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Meccanica. — *On the differential equations of the classical and relativistic dynamics for certain generalised Lagrangian functions.* Nota (*) del Socio ANTONIO PIGNEDOLI.

ABSTRACT. — One studies the differential equations of the movement of certain classical and relativistic systems for some special Lagrangian functions. One considers particularly the case in which the problem presents cyclic coordinates. Some electrodynamical applications are studied.

KEY WORDS: Relativistic Lagrangian Dynamics.

RIASSUNTO. — *Sulle equazioni differenziali della Dinamica classica e relativistica per certe funzioni Lagrangiane generalizzate.* Si studiano le proprietà differenziali del movimento di certi sistemi dinamici, classici e relativistici, dotati di speciali tipi di funzioni lagrangiane.

L'attenzione rivolta a tali sistemi è legata al fatto che, in certi casi, il concetto di potenziale può essere esteso alle vicissitudini di sistemi dinamici in cui le forze agenti non dipendono soltanto dalla posizione ma anche dalle velocità e dalle accelerazioni lagrangiane.

Particolare riguardo è dato, nel presente lavoro, al caso in cui si abbiano coordinate ignorate o «cicliche».

Sono studiate applicazioni elettrodinamiche, comparando la situazione relativistica con quella classica.

1. Let us consider first of all the movement of a classical dynamical system with a special type of Lagrangian function. In fact in certain cases the conception of a potential-energy function can be extended to dynamical systems in which the acting forces depend not only on the position but on the velocities and accelerations of the bodies. If the Lagrangian force Q_r can be expressed in the form

$$Q_r = \frac{\partial V}{\partial q_r} - \frac{d}{dt}\frac{\partial V}{\partial \dot{q}_r} , \qquad (r = 1, 2, 3, \ldots, n) \tag{1}$$

where V is a given function of $q_1, \ldots, q_n ; \dot{q}_1, \ldots, \dot{q}_n$, that is a «velocity-dependent potential», the Lagrangian equations of motion are:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_r} - \frac{\partial T}{\partial q_r} = \frac{\partial V}{\partial q_r} - \frac{d}{dt}\frac{\partial V}{\partial \dot{q}_r} , \quad (r = 1, 2, 3, \ldots, n). \tag{2}$$

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Introducing a kinetic potential defined by the equation $L = T + V$, one obtains the differential equations of movement in the customary form:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_r} - \frac{\partial L}{\partial q_r} = 0 , \qquad (r = 1, 2, 3, \dots, n). \tag{3}$$

We can regard the function V as a generalised potential-energy function· If the Q_r are derivable from a generalised potential function like V, they must be linear functions of the accelerations $\ddot{q}_1, \dots, \ddot{q}_n$ satisfying the $n(2n-1)$ relations:

$$\begin{cases} \dfrac{\partial Q_i}{\partial \ddot{q}_k} = \dfrac{\partial Q_k}{\partial \ddot{q}_i} , \\ \dfrac{\partial Q_i}{\partial \dot{q}_k} + \dfrac{\partial Q_k}{\partial \dot{q}_i} = \dfrac{d}{dt}\left(\dfrac{\partial Q_i}{\partial \ddot{q}_k} + \dfrac{\partial Q_k}{\partial \ddot{q}_i}\right), \\ \dfrac{\partial Q_i}{\partial q_k} - \dfrac{\partial Q_k}{\partial q_i} = \dfrac{1}{2}\dfrac{d}{dt}\left(\dfrac{\partial Q_i}{\partial \dot{q}_k} - \dfrac{\partial Q_k}{\partial \dot{q}_i}\right). \end{cases} \tag{4}$$

Let us consider now a generalised potential of the following form:

$$\begin{aligned} V = {} & U(q_1, \dots, q_n, t) + \sum_1^n {}_s U_s(q_1, \dots, q_n, t)\, \dot{q}_s + \\ & + \frac{1}{2}\sum_1^n {}_{sj} U_{sj}(q_1, \dots, q_n, t)\, \dot{q}_s \dot{q}_j , \; (U_{sj} = U_{js}) . \end{aligned} \tag{5}$$

