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Geometria algebrica. — *On a linearity criterion for algebraic systems of divisors on a projective variety* (*). Nota di UMBERTO BARTOCCI (**) e LUCIO GUERRA (**), presentata (***) dal Socio E. MARTINELLI.

ABSTRACT. — In the present paper, it is established in any characteristic the validity of a classical theorem of Enriques', stating the linearity of any algebraic system of divisors on a projective variety, which has index 1 and whose generic element is irreducible, as soon as its dimension is at least 2.

KEY WORDS: Divisors; Algebraic/linear systems; Chow varieties.

RIASSUNTO. — *Su un criterio di linearità per sistemi algebrici di divisori su una varietà proiettiva*. Nel presente lavoro, viene stabilita la validità in ogni caratteristica di un classico teorema di Enriques, che stabilisce la linearità di ogni sistema algebrico di divisori di una varietà proiettiva, il quale abbia indice 1 ed il cui elemento generico sia irriducibile, non appena detto sistema abbia dimensione almeno 2.

INTRODUCTION

For an irreducible algebraic system Σ^δ of hypersurfaces (of degree m) in a complex projective space $\mathbf{P}^r_c$, the *index* ν of Σ is defined as the number of *distinct* hypersurfaces of Σ passing through a *generic* set of δ points of $\mathbf{P}^r$, where $\delta =$ dimension of Σ.

The classical Index theorem (1) (see for instance [1], p. 188) states that the index ν of Σ equals the degree of the parameter variety T of Σ, which is embedded in the projective space $\mathbf{P}^N$, $N = \binom{n+r}{r} - 1$, representing all hypersurfaces of degree m in $\mathbf{P}^r$, provided that:

(i) *the generic element of* Σ *has no multiple components* (in this case, Σ is also often said to have no « variable » multiple components).

From this Index theorem one obtains the following simple characterization of *linear systems* of hypersurfaces in $\mathbf{P}^r$, namely that: *an irreducible alge-*

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(1) Not to be confused with the more widely known Index theorem of Hodge-Hirzebruch-Atiyah-Singer!

braic system Σ *as before, which satisfies assumption* (i), *is a linear system if, and only if, its index is equal to* 1.

This characterization can easily be extended to algebraic systems Σ of *positive Weil divisors* on a normal projective variety X^d of $\mathbf{P}^r$ (d = dimension of X), provided that the following (necessary) condition is considered in addition to the previous ones:

(ii) Σ *is totally contained within a linear system.*

Let us explicitly state this result:

PROPOSITION 1. *An irreducible algebraic system* Σ *of divisors on* X, *satisfying both conditions* (i) *and* (ii), *is a linear system if, and only if, its index is equal to* 1.

As a matter of fact, under assumption (ii), the system Σ is cut out on X by some algebraic system Σ' of hypersurfaces of $\mathbf{P}^r$ (up to some possible fixed divisor Δ: $\Sigma + \Delta = \Sigma' \cdot X$), and what one can actually prove is that the index ν of Σ equals the index ν' of Σ', whence the conclusion.

A deep improvement of Proposition 1 is given by the following:

THEOREM (Enriques, [2]). *An irreducible algebraic system* Σ^δ *of divisors on* X^d, *where* $d \geq 2$, *having index* $\nu = 1$ *and dimension* $\delta \geq 2$, *is a linear system provided that it only satisfies the following further condition*:

(i') *the generic element of* Σ *is irreducible.*

Condition (i') is a stronger version of condition (i); the assumption that $\nu = 1$ is the obvious necessary condition for an algebraic system to be linear. What is remarkable in Enriques' theorem is that assumption (ii) is not needed any more as in Proposition 1; as a matter of fact, we shall see that in the present set-up it is automatically satisfied as soon as the dimension of Σ is at least 2.

Of course, both conditions (i') and $\delta \geq 2$ are *necessary* (2). As a matter of fact, it is well known that on some variety X one can find a non-linear *pencil* (that is to say, a 1-dimensional algebraic system of index 1) whose generic element is irreducible; for smooth surfaces in characteristic zero, this happens precisely when X is *irregular* ([3], p. 98). Furthermore, if X carries a non-linear pencil Σ, then X also carries a non-linear system of index 1 and dimension $\delta \geq 2$ for each value of δ, since it is enough then to consider the system made up of all possible sums of δ elements of Σ.

