) \equiv \tilde{\gamma}[u_n(t)]$. Furthermore a certain weak version of (a part of) conditions (*a*) to (*c*) is proved to imply the linearity of $(\mathscr{M})$. Conversely this linearity implies a strong version of conditions (*a*) to (*c*); and when this version holds, one can say that $(\mathscr{D})$ the (M-*dimensional*) *parameter* γ *in* $(\mathscr{M})$ *is fit to suffer* (1-*dimensional first order*) *discontinuities*.

As far as the triplet $(\Sigma, \chi, \mathrm{M})$—see Summary in Nota I—is concerned, for $m = 2\mathrm{N}$, $\chi = (q, \gamma)$, and $z = (q, p)$ with $p_h = \partial T / \partial \dot{q}_h$ $(h = 1, \ldots, \mathrm{N})$, the differential system $(\mathscr{M})$ can be identified with the dynamic equations of $\Sigma_{\hat{\gamma}}$ in semi-hamiltonian form. Then its linearity in $\dot{\gamma}$ is necessary and sufficient for *the co-ordinates* χ of Σ *to be* M-*fit for* (1-*dimensional*) *hyper-impulses*, in the sense that $(\mathscr{D})$ holds.

KEY WORDS: Mathematical physics; Feed-back theory.

RIASSUNTO. — *Sulla teoria dei controlli e le sue applicazioni a certi problemi per sistemi Lagrangiani. Sui moti iperimpulsivi di questi.* II. *Alcune considerazioni generali puramente matematiche sui moti iper-impulsivi. Applicazioni ai sistemi Lagrangiani.* Sulla

(*) Presentata nella seduta del 19 giugno 1987.

base di risultati ottenuti in [2] o [5] – quali i Lemmi 8.1 e 10.1 – si mostra come applicare ai sistemi Lagrangiani i teoremi generali sulla controllabilità, considerati nella Nota I. Nel caso che Σ, χ ed M verifichino le condizioni (11.7) con $\mathscr{Q}$ polinomio nelle $\dot{\gamma}$, si mostra che le condizioni (C) – ossia le (11.8) e le (11.7) con $\mathscr{Q} \equiv 0$ – sono necessarie per poter trattare soddisfacentemente moti iper-impulsivi di Σ (in cui anche le posizioni posson subire discontinuità di 1ª specie)

Si fa quanto sopra da un punto di vista generale, riferendosi dapprima ad un sistema matematico $(\mathscr{M})\ \dot{z} = F(t, \gamma, z, \dot{\gamma})$ polinomiale in $\dot{\gamma}$ e con $z \in \mathbf{R}^m$ e $\gamma \in \mathbf{R}^M$. La suddetta trattazione si considera soddisfacente se, ad un generico istante t, (i) i valori anteriori z^- e γ^- di z e γ, e γ^+, determinano z^+ in un certo modo fisicamente naturale e basato su successioni $\{\gamma_n(\cdot)\}$ e $\{z_n(\cdot)\}$ di funzioni regolari approssimanti le discontinuità (γ^-, γ^+) e (z^-, z^+) di $\gamma(\cdot)$ e $z(\cdot)$, (ii) fissati z^- e γ^-, z^+ risulta funzione continua di γ^+ e (iii) quando γ^+ tende a γ^-, $z(\cdot)$ tende ad una funzione continua e le $z_n(\cdot)$ si comportano in un certo modo naturale per certe semplici scelte delle $\gamma_n(\cdot)$.

Se $M > 1$, le (i)-(iii) son verificate solo in casi molto eccezionali. Allora si considerano le loro versioni (1-dimensionali) (*a*)-(*c*) in cui le (i)-(iii) valgono, per così dire, lungo una traiettoria $\tilde{\gamma}\ (\in C^3)$ di $\gamma(\cdot)$ con estremi γ^- e γ^+ e prefissata ad arbitrio, nel senso che $\tilde{\gamma}$ è la traiettoria di tutte le $\gamma_n(\cdot)$, ossia $\gamma_n(t) = \tilde{\gamma}[u_n(t)]$. Inoltre si mostra che una certa versione debole (di una parte) delle condizioni (*a*)-(*c*) implica la linearità di $(\mathscr{M})$ in $\dot{\gamma}$. Viceversa questa linearità implica una certa versione forte delle (*a*)-(*c*), nel qual caso dico che ($\mathscr{D}$) *il parametro (M-dimensionale)* γ *di* $(\mathscr{M})$ *è adatto a subire discontinuità (1-dimensionali, di* 1ª *specie)*.

Riguardo alla terna (Σ, χ, M) — v. Sommario in Nota I —, per $m = 2N$, $\chi = (q, \gamma)$ e $z = (q, p)$ con $p_h = \partial T / \partial \dot{q}^h\ (h = 1, \dots, N)$, il sistema differenziale $(\mathscr{M})$ può identificarsi con le equazioni dinamiche di Σ_γ in forma hamiltoniana. Allora la loro linearità rispetto alle $\dot{\gamma}$ risulta condizione necessaria e sufficiente affinché le co-ordinate χ di Σ siano M-*adatte a (subire) iper-impulsi (1-dimensionali)* nel senso che valga ($\mathscr{D}$).

N. 8. Introduction to II. Some general mathematical considerations for hyper-impulsive motions. On the applications to Lagrangian systems of these considerations and those on controllizability made in I (*)

In the applications of control theory to Lagrangian systems appeared so far, the control $u = u(\cdot)$ is generally a force-valued function—see e.g. [1], [8], [9], and [11] to [13]— but there are many interesting problems, in which controls take Lagrangian co-ordinates as values. E.g. I have studied some problems of this kind in connection with the swing and the ski—see [7]. This research work naturally pushed me to consider the following general problem.

Given any Lagrangian system Σ referred to a system

$$(8.1) \qquad \chi = (q, \gamma) = (q^1, \dots, q^N, \gamma^1, \dots, \gamma^M) = (x_1, \dots, x_{\mathscr{N}}) \quad (\mathscr{N} = N + M)$$

of $\mathscr{N}$ (independent) Lagrangian co-ordinates, is it possible to satisfactorily identify γ with a control $\hat{\gamma}(t)$ when condition (α) below holds?