Here the forces Q_r are linear functions of the accelerations and quadratic of the velocities. The Lagrangian equations become:

$$\begin{aligned} \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_r} - \frac{\partial T}{\partial q_r} = {} & \frac{\partial U}{\partial q_r} + \sum{}_s \left(\frac{\partial U_s}{\partial q_r} - \frac{\partial U_r}{\partial q_s}\right)\dot{q}_s + \\ & + \frac{1}{2}\sum{}_{sj}\left(\frac{\partial U_{sj}}{\partial q_r} - \frac{\partial U_{rs}}{\partial q_j}\right)\dot{q}_s \dot{q}_j - \frac{\partial U_r}{\partial t} - \sum{}_s \frac{\partial U_{rs}}{\partial t}\dot{q}_s - \sum{}_s U_{rs}\ddot{q}_s . \end{aligned} \tag{6}$$

In the case in which $m < n$ coordinates (e.g. $q_1, \dots, q_m$) are cyclic, one obtains m integrals of the kinetic momenta and $(n-m)$ Lagrangian equations in terms of the "reduced Lagrangian function L*" as follows:

$$\begin{aligned} \frac{\partial L}{\partial \dot{q}_i} & = \frac{\partial (T+V)}{\partial \dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i} + U_i + \sum_1^n {}_s U_{si}\dot{q}_s = \\ & = \frac{1}{2}\frac{\partial}{\partial \dot{q}_i}\sum_1^n {}_{rs} a_{rs}\dot{q}_r \dot{q}_s + U_i + \sum_1^n {}_s U_{si}\dot{q}_s = \\ & = \sum_1^n {}_j (a_{ij} + U_{ij})\dot{q}_j + \sum_{s=m+l}^n (a_{is} + U_{is})\dot{q}_s + U_i = \alpha_i = \text{const} \\ & \qquad\qquad (i = 1, 2, 3, \dots, m) \end{aligned} \tag{7}$$

$$\frac{d}{dt}\frac{\partial L^*}{\partial \dot{q}_k} - \frac{\partial L^*}{\partial q_k} = 0, \quad (k = m+1, m+2, \ldots, n) \tag{8}$$

with:

$$L^* = L - \sum_1^m{}_i\, \alpha_i\, \dot{q}_i\,. \tag{9}$$

2. We shall consider now the problem of the movement of a relativistic particle of rest mass m_0 and velocity v from a non-quantum mechanical point of view. We indicate with $L_p = K$ the relativistic Lagrangian function of the particle, that is

$$L_p = K = m_o\, c^2\, [1 - (1 - \beta^2)^{1/2}],\ (\beta = v/c)\,, \tag{10}$$

and with $L_i = V$ the Lagrangian function of inter-action which depends on the existence and on the structure of the field of force acting on the particle. From the variational principle

$$\delta \int_{t_1}^{t_2} (L_p + L_i)\, dt = \delta \int_{t_1}^{t_2} (K + V)\, dt = 0\,,\ (L = L_p + L_i = K + V), \tag{11}$$

one has for the Lagrangian function L of the system, the dynamical differential equations:

$$\frac{d}{dt}\frac{\partial K}{\partial \dot{q}_r} - \frac{\partial K}{\partial q_r} = \frac{\partial V}{\partial q_r} - \frac{d}{dt}\frac{\partial V}{\partial \dot{q}_r} = Q_r\,, \qquad (r = 1, 2, 3) \tag{12}$$

where the force Q_r reduces to $\dfrac{\partial V}{\partial q_r}$ in the case in which the function V is not dependent on the Lagrangian velocities q_r.