With these premises, when one allows the base field to be any *algebraically closed field* K *of arbitrary characteristic* p, then the question arises as to which of the previous statements remain true.

(2) In [3], p. 25, the statement of Enriques' theorem is correct only on condition that with an "irreducible algebraic system", one satisfying (i') is meant, and not, as more usual, one whose parameter variety is irreducible, as we mean in this paper too.

In [4] a general version of the Index theorem is given, from which Proposition 1 follows exactly the same as in the classical case (we refer to the Bibliography given in [4] for more historical sources on the present subject).

Our task in the present paper is to prove that *Enriques' theorem also maintains its validity in all positive characteristics.*

We shall do that by bringing Enriques' proof into a rigorous algebraic form, which will then be also valid for any value of p. Essentially, the proof only rests on the extension of Proposition 1 quoted above and on a classical criterion relating linear equivalence on X to linear equivalence on a generic hyperplane section of X. The main point is however a reduction argument to a special case, which needs to be carefully developed: we shall do that by using some results from the theory of Chow varieties.

§ 1. PRELIMINARY REMARKS ON ALGEBRAIC SYSTEMS

We assume X^d embedded in some projective space $\mathbf{P}^r$. The positive k-dimensional cycles of a fixed degree m in $\mathbf{P}^r$, with support in X, are represented by the points of a closed subset $\mathrm{Ch}^{k,m}(X)$ of the *Chow variety* $\mathrm{Ch}^{k,m}(\mathbf{P}^r)$ parametrizing all positive k-dimensional cycles of degree m in $\mathbf{P}^r$ (here the word "variety" is traditionally used even when $\mathrm{Ch}^{k,m}(\mathbf{P}^r)$ is reducible).

If T is any subvariety of $\mathrm{Ch}^{k,m}(X)$, then the set $\Sigma(T)$ of all cycles of X whose Chow points lie in T is called the *algebraic system* associated with T. The fact that T is a variety is usually expressed by saying that $\Sigma(T)$ is an irreducible algebraic system. The dimension of $\Sigma(T)$ is then, by definition, the dimension of T.

Even though the algebraic structure of the Chow varieties $\mathrm{Ch}^{k,m}(X)$ may depend on the fixed projective embedding of X, nevertheless their Zariski topology does not, as follows from [5], § 2, hence all the above definitions about algebraic systems do not either.

However, an algebraic system can also be given by means of any variety U and positive cycle Z on $U \times X$, such that all the irreducible components of Z project onto U. Indeed, with a generic (here and in the following, this means "for almost all") $u \in U$ we can then associate the cycle $Z_u = pr_X(Z \cdot u \times X)$, which has constant dimension k and degree m. Of course, the same cycle may be associated with several u, and Z_u need not be defined for all u. But, because of the following Proposition, we obtain an algebraic system in the sense of the previous definition.

PROPOSITION 2. *In the present situation, the closure of the set of Chow points of cycles of the form* Z_u *is a subvariety* T *of* $\mathrm{Ch}^{k,m}(X)$, *and there exists an irreducible correspondence between* U *and* T *which associates the Chow point of* Z_u *with each* $u \in U$ *such that* Z_u *is defined.*

Proof. Is found in [5], § 2. The correspondence in question is actually the graph of a rational map $U \to T$ but, unfortunately, a complete proof of this result, valid in any characteristic, does not seem to be available in the existing literature (see [6] for some more references and remarks); however, we shall not need this more precise statement.

Conversely, for any algebraic system $\Sigma(T)$, there is an *incidence cycle* I on $T \times X$ such that $I_t =$ cycle whose Chow point is t, for almost all $t \in T$ (follows from [7], p. 107, § 8 (*b*)).