(*) The present paper has been prepared within the activity sphere of the research group N. 3 of the Consiglio Nazionale delle Ricerche, in the academic year 1986–87.

(α) *The control* $\gamma = \hat{\gamma}(\cdot)$ *is physically carried out by adding some frictionless constraints, which turn* Σ *into another holonomic system,* $\Sigma_{\hat{\gamma}}$ (1).

In [6] the dynamic equation of $\Sigma_{\hat{\gamma}}$—based on (α)—is written in a semi-Hamiltonian form, explicit with respect to the functional M-dimensional parameter γ; denote it by SHE $(\Sigma_{\hat{\gamma}}, \Sigma)$ (semi-Hamiltonian equation for $\Sigma_{\hat{\gamma}}$ obtained from Σ). In this $\gamma = \hat{\gamma}(\cdot)$ occurs at most through γ and $\dot{\gamma}$, and $\dot{\gamma}$ appears at most quadratically outside the Lagrangian components $\mathcal{Q}_1$ to $\mathcal{Q}_{\mathcal{N}}$ of the applied forces acting on Σ. The SHE $(\Sigma_{\hat{\gamma}}, \Sigma)$ has the form $(7.1)_1$ where $m = 2\,\mathrm{N}$, z_h is the h-th Lagrangian co-ordinate of $\Sigma_{\hat{\gamma}}$, and $z_{h+\mathrm{N}}$ is its conjugate momentum $(h = 1, \dots, \mathrm{N})$. Consider the case where

(a) *the* M-*dimensional* (*functional*) *parameter* γ *in the* SHE $(\Sigma_{\hat{\gamma}}, \Sigma)$ *is* 1-*dimensionally* BVC^0 *controllizable* [Def. 7.1].

This is interesting, because, first, system $\Sigma_{\hat{\gamma}}$ can be given by the two steps $(8.2)_{1-2}$ below, as follows, where for some $\mathcal{A} = \overset{\circ}{\mathcal{A}} \subseteq \mathbf{R}$ $(\mathcal{A} \neq \phi)$

$$(8.2) \qquad \gamma = \tilde{\gamma}(u) \quad , \quad u = u(t) \quad , \quad \hat{\gamma}(t) = \tilde{\gamma}[u(t)] \quad \tilde{\gamma} \in \mathrm{C}^3(\mathcal{A}, \mathbf{R}^{\mathrm{M}}) .$$

Let $\Sigma_{\tilde{\gamma}}$ be the holonomic system obtained from Σ by adding M-1 (scalar) frictionless constraints represented by $(8.2)_1$, so that $\Sigma_{\tilde{\gamma}}$ has the $\mathrm{N}+1$ (independent) Lagrangian co-ordinates q^1 to q^{N} and u. (For $\mathrm{M} = 1$ one can set $u = \gamma_1$). We can regard $\Sigma_{\hat{\gamma}}$ to be obtainable from $\Sigma_{\tilde{\gamma}}$ similarly, in connection with $(8.2)_2$. The SHE $(\Sigma_{\hat{\gamma}}, \Sigma_{\tilde{\gamma}})$ is equation $(2.2)_1$ with the control u, when definition $(7.2)_1$ holds. Condition (a) says that

(a') *for every admissible* C^3-*path* $\tilde{\gamma}$, *the* (*functional*) *scalar parameter* u *in the* SHE $(\Sigma_{\hat{\gamma}}, \Sigma_{\tilde{\gamma}})$ *is* BVC^0-*controllizable* [Def. 2.2], *which by Corollary* 6.1, *implies the strong* C^0- *and* $\mathcal{L}^1$-*controllizabilities of* u [Defs. 4.1 *and* 5.2]. These properties can be reasonably regarded as necessary and sufficient conditions for the experimental testability of the solutions of control problems obtained by the SHE $(\Sigma_{\hat{\gamma}}, \Sigma_{\tilde{\gamma}})$ (2).

(1) An important preliminary for applying the present theory to a particular problem, is to check its compliance with condition (α). This is explicitly done in e.g. [7]. In fact a kinematic relation can be carried out physically, in a way incompatible with (α), as is emphasized in [6], N 3, by an example.

(2) Maybe the reasonable necessity of these conditions—i.e. u's BVC^0-, C^0-, or $\mathcal{L}^1$-controllizability—might not be evident, at first sight, in connection with the fact that, by (11.2-4), for $\Sigma_{\hat{\gamma}}$

$$(\alpha) \quad p_h = p_h(t, q, \dot{q}, \gamma, \dot{\gamma}) \equiv \sum_{k=1}^{\mathrm{N}} a_{hk}\, \dot{q}^k + \mathrm{B}_h + \sum_{\rho=1}^{\mathrm{M}} \mathrm{A}_{h,\mathrm{N}+\rho}\, \dot{\gamma}^\rho$$
$$(\gamma = \hat{\gamma}(t) = \tilde{\gamma}[u(t)]) .$$

In fact, in case, e.g., for $a \to 0^+$, (a) $\| u_a(\cdot) \|_0 \to 0$ but (b) $\| \dot{u}_a(\cdot) \|_0 \geq 1$, it is natural to accept the requirement (c) $\| q(u_a, \cdot) - q(0, \cdot) \|_0 \to 0$ and to (correctly) doubt of the condition (d) $\| \dot{q}(u_a, \cdot) - q(0, \cdot) \|_0 \to 0$; furthermore by ($\alpha$) one could (incor-

By Theorem 7.1, condition (*a*)—or (*a'*)—holds if

(*b*) $\dot{\gamma}$ *occurs in the* SHE $(\Sigma_{\hat{\gamma}}, \Sigma)$ *at most linearly.*

In [6] the choices of $(\Sigma, \mathscr{X}, \mathrm{M})$—see (8.1)—for which (*b*) holds [$\dot{\gamma}$ fails to occur in the SHE $(\Sigma_{\hat{\gamma}}, \Sigma)$] have been characterized by means of certain algebraic conditions on the coefficients of Σ's kinetic energy, on their partial derivatives, and on the first N among the above components $\mathscr{Q}_1$ to $\mathscr{Q}_{\mathscr{N}}$. These conditions are written here, in N. 11, without proof—see (11.8) [(11.9)] and (11.7) with $\mathscr{Q} \equiv 0$.