In analogy with the case of the classical Mechanics, the Hamiltonian function is defined as follows:

$$H = \sum_1^3{}_r\, p_r \dot{q}_r - L = \sum_1^3{}_r\, p_r\, \dot{q}_r - K - V\,,\ p_r = \frac{\partial L}{\partial \dot{q}_r} \quad (r = 1, 2, 3), \tag{13}$$

and one finds obviously that a necessary and sufficient condition for the truth of the second order Lagrange equations is given by the six following first order Hamilton's equations:

$$\begin{aligned} &\dot{p}_r = -\frac{\partial H}{\partial q_r}\,,\ \dot{q}_r = \frac{\partial H}{\partial p_r}\,, \qquad (r = 1, 2, 3), \\ &p_r = p_r^{(p)} + p_r^{(i)} = \frac{\partial L_p}{\partial \dot{q}_r} + \frac{\partial L_i}{\partial \dot{q}_r} = \frac{\partial K}{\partial \dot{q}_r} + \frac{\partial V}{\partial \dot{q}_r}\,. \end{aligned} \tag{14}$$

If a Lagrangian coordinate, for example q_k, is cyclic, one obtains:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = \frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_k} + \frac{\partial V}{\partial \dot{q}_k}\right) = 0 \, , \; p_k^{(p)} + p_k^{(i)} = p_k = c_k = \text{const.} \tag{15}$$

Moreover, if the function V is not dependent on the Lagrangian velocities the Lagrangian dynamical equations reduce to:

$$\frac{d}{dt}\frac{\partial K}{\partial \dot{q}_r} - \frac{\partial K}{\partial q_r} = \frac{\partial V}{\partial q_r} \, , \qquad (r = 1 \, , 2 \, , 3). \tag{16}$$

Besides if V and K are independent from the coordinate q_k, one has:

$$\frac{d}{dt}\frac{\partial K}{\partial \dot{q}_k} = 0 \, , \; p_k^{(p)} = \text{const.} \tag{17}$$

Finally, if the Lagrangian function $L = K + V$ is not explicit time-dependent, the system of Lagrangian dynamical equations possesses the relativistic first integral of the energy:

$$L - \sum_1^3{}_r \frac{\partial L}{\partial \dot{q}_r} \dot{q}_r = K + V - \sum_1^3{}_r \frac{\partial (K+V)}{\partial \dot{q}_r} \dot{q}_r = \text{const.} \tag{18}$$

that is:

$$m_o c^2 \left[(1 - \beta^2)^{-1/2} - 1\right] - V = \text{const.} \tag{19}$$

3. Let's consider now a generalised Lagrangian function of the form:

$$L = K + V = K + U + \sum_s U_s \dot{q}_s + \frac{1}{2} \sum_{sj} U_{sj} \dot{q}_s \dot{q}_j \, , \tag{20}$$

$$(K = K(\dot{q}), \; U = U(q,t), U_s = U_s(q,t), U_{sj} = U_{js}(q,t); \; j, s = 1, 2, 3).$$

The Lagrangian equations of the motion transform as follows:

$$\begin{aligned} \frac{d}{dt}\frac{\partial K}{\partial \dot{q}_r} - \frac{\partial K}{\partial q_r} &= \frac{\partial U}{\partial q_r} + \sum_s \left(\frac{\partial U_s}{\partial q_r} - \frac{\partial U_r}{\partial q_s}\right) \dot{q}_s + \\ &+ \frac{1}{2} \sum_s \left(\frac{\partial U_{sj}}{\partial q_r} - \frac{\partial U_{rs}}{\partial q_j}\right) \dot{q}_s \dot{q}_j - \frac{\partial U_r}{\partial t} - \sum_s \frac{\partial U_{rs}}{\partial t} \dot{q}_s - \sum_s U_{rs} \ddot{q}_s . \end{aligned} \tag{21}$$