Let us recall just one more simple definition, which is used throughout the paper. If Σ is an algebraic system on X, we say that its generic element is irreducible if its Chow variety T is not contained in the closed subset of $\mathrm{Ch}^{k,m}(X)$ parametrizing reducible cycles. If Σ is given by means of the pair (U, Z), this means that Z is irreducible and, in addition, that Z_u is irreducible for a generic u.

Consider, now, algebraic systems of positive Weil divisors of X, i.e. the case $k = d - 1$. In this paper we are concerned with proving that, in this case, a certain $\Sigma(T)$ is a *linear system* (the hypothesis that X is a normal variety is set so that the concepts of the theory of linear systems of divisors may legitimately be applied). In the present context, this means that the following two conditions hold (let us denote by D_t the divisor whose Chow point is t):

(*a*) for all pairs $t, \tau \in T$, the divisors D_t and D_τ are *linearly equivalent* (Σ is *totally contained within the complete linear system* $\mathbf{P}^s = |D_t|$);

(*b*) the image of T in the natural mapping $T \xrightarrow{\phi} \mathbf{P}^s$, defined by $\phi(\theta) = D_\theta$, is a linear subspace of $\mathbf{P}^s$ (when this happens, one can prove that ϕ is actually an isomorphism of T onto the linear space $\phi(T)$, [6]).

If we assume that Σ satisfies all conditions in the statement of Enriques' theorem then, in order to prove that Σ is a linear system, *it suffices to check* (*a*), since (*b*) then follows in virtue of Proposition 1. We point out that, in order to prove (*a*), it is enough to prove:

(*a'*) for *almost all* pairs $t, \tau \in T$, the divisors D_t and D_τ are *rationally equivalent*. This is because, for divisors on a normal variety, rational equivalence is the same as linear equivalence and since then the Chow variety T of the system Σ and the Chow variety L of the linear system $|D_t|$ intersect at least in a set of points which is open as a subset of T, whence $T \subseteq L$ and $\Sigma \subseteq |D_t|$.

We end this section with quoting the following criterion for linear equivalence, which will be needed in § 2.

PROPOSITION 3. *Let* D *be a divisor on a normal variety* $X \subset \mathbf{P}^r$ *of dimension* $d \geq 3$. *Then, for a generic hyperplane* L *of* $\mathbf{P}^r$, D *is linearly equivalent to zero on* X *if and only if* $D \cdot L$ *is linearly equivalent to zero on* $Y = X \cdot L$.

Proof. [8], p. 111, Theorem 2.

In the next §§ 2–3, we will show how the proof of the general assertion of Enriques' theorem can be reduced to the proof of some special case of the assertion itself.

§ 2. REDUCTION TO THE 2-DIMENSIONAL CASE

Here we show that, in order to prove the theorem (or, as is the same, in order to prove (a')), *we may confine ourselves to the case when* $\mathrm{d} = \delta = 2$.

2.1. *It suffices to deal with the case* $\mathrm{d} = \dim(\mathrm{X}) = 2$.

In fact, if $\mathrm{d} \geq 3$, given two divisors $\mathrm{D}_t, \mathrm{D}_\tau$ of Σ, then a generic hyperplane L of the ambient space $\mathbf{P}^r$ cuts out on X and irreducible normal variety $\mathrm{Y} = \mathrm{X} \cdot \mathrm{L}$ (see [9]) and is such that both $\mathrm{D}_t \cdot \mathrm{L}$, $\mathrm{D}_\tau \cdot \mathrm{L}$ are defined and $\mathrm{D}_\theta \cdot \mathrm{L}$ is also defined for almost all θ. Thus Σ determines an algebraic system $\Sigma' = \Sigma \cdot \mathrm{L}$ supported in Y, containing both $\mathrm{D}_t \cdot \mathrm{L}$, $\mathrm{D}_\tau \cdot \mathrm{L}$, which still satisfies all the hypotheses of Enriques' theorem. By induction on d, we know that $\mathrm{D}_t \cdot \mathrm{L}$, $\mathrm{D}_\tau \cdot \mathrm{L}$ are linearly equivalent on Y, whence we deduce that D_t, D_τ are linearly equivalent on X, in virtue of Proposition 3.