In [5] the content of the following lemma has been easily proved on the basis of [2]—see (3.10) and below (3.11) in [5].

LEMMA 8.1. *The differential system* $(4.1)_1$, *which has the scalar control* u *and is linear in* $\dot{u}$, *can be locally transformed, by means of a transformation* $y = y(x, u) \in C^3(\mathbf{R}^{n+1}, \mathbf{R}^n)$, *into a similar system where* $\dot{u}$ *fails to occur, so that Pontrjagin's maximum principle can be applied.*

Hence the simple conditions (11.8) and (11.7) with $\mathscr{Q} \equiv 0$ are satisfied by the triplet $(\Sigma, \mathscr{X}, \mathrm{M})$ iff, for every admissible path $\tilde{\gamma}$, control theory [with inclusion of Pontrjagin's maximum principle] can be satisfactorily applied to the SHE $(\Sigma_{\hat{\gamma}}, \Sigma_{\tilde{\gamma}})$ [up to a change of the Lagrangian co-ordinates].

* * *

The research work leading to [6] was started to solve another general problem tightly connected with the afore-mentioned one. Rational Mechanics includes the well known theory of impulsive motions where velocities can suffer 1st order discontinuities, whereas positions are continuous. This theory has (easy and) important applications, to e.g. the ballistic pendulum. Hence it is natural to ask whether the same theory can be extended to a theory of hyper-impulsive motions. The main problem is, roughly speaking, to state the conditions by which

(*c*) *for any state* $(q^-, \dot{q}^-, \gamma^-, \dot{\gamma}^-)$ *of* Σ *at any* t^-, *and for any jumps* $\Delta\gamma = \gamma^+ - \gamma^-$ *and* $\Delta\dot{\gamma} = \dot{\gamma}^+ - \dot{\gamma}^-$ *at* t, *that are not too large, the jump* $(\Delta q, \Delta\dot{q}, \Delta\gamma, \Delta\dot{\gamma})$ *of the whole system* Σ *at* t *is determined, and is a continuous function of* $\Delta\gamma$ *and* $\Delta\dot{\gamma}$.

As the example in [6], N 10, on an astronaut practically shows, generally (*c*) does not hold for $\mathrm{M} > 1$. Hence it is natural to consider the analogue of (*c*) for any $\Sigma_{\tilde{\gamma}}$—such discontinuities are treated in [6], NN 12-13. Requirement ($\mathscr{R}$) in N 9 is a rather weak such analogue formulated for the general

rectly) doubt of the condition (*e*) $\| p(u_a, \cdot) - p(0, \cdot) \|_0 \to 0$. However $p(u_a, \cdot)$ behaves like $q(u_a, \cdot)$ (and unlike $\dot{q}(u_a, \cdot)$). In fact (*f*) *in purely impulsive motions,* p *is (necessarily) continuous (unlike* $\dot{u}$ *and* $\dot{q}$*) by the impulsive equations and assertion* (α) *in* N 8 *(of frictionlessness).* Other intuitive reasons for requiring (*e*) will be given in ftn. 3. Furthermore they will be replaced by a rigorous theorem in Part III: Theor. 15.2.

mathematical equations $(2.2)_1$ with a scalar control u. In case $\dot{u}$ has a polynomial occurrence in $(2.2)_1$, in N 9 the validity of $(\mathscr{R})$ is shown to imply the linearity of $(2.2)_1$ in $\dot{u}$, and in N 10 this linearity condition is proved to imply some stronger useful properties concerning jumps (formulated in terms of u's fitness for jumps [Def. 10.1]).

Incidentally, in [6] it is briefly remarked that any Lagrangian system with time independent constraints has controllizable (and hence fit–for–jumps) co-ordinates, in that, such are its geodesic co-ordinates.

N. 9. ON THE FITNESS OF A DIFFERENTIAL SYSTEM WITH A SCALAR CONTROL u TO TREAT THE 1st ORDER DISCONTINUITIES OF u. CONSEQUENCES ON THE FORM OF THE SYSTEM

Consider a Cauchy problem with a scalar control or scalar functional parameter u, of the form

$$(9.1)\quad \dot{x} = f(x) + \sum_{h=1}^{\nu} g_h(x)\,\dot{u}^h,\ x(\bar{t}) = \bar{x} \qquad (f, g_1, \ldots, g_\nu \in C^2(\mathscr{V}, \mathbf{R}^n),\ \mathscr{V} \subseteq \mathbf{R}^n),$$

where $x = (t, u, z)$—see (2.2–3)—; and assume that $v = v(\cdot)$ is continuous, $j \in \mathbf{R}$, $v_j(t) =_{\mathrm{D}} v(t)\ \forall t \leq \bar{t}$, and $v_j(t) =_{\mathrm{D}} v(t) + j\ \forall t > \bar{t}$, hence $\bar{u} =_{\mathrm{D}} v(\bar{t}) = v_j(\bar{t}^+) - j$. Physically this control can be carried out only approximately, through a family $\{u_{\mathrm{T}}(\cdot)\}_{\mathrm{T} > \bar{t}}$ of controls in KC^1, i.e. piece-wise C^1; for instance we can set

$$(9.2)\quad u_{\mathrm{T}}(t) = \bar{u} + j\,\frac{t-\bar{t}}{\eta}\ (\bar{t} \leq t \leq \mathrm{T});\ u_{\mathrm{T}}(t) = \begin{cases} v(t)\ (t < \bar{t}) \\ v_j(\eta + t)\ (t > \mathrm{T}) \end{cases} \qquad (\eta = \mathrm{T} - \bar{t}).$$

Assume $\mathrm{T} > \bar{t}$ and that $x(u, \cdot) \in \mathrm{KC}^1([\bar{t}, \mathrm{T}], \mathbf{R}^n)$ is the absolutely continuous) solution of (9.1) in $[\bar{t}, \mathrm{T}]$ for $u(\cdot) \in \mathrm{KC}^1$. A minimal requirement on $(9.1)_1$ often necessary to regard $(9.1)_1$ as *fit to treat jumps* (*or first order discontinuities*) of $u(\cdot)$—or in other words, to regard the parameter u in $(9.1)_1$ as *fit for jumps*—, is the following.