The generalised Lagrangian forces are here linear functions of the accelerations and linear of the velocities. If a coordinate (e.g. q_1) is cyclic, the first equation of the system (21) becomes:

$$(22)\qquad \begin{aligned}\frac{d}{dt}\frac{\partial K}{\partial \dot{q}_1} &= -\frac{dU_1}{dt} - \sum_1^3{}_s \dot{q}_s \frac{dU_{1s}}{dt} - \sum_1^3{}_s U_{1s}\ddot{q}_s = \\ &= -\left(\frac{dU_1}{dt} + \frac{d}{dt}\sum_1^3{}_s U_{1s}\dot{q}_s\right),\end{aligned}$$

and therefore one has the first integral:

$$(23)\qquad U_1 + \sum_1^3{}_s U_{s1}\dot{q}_s = \alpha_1 = \text{const.}$$

Naturally, for the whole description of the movement, one must associate the two following Lagrangian equations:

$$(24)\qquad \frac{d}{dt}\frac{\partial L^*}{\partial \dot{q}_j} - \frac{\partial L^*}{\partial q_j} = 0\,,\ (j = 2, 3)\,.$$

From the differential equations (12) it descends:

$$(25)\qquad \frac{dH}{dt} + \frac{\partial(K+V)}{\partial t} = 0\,.$$

If the Lagrangian function $K + V$ is not explicitly time-dependent, one has the generalised first integral of the energy:

$$(26)\qquad H = \sum_1^3{}_r p_r \dot{q}_r - K - V = \sum_1^3{}_r \frac{\partial K}{\partial \dot{q}_r}\dot{q}_r + \frac{\partial V}{\partial \dot{q}_r}\dot{q}_r - K - V = \text{const.}$$

In the case in which the function V is dependent on the q_r only, the first integral (26) becomes:

$$(27)\qquad \sum_1^3{}_r \frac{\partial K}{\partial \dot{q}_r}\dot{q}_r - K - V = \text{const.}$$

Now we make some remarks. First of all in the case of the existence of a cyclic coordinate, if the Lagrangian function

$$(28)\qquad L^* = K + V - \alpha_1\,\dot{q}_1\,,\ (\alpha_1 = \text{const.})$$

is not explicitly time-dependent, one has again the first energy integral:

$$(29)\qquad H^* = \sum_2^3{}_j \frac{\partial L^*}{\partial \dot{q}_j}\dot{q}_j - L^* = \text{const},$$

which takes the form:

$$\sum_{2}^{3}{}_{j}\left(\frac{\partial K}{\partial \dot{q}_j}\dot{q}_j + U_j\dot{q}_j\right) + \sum_{2}^{3}{}_{j}\sum_{1}^{3}{}_{s} U_{sj}\dot{q}_s\dot{q}_j - K - U - \sum_{1}^{3}{}_{s}U_s\dot{q}_s - \tag{30}$$
$$- \frac{1}{2}\sum_{1}^{3}{}_{sj} U_{sj}\dot{q}_s\dot{q}_j + \alpha_1\dot{q}_1 = \text{const.}$$

A second remark is the following. Let us consider the Lagrangian function of the particle K and the developments in series of the relativistic kinetic energy W_c, the total energy W and the Lagrangian function of the particle K:

$$W_c = \frac{1}{2} m_o v^2 + \frac{3}{8} m_o c^2 \beta^4 + \frac{5}{16} m_o c^2 \beta^6 + \ldots \tag{31}$$

$$W = m_o c^2 + \frac{1}{2} m_o v^2 + \frac{3}{8} m_o c^2 \beta^4 + \frac{5}{16} m_o c^2 \beta^6 + \ldots \tag{32}$$

$$K = -m_o c^2 (1-\beta^2)^{-1/2} = -m_o c^2 + \frac{1}{2} m_o v^2 + \frac{3}{8} m_o c^2 \beta^4 + \frac{5}{16} m_o c^2 \beta^6 + \ldots \tag{33}$$

These three quantities tend to the same limit if we keep the quadratic terms in the velocity only without consideration of the energy at rest.