2.2. *We may also suppose without loss of generality that* $\delta = \dim(\Sigma) = 2$.

Indeed, if $\delta \geq 3$, then two generic irreducible divisors $\mathrm{D}_t, \mathrm{D}_\tau$ of Σ both contain such a point $x \in \mathrm{X}$ that the subsystem $\Sigma_x = \Sigma(\mathrm{T}_x)$, where $\mathrm{T}_x = \{t \in \mathrm{T} \mid \mathrm{Supp}(\mathrm{D}_t) \ni x\}$, is irreducible of dimension $\delta - 1$ and index 1. The generic element of Σ_x being clearly irreducible, we know therefore, by induction on δ, that $\mathrm{D}_t, \mathrm{D}_\tau$ are linearly equivalent divisors. There is one basic fact which was implicitely used in the argument above. Let us state it also for future reference, as a:

Remark 2.3. Let Σ be an irreducible algebraic system on a normal variety X. If $x \in \mathrm{X}$ is not a base point of Σ, then Σ_x is an algebraic system whose Chow variety T_x is of pure dimension $\delta - 1$ and the index of Σ_x (which is defined in the same way as in the introduction) is still equal to ν. Furthermore, if $\nu = 1$, then Σ_x is necessarily irreducible.

§ 3. FURTHER REDUCTION TO A MORE SPECIAL CASE

From now on we assume that X is a (normal) surface and $\Sigma(\mathrm{T})$ is a 2-dimensional system on X satisfying all the hypotheses of Enriques' theorem.

There is another important invariant of the system Σ which we are going to introduce. To this purpose, consider the following incidence correspondence between X and $\mathrm{T} \times \mathrm{T}$:

$$\Gamma = \{(x, t, \tau) \mid x \in \mathrm{Supp}(\mathrm{D}_t) \cap \mathrm{Supp}(\mathrm{D}_\tau)\}.$$

It is not difficult to see, by some easy dimension argument (and using Remark 2.3), that Γ possesses just one irreducible component G whose projection into X contains some non-base point of Σ, and that G then projects onto X, with (irreducible) fibre $x \times T_x \times T_x$ over any non-base point of Σ. The remaining components of Γ are therefore the $x \times T \times T$, for each base point x of Σ. Thus, we have the following decomposition:

$$\Gamma = G \cup B \times T \times T \text{, where } B = \text{base locus of } \Sigma .$$

It is known that, denoting by $q = p^{e}$ $(\epsilon \geq 0)$ the inseparability degree, and by n the separability degree, of G over $T \times T$, then for almost all pairs (t, τ), the intersection cycle

$$G_{t\tau} = \mathrm{pr}_X (G \cdot X \times t \times \tau)$$

is defined and is q times a group of n distinct points of X. For such a pair (t, τ), $G_{t\tau}$ contains every intersection point of D_t, D_τ apart from the base points of Σ, each being counted with multiplicity q (of course, a base point of Σ may appear in some special $G_{t\tau}$, but not in the generic one). The integer n is classically called the *degree of the system* Σ.

The degree of Σ can also be seen from a different point of view. If we consider the incidence divisor I on $X \times T$ then, in the same way as it defines the system Σ on X, it also defines an *inverse system* Σ' of divisors of T, parametrized by X, whose generic member has support equal to T_x. Σ' is still 2-dimensional and its index is equal to the degree of Σ.

Remark 3.1. In general, the numerical invariants of an inverse system Σ' are related to those of Σ by the following formulas:

$$\delta' = \delta , \quad \nu' = n , \quad n' = \nu ,$$

with obvious notations. Furthermore, when $\nu = 1$, if the generic element of Σ is irreducible, then the generic element of Σ' is q' times a variety T_x, where q' = inseparability degree of the projection $I \to X$. Let us say, in this situation, that the generic element of Σ' is *quasi-irreducible*.

The aim of this section is to justify and develop the main point of Enriques' proof, that is, to show that:

3.2. *We may confine ourselves to the case when both* ν *and* n *are equal to* 1.