$(\mathscr{R})$ *For* $\bar{x} = (\bar{t}, \bar{u}, \bar{z}) \in \mathscr{V}$ *there are some* $\delta_1 > 0$ *and* $\mathrm{T}_1 > 0$ *such that*

(i) *for* $|j| < \delta_1$ *and* $\bar{t} < \mathrm{T} < \mathrm{T}_1$, *the solution* $x_{\mathrm{T}}(\cdot) = x(u_{\mathrm{T}}, \cdot)$ *of* (9.1) (*for* $u = u_{\mathrm{T}} = u_{\mathrm{T}}(\cdot)$—*see* (9.2)—)*in* $[\bar{t}, \mathrm{T}]$ *exists and* $x^+ = \lim_{\mathrm{T} \to \bar{t}} x_{\mathrm{T}}(\mathrm{T})$ *also exists and is finite*; *furthermore*

(ii) *chosen any* r_1 *with* $\overline{\mathrm{B}}(\bar{x}, r_1) \subset \mathrm{V}$, *there are some* $\delta \in (0, \delta_1)$ *and some* $\mathrm{T}_2 \in (\bar{t}, \mathrm{T}_1)$ *such that*

$$(9.3)\qquad |j| < \delta \Rightarrow x_{\mathrm{T}}(t) \in \mathrm{B}(\bar{x}, r_1)\ \forall t \in [\bar{t}, \mathrm{T}]\ \forall \mathrm{T} \in (0, \mathrm{T}_2).$$

In fact the existence condition (i) is obvious; and condition (ii) practically follows from (the second of) the natural requirements [3]

(*a*) *the jump* $J = \Delta x = x(\bar{t}^+) - \bar{x}$ *of the solution* $x(u, \cdot)$ (*in* KC^1) *must be a continuous function* $J(j)$ *of* j—otherwise the jumps forecast by our theory cannot be compared with experiments—and

(*b*) *when* j *tends to zero,* $x(v_j, \cdot)$ *must tend* (*in* C^0) *to a* C^0*-function.*

THEOREM 9.1. *The* (*local*) *compatibility of problem* (9.1) *with requirement* ($\mathscr{R}$) *implies the linearity of* $(9.1)_1$ *in* $\dot{u}$.

Indeed assume requirement $(\mathscr{R})$, $\nu \geq 1$, and $g_\nu(x) \not\equiv 0$. Then $g_\nu(\bar{x}) \neq 0$ for some $\bar{x} \in V$. Now choose δ_1 and $T_1 > \bar{t}$ in a way by which the conditions (*i*) to (*ii*) in requirement $(\mathscr{R})$ are satisfied. For some $r \in (0, r_1)$, $\mathscr{M} > 0$, and $\mathscr{N} > 0$

$$(9.4) \qquad |g_\nu(x)| > \mathscr{N}, \quad |f(x)| < \mathscr{M}, \quad |g_h(x)| < \mathscr{M} \; (1 \leq h < \nu) \; \forall x \in \bar{B}(\bar{x}, r).$$

By condition (ii) in requirement $(\mathscr{R})$, for some $\delta \in (0, \delta_1)$ and $T_2 \in (\bar{t}, T_1)$ implication (9.3) holds. Hence, chosen $|j| \in (0, \delta)$ and $T \in (0, T_2)$, we have that $x_T(s) \in B(\bar{x}, r) \; \forall s \in [\bar{t}, T]$. In addition $\dot{u} = j/\eta$ by $(9.2)_1$. Then by (9.1),

$$(9.5) \qquad |x_T(T) - \bar{x}| = \left| \int_{\bar{t}}^{T} \{ f[x_T(s)] + \sum_{h=1}^{\nu} g_h[x_T(s)] \, j^h \eta^{-h} \} \, ds \right| \geq L_\eta$$

were $\eta = T - \bar{t}$ and —see (9.4)

$$(9.6) \qquad L_\eta = \eta \left(\frac{\mathscr{N}}{\eta_\nu} - \mathscr{M} \sum_{h=1}^{\nu-1} j^h \eta^{-h} \right), \quad \text{hence} \quad \lim_{\eta \to 0^+} L_\eta = +\infty \quad (\text{for } \nu \geq 2).$$

Since (9.5) and $(9.6)_2$ contrast with (the consequent of) (9.3), we conclude that (9.1) holds for $\nu = 1$.

q.e.d.

(3) In spite (9.3) may appear too strong at first sight, and one may wish to replace « $x_T(t) \in B(\bar{x}, r)$ » in it with « $q_T(t) \in B(\bar{q}, r)$ », it is reasonable to keep the actual version of (9.3) by the following intuitive considerations. First, in case $\nu = 1$, conditions (i), (ii), and (*b*) below hold for problem (9.1), as can be proved on the basis of [2] or [5]. Second, in accordance with assertion (*f*) in ftn. 2, for $u_{Tj}(\cdot) = u_T(\cdot)$ —see (9.2)— $p(u_{Tj}, \cdot)$ is continuous at both $t = \bar{t}$ and $t = T$. Hence, by (α) in ftn. 2, $p(u_{Tj}, \bar{t}) = p[\bar{t}, q(\bar{t}), \dot{q}(\bar{t}^-), \gamma(t), \dot{\gamma}(\bar{t}^-)]$ and $p(u_{Tj}, T) = p[T, q(T), \dot{q}(T^+), \gamma(T), \dot{\gamma}(T^+)]$ where e.g. $q(\cdot) = q(u_{Tj}, \cdot)$. Hence $p(u_{Tj}, \bar{t}) \to p(u_{Tj}, \bar{t}) (= p(v, \bar{t}))$ for $T \to t^+$ and $|j| \to 0$. Let us add that $u_{Tj}(\cdot)$ is monotone in $[\bar{t}, T]$ and has the increment j. Therefore, along $[\bar{t}, T]$, $p(u_{Tj}, \cdot)$ varies very little for $T - \bar{t} (> 0)$ and $|j|$ small.