In general the Lagrangian function of interaction of the particle is of the form:

$$V = -(A - \boldsymbol{v}\cdot\mathbf{A}) \tag{34}$$

where A is a scalar function and $\mathbf{A}$ is a certain vector whose meaning is a priori not determined. In the problem of the movement of a particle of electric charge e in an electromagnetic field $(\mathbf{E}, \mathbf{H})$ the Lagrangian function can be written in the well known form:

$$L = K + V = -m_o c^2 (1-\beta^2)^{-1/2} - e V_1 + \frac{e}{c}\boldsymbol{v}\cdot\mathbf{A}, \tag{35}$$

with V scalar potential and $\mathbf{A}$ vector potential ($\mathbf{H} = \text{rot}\,\mathbf{A}$, $\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \text{grad}\, V_1$, V_1 = electric potential).

Let us consider now the vector of position of the particle

$$(P - O) = r \exp(i\varphi)\,\mathbf{I} + z\,\mathbf{K} \tag{36}$$

with $Oxyz$ cartesian system of reference, $\mathbf{I}, \mathbf{J}, \mathbf{K}$ unitary vectors of the axes and r, φ, z cylindrical coordinates, associated with $Oxyz$. Moreover consider the form of the vector potential $\mathbf{A}$ in cylindrical coordinates:

$$\mathbf{A} = A_1 e^{i\varphi} \mathbf{I} + A_2 i e^{i\varphi} \mathbf{I} + A_3 \mathbf{K}, \quad (i^2 = -1). \tag{37}$$

We make the following hypotheses: V, A_1, A_2, A_3 are not explicitly time-dependent, φ is cyclic; moreover we put

$$\mathbf{H} = \operatorname{rot} \mathbf{A} = \operatorname{grad} \Phi(r, z) \tag{38}$$

with Φ harmonical function and indicate with $G(r, z)$ the associated function with Φ and we put again:

$$A_1 = A_3 = 0, \; A_2 = \psi(r, z), \; \mathbf{A} = r \psi \operatorname{grad} \varphi. \tag{39}$$

One has:

$$\frac{\partial G}{\partial r} = -r \frac{\partial \Phi}{\partial z}, \; \frac{\partial G}{\partial z} = r \frac{\partial \Phi}{\partial r}. \tag{40}$$

Therefore one obtains the first integral:

$$m_0 r^2 \dot{\varphi} \left(1 - \frac{\dot{r}^2 + r^2 \dot{\varphi}^2 + \dot{z}^2}{c^2}\right)^{-1/2} + G = \text{const}. \tag{41}$$

Analogically if V, A_1, A_2, A_3 are independent from z, if we suppose

$$A_1 = A_2 = 0, \quad A_3 = A(x, y), \quad \mathbf{A} = A(x, y) \operatorname{grad} z, \tag{42}$$

we obtain the first integral:

$$m_0 \dot{z} [1 - (\dot{r}^2 + r^2 \dot{\varphi}^2 + \dot{z}^2)/c^2]^{-1/2} + \frac{e}{c} A = \text{const}, \tag{43}$$

with $A(x, y)$ harmonical function coniugated with Φ such that

$$\frac{\partial A}{\partial y} = \frac{\partial \Phi}{\partial x}, \; \frac{\partial A}{\partial x} = -\frac{\partial \Phi}{\partial y}. \tag{44}$$

The first integrals (41) and (43) extend previous integrals of Agostinelli and Boggio.