More precisely, we show how we can always replace the surface X with another (normal) surface $\bar{X}$, and the system Σ on X with a new system $\bar{\Sigma}$ on $\bar{X}$, having both $\bar{\nu}$ and $\bar{n}$ equal to 1, so that $\bar{\Sigma}$ being a linear system on $\bar{X}$ implies Σ being a linear system on X.

From Proposition 2, we know that there exist a subvariety V of $\mathrm{Ch}^{0,n'}(\mathrm{X})$, $n' = qn$, and an irreducible correspondence between $\mathrm{T} \times \mathrm{T}$ and V which associates the Chow point $g_{t\tau}$ of the 0-cycle $\mathrm{G}_{t\tau}$ with each pair (t, τ) such that $\mathrm{G}_{t\tau}$ is defined. Let us consider the algebraic system $\Lambda = \Sigma(\mathrm{V})$ of 0-cycles of X, whose generic element is of the form $\mathrm{G}_{t\tau}$.

It is not difficult to see that, for almost all pairs (t, τ), each one of the n distinct points $x_1, \ldots, x_n$ which appear in $\mathrm{G}_{t\tau}$ is a generic point of X, in the sense that T_{x_i} is irreducible and D_t is irreducible for almost all $t \in \mathrm{T}_{x_i}$. Then, for $i \neq j$, $\mathrm{T}_{x_i} \cap \mathrm{T}_{x_j}$ contains two distinct points t, τ. But, if through two distinct points $x, y \in \mathrm{X}$ there are at least two distinct divisors of an algebraic system Σ having index 1, then there are infinitely many divisors of Σ passing through x, y. Therefore, in the present case, $\mathrm{T}_{x_i} = \mathrm{T}_{x_j}$ – denote it by T_0 – or, in other words, every D_θ through some one of the x_i must contain every $x_1, \ldots, x_n$. Hence, for almost all pairs $(\theta, \xi) \in \mathrm{T}_0 \times \mathrm{T}_0$, one has $\mathrm{G}_{\theta\xi} = \mathrm{G}_{t\tau}$. Now, a generic $g \in \mathrm{V}$ represents a 0-cycle of the form $\mathrm{G}_{t\tau} \in \Lambda$. If $(\theta, \xi) \in \mathrm{T} \times \mathrm{T}$ corresponds with g, then clearly $\mathrm{D}_\theta \cap \mathrm{D}_\xi$ contains the support of $\mathrm{G}_{t\tau}$, hence $\theta, \xi \in \mathrm{T}_0$. On the other hand, we have seen that almost all pairs $(\theta, \xi) \in \mathrm{T}_0 \times \mathrm{T}_0$ correspond with g. Thus, with a generic $g \in \mathrm{V}$, there is precisely a 2-dimensional variety $\mathrm{T}_0 \times \mathrm{T}_0$ which corresponds in the way previously defined. This implies that $\mathit{dim}(\Lambda) = \mathit{dim}(\mathrm{V}) = 2$.

Furthermore, the 2-dimensional system Λ *is an involution on* X, i.e. a generic point $x \in \mathrm{X}$ belongs to a unique 0-cycle of Λ. In fact, a generic element of Λ is of the form $\mathrm{G}_{t\tau} = q(x_1 + \ldots + x_n)$, where the x_i are different from each another and are such that $\mathrm{T}_{x_i} = \mathrm{T}_{x_j} = \mathrm{T}_0$ is irreducible, as we already saw. Moreover, through a generic point $x \in \mathrm{X}$ only there are generic members of Λ. If $\mathrm{G}_{t\tau} = q(\Sigma x_i)$, $x_1 = x$, and $\mathrm{G}_{\theta\xi} = q(\Sigma y_i)$, $y_1 = x$, are two such 0-cycles, then $\mathrm{T}_x = \mathrm{T}_{x_1} = \mathrm{T}_{y_1} = \mathrm{T}_0$ and hence $\mathrm{G}_{t\tau} = \mathrm{G}_{\theta\xi}$.