N. 10. STRENGHTENED CONVERSE OF THE PRECEDING THEOREM ON FITNESS TO TREAT CONTROL JUMPS. FITNESS TO TREAT 1-DIMENSIONAL JUMPS OF VECTOR-VALUED CONTROLS

The requirements (B) to (C) (and (A)) in Def. 10.1 below, on the ODEs $(9.1)_1$—or $(2.2)_1$ and $(2.3)_1$—are stronger than $(\mathscr{R})$ in N. 9. They appear both necessary and sufficient for $(9.1)_1$ to be fit to treat control jumps.

DEFINITION 10.1. *I shall say that the* ODEs $(9.1)_1$*—or* $(2.2)_1$*—with the control* u*, are fit to treat control jumps—or that the (functional scalar) parameter* u *in them is fit for jumps—in case, for all* $\bar{x} = (\bar{t}, \bar{u}, \bar{z}) \in \mathrm{V}$, *requirements* (A) *to* (C) *below hold.*

(A) *For some compact set* $\mathrm{K} \subset \mathrm{V}$ *and some* $\tau > 0$, $b > 0$, K', *and* Z, *relations* $(4.3)_{3-5}$ *and condition* (H) *in Lemma* 4.1 *hold.*

(B) *If* (i) $(4.3)_{3-5}$ *and* (H) *in Lemma* 4.1 *hold for some* K *to* Z, (ii) $u(\cdot) \in \overline{\overline{\mathscr{U}}}_{\mathscr{L}_1^\rho}$*—see* (5.2)—, (iii) $\bar{t} < d < \bar{t} + \tau$, *and* (iv) $u(\cdot)$ *has a first order discontinuity at* d *with the jump* j, *then* (β_1) *the significative solution* $x(\cdot) = x(u, \cdot)$ *of* (9.1) *in* $[\bar{t}, \bar{t} + \tau]$ [Def. 5.1] *exists,* (β_2) *it has a first order discontinuity at* d *with the jump* J, *and* (β_3) J *is a continuous function* $\mathrm{J}(j)$ *of* j, *or more precisely* $u(\mathrm{d}^+)$.

(C) *For any* $\varepsilon > 0$ *with* $\overline{\mathrm{B}}(\bar{x}, \varepsilon) \subset \mathrm{V}$ *and any* $\varphi \in \mathrm{C}^1[0, 1]$ *with* $\varphi(0) = 0$ *and* $\varphi(1) = 1$, *there are some* $\delta_1 > 0$ *and* $\mathrm{T}_1 > \bar{t}$ *such that*

(γ_1) *for* $|j| < \delta_1$ *and* $\mathrm{T} \in (\bar{t}, \mathrm{T}_1)$ *the solution* $x_\eta(\cdot) = x(u_\eta, \cdot)$ *of problem* (9.1)*—or* (2.3)*—in* $[\bar{t}, \mathrm{T}]$, *where*

$$u_\eta(t) = \bar{u} + j\varphi\left(\frac{\bar{t} - t}{\eta}\right) \qquad (\eta = \mathrm{T} - \bar{t}), \tag{10.1}$$

satisfies the condition $x_\eta(t) \in \mathrm{B}(\bar{x}, \varepsilon)\ \forall t \in [\bar{t}, \mathrm{T}]$.

Note that (i) [(ii)] in requirement $(\mathscr{R})$, in N. 9, follows from (A) and (B) (without (β_2)) [from (C)] in Def. 10.1; furthermore this condition (B) [(C)] is a technical counterpart of the intuitive natural requirement (*a*) [(*b*)] in N. 9.

THEOREM 10.1. (α) *The* ODEs $(9.1)_1$ *with the scalar control* u *are fit to treat control jumps, iff* (β) *they are linear in* $\dot{u}$.

Indeed fix any $\bar{x} = (\bar{t}, \bar{u}, \bar{z}) \in \mathrm{V}$. First assume (α), which by Def. 10.1 implies requirement $(\mathscr{R})$ in N. 9. Then by Theorem 9.1 (and the arbitrariness of $\bar{x} \in \mathrm{V}$) (β) holds. Thus $(\alpha) \Rightarrow (\beta)$.

Now assume (β), so that $(9.1)_1$ can be identified with $(4.1)_1$. Fr m (the last part of the) proof of Theorem 4.1 remember that, for some $\mathrm{K}_i = \overline{\mathrm{B}}(\bar{x}, r_i)$ $(i = 0,1)$ (4.2) holds. Then, by Lemma 4.1, for some K, $\tau > 0$, $b > 0$, K', and Z, we have (4.3) and condition (H) in that lemma. Thus (A) in Def. 10.1 holds.

To prove (B) in Def. 10.1, assume (i) to (iv) in it. By (i) and (ii), Lemma 6.1 yields (through (*c*)) (β_1) in Def. 10.1 (B). Now, that (iii) to (iv) in Def. 10.1 (B) imply (β_2) in Def. 10.1 (B), substantially constitutes a remark made in [2], sect. 3.

In order to deduce (β_3) in Def. 10.1 (B), let us consider the following lemma, substantially contained in [5], Theor. 11.2, and proved there on the basis of [2].

LEMMA 10.1. *Let assumptions* (i) *to* (iv) *in Def.* 10.1 (B) *imply* (β_1) *and* (β_2) *there. Then there is a function* $\varphi \in C^2(\mathbf{R}^{n+2}, \mathbf{R}^n)$ *such that, for all* $u(\cdot)$ *and* d, *conditions* (ii) *to* (iv) *in Def.* 10.1 (B) *imply*

$$x(\mathrm{d}^+) = \varphi\{u(\mathrm{d}^+), \sigma, \varphi[\sigma, u(\mathrm{d}^-), x(\mathrm{d}^-)]\}, \qquad \textit{e.g. for } \sigma = 0. \tag{10.2}$$

Since the implication assumed in Lemma 10.1 has been proved, the consequent of this lemma holds.