4. Now we shall give an application. Let us consider the case of analytical interest, of a Lagrangian function of the following form (in polar coordinates r, θ in the plane; a and b constants, $\Phi(r)$ given function, $c =$ velocity

of light):

$$\text{(45)} \qquad K + V = - m_0 c^2 \left(1 - \frac{\dot{r}^2 + r^2 \dot{\theta}^2}{c^2}\right)^{1/2} + \\ + \Phi(r)\left(a + b\frac{\dot{r}^m}{c^2}\right), \; [m \text{ integer}, \, m \geq 2]$$

The Lagrangian equations of the movement are:

$$\text{(46)} \qquad \frac{d}{dt}\left[\frac{m_0 \dot{r}}{\sqrt{1 - \frac{\dot{r}^2 + r^2\dot{\theta}^2}{c^2}}} + m\,\Phi(r)\,b\,\frac{\dot{r}^{m-1}}{c^2}\right] - \frac{m_0 r \dot{\theta}^2}{\sqrt{1 - \frac{\dot{r}^2 + r^2\dot{\theta}^2}{c^2}}} - \\ - \Phi'(r)\left(a + \frac{b\dot{r}^m}{c^2}\right) = 0$$

$$\text{(47)} \qquad \frac{d}{dt}\left[\frac{m_0 r^2 \dot{\theta}}{\sqrt{1 - \frac{\dot{r}^2 + r^2\dot{\theta}^2}{c^2}}}\right] = 0 .$$

The equation (47) gives immediately the first relativistic integral of the momentum:

$$\text{(48)} \qquad m_0 r^2 \dot{\theta}\left[1 - \frac{\dot{r}^2 + r^2\dot{\theta}^2}{c^2}\right]^{-1/2} = m r^2 \dot{\theta} = A = \text{const.}$$

But one has also the first relativistic integral of the energy, that is:

$$\text{(49)} \qquad \frac{\partial(K+V)}{\partial\dot{r}}\dot{r} + \frac{\partial(K+V)}{\partial\dot{\theta}}\dot{\theta} - K - V = h, \; (h = \text{const}).$$

Eliminating $\dot{\theta}$ between the two integrals, one has the possibility of characterizing the allowed motions. We summarize here the question in the case of the problem of motion of an electric charge e in the field of an equal charge e. The problem of the interaction between the two charges was studied by Gauss, Weber, Riemann, Helmoltz, Ritz, Boggio, Warburton, Arzeliès. The theory of Arzeliès gives the interaction according with the experience. But the difficulty of the problem from the analytical point of view advises the study of the relativistic equations of motion in the Weber's approximating interaction. The relativistic differential equation of the relative motion of the two particles is the following:

$$\text{(50)} \qquad \frac{d}{dt}\left[m\frac{d(P_1 - P_2)}{dt}\right] = -\frac{2e^2}{r^3}\left[1 - \frac{\dot{r}^2 - 2r\ddot{r}}{c^2}\right](P_1 - P_2),$$

where $m = m_0 \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$ is the maupertuisian mass at the velocity $v = \frac{d(P_1 - P_2)}{dt}$ of the particle P_1 in the movement relatively to the particle P_2, which is considered at rest. One obtains for the problem the two relativistic integrals:

$$m_o r^2 \dot{\theta} [1 - (\dot{r}^2 + r^2 \dot{\theta}^2)/c^2]^{-1/2} = A, \; A = \text{constant} \tag{51}$$

$$m_o c^2 (1 - (\dot{r}^2 + r^2 \dot{\theta}^2)/c^2)^{-1/2} - \frac{2e^2}{r}\left(1 + \frac{\dot{r}^2}{c^2}\right) = h, \; (h = \text{const}). \tag{52}$$

Eliminating $\dot{\theta}$ between (51), (52) one obtains, with the condition $0 \leq \dot{r}^2 \leq \leq c^2$, a certain equation

$$\dot{r}^2 = R(r), \; (R(r) = \text{known function of } r) \tag{53}$$

for the possible movements of the particle. Therefore one has:

$$t - t_0 = \int \frac{dr}{\pm \sqrt{R(r)}}, \tag{54}$$

$$\theta - \theta_0 = \int \pm \sqrt{\frac{c^2 - R(r)}{r^2 R(r) + m_o^2 c^2 r^4 R(r)/A^2}} \, dr. \tag{55}$$

These give the time t and the angle θ in function of r.