If we now consider the natural incidence correspondence between X and V:

$$\mathrm{Z} = \{(x, g) \mid x \text{ belongs to the 0-cycle G of } \Lambda \text{ whose Chow point is } g\},$$

then Λ being an involution implies that Z possesses a unique irreducible component Δ projecting onto X and then, clearly, onto V:

$$\begin{tikzcd} & \Delta \arrow[dl, "\pi"'] \arrow[dr, "\rho"] & \\ \mathrm{X} & & \mathrm{V} \end{tikzcd}$$

With a generic $x \in \mathrm{X}$ there corresponds the Chow point $g \in \mathrm{V}$ of the unique 0-cycle $\mathrm{G} \in \Lambda$ passing through x and, conversely, with a generic $g \in \mathrm{V}$ there correspond the n distinct points belonging to the support of the 0-cycle $\mathrm{G} = q(x_1 + \ldots + x_n) \in \Lambda$ whose Chow point is g. In other words, the separability degree of π is 1, the one of ρ is n.

In the present situation, it follows from standard intersection theory that the pull-back $\pi^* (D_t)$ is defined for almost all t, and is then of the form a times a simple divisor E_t, where the multiplicity a is a power of the characteristic p. Actually, E_t is an irreducible divisor, because of π being generically $1-1$. Thus, if we write $\pi_* (E) = a' D_t$, a' the (inseparable) degree of the restriction $E_t \xrightarrow{\pi} D_t$, then $\pi_* \pi^* (D_t) = a\, a' D_t$ and, therefore, $a\, a'$ = (inseparable) degree of π. Let us remark that, a priori, these a, a' might be depending on t; however, one sees that they are constant for almost all t.

From Proposition 2, we know that there exists an algebraic system $\pi^* (\Sigma)$ on Δ, whose generic member is of the form $a E_t$. It is clear, moreover, that we can then write $\pi^* (\Sigma) = a \Sigma'$, where Σ' is an algebraic system whose generic member is of the form E_t (look at the injective morphism $E \mapsto a E$ between the suitable Chow varieties of Δ). *The index of Σ' is easily seen to be 1* (see the last comma before Remark 3.3 below).

On the other hand, the push-forward $\rho_* (E_t)$ is defined for almost all t, and is of the form $b F_t$, where b = degree of the restriction $E_t \xrightarrow{\rho} F_t$ and F_t = = irreducible divisor of V. Then, we can write $\rho^* (F_t) = b' E_t$, the multiplicity b' being a power of the characteristic p. This happens because the generic E_t is a disjoint union of supports of 0-cycles belonging to Λ. We thus have $\rho_* \rho^* (F_t) = b\, b' E_t$ and, therefore, $b\, b'$ = degree of ρ.

There is, then, an algebraic system $\overline{\Sigma}$ on V, whose generic member is o the form F_t, hence irreducible. *Both the index and the degree of $\overline{\Sigma}$ are equal to 1.*

The two marked assertions above, concerning the numerical invariants of Σ', $\overline{\Sigma}$, are easily checked by a purely set-theoretic argument, if one only keeps in mind the following:

Remark 3.3. When, under some irreducible correspondence Z between two varieties X and Y, there is only a finite set of points $y \in Y$ which correspond with a generic point $x \in X$, whatever non-empty open subset V of Y be fixed, then there is a non-empty open subset U of X such that each one of the finite number of y, which corresponds with $x \in U$, actually lies in V. In particular, the index of an algebraic system $\Sigma (T)$ can be also computed as the number of *generic* divisors D_t (i.e. for t belonging to some non-empty open subset $U \subseteq T$, which can be arbitrarily chosen) passing through a generic set of δ points of the carrier variety X. An analogous remark also holds for computing the degree of Σ.