Consider $u(\cdot)$ as given on $[\bar{t}, \mathrm{d})$, hence $u(\mathrm{d}^-)$ and $x(\mathrm{d}^-)$ are known. Then, by (10.2), $x(\mathrm{d}^+)$ is a continuous function of $u(\mathrm{d}^+)$, i.e. J is such a function of j. Thus (β_3) in Def. 10.1 (B) holds. We conclude that condition (B) in Def. 10.1 holds.

To prove (C) in Def. 10.1, assume that $\varepsilon > 0$, $\overline{\mathrm{B}}(\bar{x}, \varepsilon) \subset \mathrm{V}$, $\varphi \in C^1[0, 1]$, $\varphi(0) = 0$, and $\varphi(1) = 1$. Furthermore set (10.1) and consider the corresponding solution $x_\eta(t) = (t, u_\eta(t), z(t)) = x(u_\eta, t)$ of problem (9.1), where $\nu = 1$ is assumed. Thus this problem has the form (2.3) with $f_{vv}(x, v) = 0$, and hence the form (2.2) $\cup$ $(2.4)_{1-4}$ where

$$\mathrm{F}(t, u, z, v) \equiv \mathscr{F}(t, u, z) + \mathscr{G}(t, u, z)\, v. \tag{10.3}$$

Let us set $\xi = (t - \bar{t})/\eta$ and $\zeta_\eta(\xi) = z_\eta(t)$, hence $\zeta'_\eta = \mathrm{d}\zeta_\eta/\mathrm{d}\xi = \eta\, \dot{z}_\eta$. Then, by definitions (10.3) and (10.1), problem (2.2) for $u = u_\eta(\cdot)$ becomes

$$\begin{aligned} \xi'_\eta = \eta \mathscr{F}[\bar{t} + \eta\xi, \bar{u} + j\varphi(\xi), \zeta_\eta(\xi)] + j\mathscr{G}[\bar{t} + \eta\xi, \bar{u} + {} \\ + j\varphi(\xi), \zeta_\eta(\xi)]\, \varphi'(\xi), \ \zeta_\eta(0) = \bar{z}. \end{aligned} \tag{10.4}$$

For some $\mu > 0$ and $\mathscr{M} \geq 1$,

$$\begin{aligned} &\mu = \sup\{|\varphi(\xi)| + |\varphi'(\xi)| \;\big|\; \xi \in [0, 1]\}, \\ &|\mathscr{F}(x)| < \mathscr{M}, \quad |\mathscr{G}(x)| < \frac{\mathscr{M}}{\mu} \quad \forall x \in \overline{\mathrm{B}}(\bar{x}, \varepsilon). \end{aligned} \tag{10.5}$$

Now fix any η and j with

$$\eta, \mu|j| \in (0, \varepsilon/8\mathscr{M}) \qquad (\mathscr{M} \geq 1). \tag{10.6}$$

Consider the maximal solution $x_\eta(\cdot) = (\cdot, u_\eta(\cdot), z_\eta(\cdot))$ of problem (2.3), so that $\zeta_\eta(\cdot)$ is its analogue for problem (10.4). Let $\mathscr{D}_\eta$ be the domain of $\zeta_\eta(\cdot)$ and set $\xi' = \sup\{\xi \leq 1 \mid [0, \xi] \subseteq \mathscr{D}_\eta\}$. Then, by (10.4-6), for $\xi \in [0, \xi')$

$$(10.7) \qquad |x_\eta(t) - \bar{x}| \leq |\eta\xi| + |j\varphi(\xi)| + |\zeta_\eta(\xi) - \bar{z}| \leq \frac{\xi' + 1}{8\mathscr{M}}\varepsilon + \frac{\mathscr{M}\varepsilon}{4\mathscr{M}} < \frac{\varepsilon}{2}.$$

Hence, since the solution $\zeta_\eta(\cdot)$ is absolutely continuous, $\zeta_\eta(\xi')$ exists and $x_\eta(\eta\xi') \in \mathring{B}(\bar{x}, \varepsilon)$. Therefore, $\zeta_\eta(\cdot)$ (and $x_\eta(\cdot)$) being maximal, we must have $\xi' = 1$, so that $x_\eta(t)$ is defined for $\bar{t} \leq t \leq T = \bar{t} + \eta$. Thus by (10.6), for $\delta_1 = \varepsilon/8\mathscr{M}\mu$ and $0 < T - \bar{t} = \eta < \varepsilon/8\mathscr{M}\mu$, condition (γ_1) in Def. 10.1 holds; hence (C) in Def. 10.1 also holds. Then, by Def. 10.1, condition (α) holds. Thus $(\beta) \Rightarrow (\alpha)$. q.e.d.

* * *

Let us consider some analogues of the notions of 1-dimensional controllizability (introduced by Def. 7.1) for fitness for jumps.

DEFINITION 10.2. *Let us say that the* M-*dimensional* (*functional*) *parameter* γ *in the* ODEs $(7.1)_1$ *is* 1-*dimensionally fit for* (*or to suffer*) *jumps—or that* $(7.1)_1$ *are fit to treat* 1-*dimensional jumps of* γ *if, for every* (*admissible*) *path* $\tilde{\gamma} \in C^3(\mathscr{A}, \mathbf{R}^M)$ *where* $\phi \neq \mathscr{A} = \mathring{\mathscr{A}} \subseteq \mathbf{R}$, *the scalar parameter* u *in the* ODEs $(2.2)_1$ *is fit for jumps when identity* $(7.2)_1$ *holds.*

By the consequence $(7.2)_2$ of $(7.2)_1$, Def. 10.2 and Theorem 10.1 imply the following analogue of Theorem 7.1.