5. Let us consider now, in classical mechanics, the relative movement (in a plane r, θ) of two unitarian masses P_1 and P_2, under the action of a force depending on a potential function V (of analytical interest) of the general type:

$$V = \frac{\mu}{r^n}(1 + K \dot{r}^m) \tag{56}$$

where μ and K are real positive constants and n and m are positive integers, $n \geq 1$, $m \geq 2$. The Lagrangian non-relativistic function of the problem is the following:

$$L = \frac{1}{2}(\dot{r}^2 + r^2 \dot{\theta}^2) + \mu r^{-n}(1 + K \dot{r}^m). \tag{57}$$

Consequently the Lagrangian equations of the considered movement are:

$$\ddot{r} - r\,\dot{\theta}^2 = -\frac{n\,\mu}{r^{n+1}}\,(1 + \mathrm{K}\,\dot{r}^m) - \frac{\mathrm{K}m\mu\,(m-1) r\,\dot{r}^{m-2}\,\ddot{r}}{r^{n+1}} + \frac{n\,\mathrm{K}\,m\,\mu\,\dot{r}^m}{r^{n+1}}\,, \tag{58}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\,(r^2\,\dot{\theta}) = 0,\; r^2\,\dot{\theta} = \gamma\,,\; (\gamma = \text{const})\,. \tag{58}$$

The function L is not explicitely time dependent and therefore the energy first integral exists

$$\mathrm{H} = \frac{\partial \mathrm{L}}{\partial \dot{r}}\,\dot{r} + \frac{\partial \mathrm{L}}{\partial \dot{\theta}}\,\dot{\theta} = \mathrm{h},\; (\mathrm{h} = \text{constant}) \tag{59}$$

and explicitely:

$$\frac{1}{2}\,(\dot{r}^2 + r^2\,\dot{\theta}^2) + \frac{(m-1)\,\mu\,\mathrm{K}\,\dot{r}^m}{r^n} - \frac{\mu}{r^n} = \mathrm{h}\,; \tag{59'}$$

that is, in general for $m > 2$ and $n > 1$ the problem admits two first integrals.

In the case $m = 2$ and $n = 1$ one can analytically give the equation of the trajectory and the laws of dependance of r and θ from the time t. In fact, in this case, the first integral of the energy becomes:

$$\mathrm{H} = \frac{\partial \mathrm{L}}{\partial \dot{r}}\,\dot{r} + \frac{\partial \mathrm{L}}{\partial \dot{\theta}}\,\dot{\theta} - \mathrm{L} = \frac{1}{2}\,(\dot{r}^2 + r^2\dot{\theta}^2) + \mathrm{K}\,\mu\,\frac{\dot{r}^2}{r} - \frac{\mu}{r} = \mathrm{h},\; (\mathrm{h} = \text{const}). \tag{60}$$

Eliminating $\dot{\theta} = \gamma/r^2$, one obtains:

$$\frac{1}{2}\left(\dot{r}^2 + \frac{\gamma^2}{r^2}\right) + \mathrm{K}\,\mu\,\frac{\dot{r}^2}{r} - \frac{\mu}{r} = \mathrm{h} \tag{61}$$

and therefore one has the time in terms of r as follows:

$$t - t_0 = \pm \int \sqrt{\frac{1 + 2\,\mu\,\mathrm{K}\,\dfrac{1}{r}}{2\,\mathrm{h} + \dfrac{2\,\mu}{r} - \dfrac{\gamma^2}{r^2}}}\;\mathrm{d}r. \tag{62}$$