THEOREM 10.2. *Assume that* $\phi(t, \gamma, z, \dot{\gamma})$ *has a polynomial dependence on* $\dot{\gamma}$. *Then the functional parameter* γ *in the* ODEs $(7.1)_1$ *is* 1-*dimensionally fit for jumps, iff* $\dot{\gamma}$ *appears in* $(7.1)_1$ *at most linearly.*

N. 11. SIMPLE CONDITIONS FOR A SUBSET OF CO-ORDINATES OF A LAGRANGIAN SYSTEM TO BE 1-DIMENSIONALLY CONTROLLIZABLE OR FIT FOR JUMPS

Let us express Σ's kinetic energy by

$$(11.1) \qquad \mathscr{T} = \frac{1}{2}\sum_{\mathscr{R},\mathscr{S}=1}^{\mathscr{N}} A_{\mathscr{R}\mathscr{S}}\dot{\chi}^{\mathscr{R}}\dot{\chi}^{\mathscr{S}} + \sum_{\mathscr{R}=1}^{\mathscr{N}} B_{\mathscr{R}}\dot{\chi}^{\mathscr{R}} + C,$$

where $A_{\mathscr{R}\mathscr{S}}$, $B_{\mathscr{R}}$, and C are C^3-functions of $t, q, \gamma, \dot{q}$, and $\dot{\gamma}$. Then the kinetic energy T of the Lagrangian system $\Sigma_{\hat{\gamma}}$—see N 8—with $\hat{\gamma} \in C^2(\mathbf{R}, \mathbf{R}^M)$ has the expression

$$(11.2) \qquad T = \frac{1}{2}\sum_{h,k=1}^{\mathscr{N}} a_{hk}\dot{q}^h\dot{q}^k + \sum_{h=1}^{\mathscr{N}} b_h\dot{q}^h + c$$

where a_{hk}, b_h, and c are the following functions of t and q: (4)

$$(11.3)\quad a_{hk} = \mathrm{A}_{hk}[t, q, \hat{\gamma}(t)] \quad , \quad b_h = [\mathrm{B}_h + \sum_{\rho=1}^{\mathrm{M}} \mathrm{A}_{h,\mathrm{N}+\rho}\, \dot{\gamma}^\rho]_{\gamma=\hat{\gamma}(t),\, \dot{\gamma}=\dot{\hat{\gamma}}(t)} ,$$

$$c = \left[\mathrm{C} + \frac{1}{2}\sum_{\rho,\sigma} \mathrm{A}_{\mathrm{N}+\rho,\mathrm{N}+\sigma}\, \dot{\gamma}^\rho \dot{\gamma}^\sigma + \sum_\rho \mathrm{B}_{\mathrm{N}+\rho}\, \dot{\gamma}^\rho\right]_{\gamma=\hat{\gamma}(t),\, \dot{\gamma}=\dot{\hat{\gamma}}(t)} \qquad (\hat{\gamma} \in \mathrm{C}^2) .$$

Furthermore set

$$(11.4)\qquad p_h = \frac{\partial \mathrm{T}}{\partial \dot{q}^h} , \mathrm{H} = \mathrm{H}(t, \dot{q}, p) = p_h \dot{q}^h = \frac{1}{2} a_{hk}\, \dot{q}^h \dot{q}^k - \mathrm{C} ;$$

and let the Lagrangian components of the applied forces acting on Σ have the C^2–expressions $\mathcal{Q}_{\mathcal{R}} = \mathcal{Q}_{\mathcal{R}}(t, q, \gamma, \dot{q}, \dot{\gamma})$; hence those for $\Sigma_{\hat{\gamma}}$ are $\mathcal{Q}_h = \mathcal{Q}_h[t, q, \hat{\gamma}(t), \dot{q}, \dot{\hat{\gamma}}(t)]$. It can be asserted that *the* C^2*–functions* $q^h = q^h(t)$ *solve the Lagrangian equations for* $\Sigma_{\hat{\gamma}}$.

$$(11.5)\qquad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathrm{T}}{\partial \dot{q}^h} = \frac{\partial \mathrm{T}}{\partial q^h} + \mathcal{Q}_h \qquad (h = 1, \dots, \mathrm{N})$$

if, after having defined $p_h(t)$ *in terms of* $q_h(t)$ *and* $\dot{q}_h(t)$ *by* $(11.4)_1$, *the functions* $q_h(t)$ *and* $p_h(t)$ *solve the, so to say, semi–Hamiltonian equations*

$$(11.6)\ \dot{q}^h = \frac{\partial \mathrm{H}}{\partial q^h} , \dot{p}_h = -\frac{\partial \mathrm{H}}{\partial q^h} + \tilde{\mathcal{Q}}_h , \textit{ where } \tilde{\mathcal{Q}}_h = \mathcal{Q}_h\left[t, q, \frac{\partial \mathrm{H}}{\partial p_h}(t, q, p)\right] .$$

Now replace $\hat{\gamma}(t)\,[\dot{\hat{\gamma}}(t)]$ with $\gamma\,[\dot{\gamma}]$ in (11.3-4), and let $\hat{\mathcal{Q}}_{\mathcal{R}}(t, q, \gamma, p, \dot{\gamma})$ be $\mathcal{Q}_{\mathcal{R}}(t, q, \gamma, \dot{q}, \dot{\gamma})$ for $\dot{q}^h = \partial \mathrm{H}/\partial p_h\,(h = 1, \dots, \mathrm{N})$. Then, by $(11.6)_3$, we have the MacLaurin expansion

$$(11.7)\qquad \mathcal{Q}_h = \tilde{\mathcal{Q}}_h(t, q, \gamma, p, \dot{\gamma}) = \tilde{\mathcal{Q}}_{h0} + \tilde{\mathcal{Q}}_{h\rho}\, \dot{\gamma}^\rho + \frac{1}{2}\tilde{\mathcal{Q}}_{h\rho\sigma}\, \dot{\gamma}^\rho \dot{\gamma}^\sigma + \mathcal{Q} \qquad (\mathcal{Q}/|\gamma|^2 \to 0) ,$$

where $\tilde{\mathcal{Q}}_{h0}$, $\tilde{\mathcal{Q}}_{h\rho}$, and $\tilde{\mathcal{Q}}_{h\rho\sigma}$ are functions of t, q, γ, and p.

The content of the Theorems 11.1-3 below is proved in [6], N 4.

THEOREM 11.1. (α) *equations* (11.6) *involve* $\dot{\gamma}$ *at most linearly iff* (α') *identity* (11.7) *holds with* $\mathcal{Q} \equiv 0$, *together with the* $\mathrm{NM}(\mathrm{M}+1)/2$ *identities*

$$(11.8)\qquad \frac{\partial}{\partial q^h}(a^{rs}\, \mathrm{A}_{r,\mathrm{N}+\rho}\, \mathrm{A}_{s,\mathrm{N}+\sigma} - \mathrm{A}_{\mathrm{N}+\rho,\mathrm{N}+\sigma}) \equiv \tilde{\mathcal{Q}}_{h\rho\sigma} \qquad (a^{rh} a_{hs} = \delta_{rs})\text{—see ftn. 2.}$$

(4) The indices h, k, r, and s run from 1 to N; ρ and σ run from 1 to M. Furthermore Einstein's convention is used: e.g. $p_h \dot{q}^h = \sum_{h=1}^{\mathrm{N}} p_h \dot{q}^h$.

Thus, by Corollary 6.1, the coordinates $\gamma^\rho = \chi^{N+\rho}$ $(\rho = 1, \dots, M)$ of Σ are 1-dimensionally controllizable iff (11.8) and (11.7) with $\mathscr{Q} \equiv 0$ hold; furthermore—see Theorems 10.1 and 11.1—, in case (11.7) holds with $\mathscr{Q} \equiv 0$, those co-ordinates are 1–dimensionally fit for jumps iff (11.8) holds. In this case I say that *the last* M *co-ordinates in* χ *are* (*1–dimensionally*) *fit* (*for hyper-impulses*), or that *the co-ordinates* χ *for* Σ *are* M*–fit for jumps.*

THEOREM 11.2. (β) *Equations* (11.6) *do not involve* $\dot{\gamma}$, *iff* (β') *we have both identities* (11.7) *with* $\mathscr{Q} \equiv 0$ *and the* $2(N + M) + NM(M + 1)/2$ *identities in* t, q, p, *and* γ

$$(11.9) \qquad A_{h,N+\rho} = 0\,, \frac{\partial B_{N+\rho}}{\partial q^h} + \tilde{\mathscr{Q}}_{h\rho} = 0 = \frac{\partial A_{N+\rho,N+\sigma}}{\partial q^h} + \tilde{\mathscr{Q}}_{h\rho\sigma}\,.$$

THEOREM 11.3. *Condition* (α) *in Theorem* 11.1 [(β) *in Theorem* 11.2] *renders equations* $(11.6)_{1-2}$ *equivalent to*

$$(11.10) \quad \dot{q}^h = a^{hk}(p_k - b_k)\,;\; \dot{p}^h = \frac{\partial}{\partial q^h}\left[\frac{a^{rs}}{2}(p_r - B_r)(p_s - B_s) - C\right] + \\ + (\tilde{\mathscr{Q}}_{h_0} + \mathscr{P}_h)_{\gamma = \hat{\gamma}(t),\, \dot{\gamma} = \dot{\hat{\gamma}}(t)}\,;$$

where

$$(11.11) \quad \mathscr{P}_h = \left\{\frac{\partial}{\partial q^h}[a^{rs}(B_r - p_r)A_{s,N+\rho} + B_{N+\rho}] + \tilde{\mathscr{Q}}_{h\rho}\right\}\dot{\gamma}^\rho \qquad [\mathscr{P}_h = 0]\,.$$

REFERENCES

[1] AILON A. and LANGHOLZ G. (1985) – *On the existence of time optimal control of mechanical manipulators.* « Journal of optimization theory and applications », *46*, 3-21.

[2] BRESSAN ALBERTO (1987) – *On differential systems with impulsive controls,* « Sem. Mat. Univ. Padova », *78*, 227.

[3] BRESSAN ALBERTO and RAMPAZZO F. – *Differential systems with vector-valued impulsive controls,* being printed on Bollettino dell'Unione Matematica Italiana.

[4] BRESSAN ALDO (1988) – *On the application of control theory to certain problems for Lagrangian systems, and hyper-impulsive motions for these.* I. *On some general mathematical considerations on controllizzable parameters,* « Atti Accad. dei Lincei », *82*, 91-105.

[5] BRESSAN ALDO – *On some recent results in control theory for their application to Lagrangian system,* being printed as a memoir on Atti Accad. dei Lincei.

[6] BRESSAN ALDO – *Hyper-impulsive motions and controllizable co-ordinates for Lagrangian systems,* being printed as a memoir on « Atti Accad. Lincei ».

[7] BRESSAN ALDO – *On some control problems concerning the ski or swing* (in preparation).

[8] BROCKETT R.W. – *Control theory and analytical mechanics.* « Lie groups: History frontiers and application », Vol. VII. The 1976 NASA Conference on Control theory Edited by Martin Hermann (Math. Sci. Press).

[9] KARIHALOO B.L. and PARBERY L.D. (1982) - *Optimal control of a dynamic system representing a gantry crane*, «Journal of optimization theory and applications», *36*, 409-417.

[10] SUSSMANN H.J. (1978) - *On the gap between deterministic and stocastic ordinary differential equations*, «Ann. of Probability», *6*, 19-41.

[11] VAN DER SCHAFT A.J. (1982) - *Hamiltonian dynamics with external forces and observation*, «Math. Systems theory», *15*, 145-168.

[12] VAN DER SCHAFT A.J. (1982) - *Observability and controllability for smooth nonlinear systems*, «SIAM J. Control and Optimization», *20*, 338-354.

[13] VAN DER SCHAFT A.J. (1983) - *Symmetries, conservation laws, and time reversibility for Hamiltonian systems with external forces.* «J. Math. Phys.», *24*, 2085-2101.