But one has $\dot{r} = \dfrac{\mathrm{d}r}{\mathrm{d}\theta}\,\dot{\theta} = -\gamma\,\dfrac{\mathrm{d}\frac{1}{r}}{\mathrm{d}\theta}$ and, putting $\dfrac{1}{r} - \dfrac{\mu}{\gamma^2} = \xi\,,\; 2\,\mathrm{K}\mu = \varepsilon\,,$ $\dfrac{2\,h}{\gamma^2} - \dfrac{\mu^2}{\gamma^4} = \xi_0\,,$ one can write the equation of the trajectory in the following form:

$$\left(\frac{d\xi}{d\theta}\right)^2 = \frac{\xi_0^2 - \xi^2}{1 + \varepsilon\left(\frac{\mu}{\gamma^2} + \xi\right)}, \quad (\xi_0 = \text{const}), \tag{63}$$

If one supposes that r is initially increasing with θ, such that one has initially

$$\frac{d\frac{1}{r}}{d\theta} = \frac{d\xi}{d\theta} < 0, \tag{64}$$

one obtains

$$\theta - \theta_0 = \int - \sqrt{\frac{1 + \varepsilon\left(\frac{\mu}{\gamma^2} + \xi\right)}{\xi_0^2 - \xi^2}}\, d\xi \tag{65}$$

that is θ in terms of ξ by elliptic integrals. If $\varepsilon = 2\,K\mu$ is a very small quantity, such that one can write:

$$\frac{d\xi}{d\theta} = -\sqrt{\xi_0^2 - \xi^2}\left[1 + \frac{\varepsilon}{2}\left(\frac{\mu}{\gamma^2} + \xi\right)\right], \tag{66}$$

one obtains θ in the form of a well-known integral, that is

$$\theta - \theta_0 = -\int \frac{d\xi}{\sqrt{\xi_0^2 - \xi^2}\left[1 + \frac{\varepsilon}{2}\left(\frac{\mu}{\gamma^2} + \xi\right)\right]}. \tag{67}$$

By simple calculations one obtains the equation of the trajectory of the motion of a particle relative to the other:

$$\frac{1}{r} = \frac{\mu}{\gamma^2} + \xi_0 \cos(\theta - \theta_0) + \frac{\varepsilon\xi_0}{2}\sin(\theta - \theta_0)\left[\frac{\mu\,\theta}{\gamma^2} + \xi \sin(\theta_0 - \theta)\right] \tag{68}$$

(This equation reduces to the equation of an ellipse in the case $\varepsilon = 0$:

$$\frac{1}{r} = \frac{\mu}{\gamma^2} + \xi_0 \cos(\theta - \theta_0)). \tag{69}$$

In the same approximation for $\varepsilon \neq 0$, one has for the time t the following expression containing elementary integrals:

$$t - t_0 = \int \frac{r\,\mathrm{d}r}{\sqrt{2hr^2 + 2\mu r - \gamma^2}} + \frac{\varepsilon}{2} \int \frac{\mathrm{d}r}{\sqrt{2hr^2 + 2\mu r - \gamma^2}} \tag{70}$$

Putting

$$-\frac{\mu}{2h} = a, \quad -\frac{\gamma^2}{2h} = a^2(1 - e^2),$$

with a semi-major axis and e eccentricity of the ellipse and assuming as inf for the integrals the value $r_1 = a(1 - e)$, one obtains

$$t - t_0 = \frac{1}{\sqrt{-2h}} \int_{r_1}^{r} \frac{r\,\mathrm{d}r}{a^2 e^2 - (a - r)^2} + \frac{1}{2} \frac{\varepsilon}{\sqrt{-2h}} \cdot \int_{r_1}^{r} \frac{\mathrm{d}r}{\sqrt{a^2 e^2 - (a - r)^2}}. \tag{71}$$

Finally, putting $r = a(1 - e \cos u)$, one has for the time in the considered relative movement:

$$t - t_0 = \frac{a}{\sqrt{-2h}} (u - \mathrm{e} \sin u) + \frac{\varepsilon}{2\sqrt{-2h}} u. \tag{72}$$

For $\varepsilon = 0$, the (72) reduces to the

$$t - t_0 = \frac{a}{\sqrt{-2h}} (u - e \sin u) \tag{73}$$

which is the Keplerian classical formula.

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