We may now replace V by its normalization, which is the variety $\overline{X}$ we claimed at the beginning, and Δ by its normalization, let us denote it by the same symbol Δ, and still we have a diagram

$$\begin{tikzcd} & \Delta \arrow[dl, "\pi"'] \arrow[dr, "\rho"] & \\ X & & \overline{X} \end{tikzcd}$$

(we allow ourselves to use here the same symbols as before, for simplicity of notation). Also, we obtain a system Σ' on the normalized Δ, and a system $\overline{\Sigma}$ on $\overline{X}$, which still share the same properties with their homonymous ones defined before. We have the following chain of implications:

$$\overline{\Sigma} \text{ a linear system} \Rightarrow \rho^*(\overline{\Sigma}) = b'\,\Sigma' \text{ linear} \Rightarrow \Sigma' \text{ linear} \Rightarrow$$

$$\Rightarrow \pi_*(\Sigma') = a'\,\Sigma \text{ totally contained within a linear system} \Rightarrow$$

$$\Rightarrow a'\,\Sigma \text{ a power of a linear system} \Rightarrow \Sigma \text{ a linear system.}$$

The first and third arrows in this sequence are standard. The second and fifth depend on the following easy.

Remark 3.4. Let Σ be an algebraic system on a normal variety X, and let a be a positive multiplicity. Then a Σ is never a linear system if Σ is non-linear itself. Furthermore, if Σ is linear, then a Σ is linear if, and only if, the multiplicity a is a power of the characteristic p.

Finally, the fourth arrow follows from a more general version of Proposition 1, which can be easily proved in the same way as Proposition 1, with the same arguments used in [4]. We state this Proposition as:

Remark 3.5. An algebraic system Σ of index 1, satisfying condition (ii), either is a linear system itself, or is a *power of a linear system* L, plus some possible fixed divisor Δ (that is to say, Σ is made up of all divisors of kind a $D + \Delta$, with $D \in L$, for some positive multiplicity a).

§ 4. CONCLUSION

Because of the results of the last two sections, it is enough now to prove Enriques' theorem in the case when X is a (normal) surface, and $\Sigma(T)$ is a 2-dimensional system on X, whose generic element D_t is irreducible, and both whose index and degree are equal to 1. Thus, the present situation is symmetric with respect to Σ and its inverse system Σ', both whose index and degree also are equal to 1, the only difference being that the generic element of Σ' is a priori only quasi-irreducible (Remark 3.1). We shall now prove that:

4.1. *Both the generic element D_t of Σ and the support T_x of the generic element of Σ' are rational curves.*

Property (a') stated in § 1 then follows, by only taking into account the second comma after Proposition 2 in § 1, and therefore Enriques' theorem results completely demonstrated. Furthermore, in order to verify assertion 4.1, note that:

4.2. *It is enough to prove* D_t *a rational curve.*

Indeed, we might well have made the situation wholly symmetric by simply allowing from the beginning the generic element of the system Σ to be quasi-irreducible (it is clear a priori, for instance because of Remark 3.4, that Enriques' theorem still holds in this slightly more general form); moreover, under this hypothesis, a direct proof completely similar to the following one would serve to prove the rationality of the support of the generic D_t. On the other hand, in the present situation, we may: first, observe that proving the rationality of the support T_x of the generic element of the inverse system Σ' is completely similar to proving the rationality of the generic D_t and, second, apply the already mentioned Remark 3.4 to the system on T made up of all these T_x (to be precise, we have to consider the induced system on the normalization of T).

Thus, let us come to the proof of 4.2, which will need just a few more lemmas.

For every two irreducible D_t, $D_\tau \in \Sigma$ such that $G_{t\tau}$ is q times a single point $0 \in X$, not a base point of Σ, let us consider the following incidence subset of $X \times D_t \times D_\tau$:

$$\Gamma = \{(x, A, B) \mid \exists\, \theta \in T \mid D_\theta \ni x, A, B\},$$

giving rise to a family of correspondences between D_t and D_τ:

$$\Gamma_x = \{(A, B) \in D_t \times D_\tau \mid (x, A, B) \in \Gamma\}, \qquad \text{for every } x \in X.$$

LEMMA 4.3. *If x is a generic point of* X, *then* Γ_x *is a generically* $1-1$ *correspondence.*

Proof. It is clear that almost all divisors $D_\theta \in \Sigma$ are such that both $G_{\theta t}$ and $G_{\theta\tau}$ (are defined and) consist of q times a single point which is not a base point of Σ. Fix one such D_θ, and choose $x \in D_\theta$ not belonging to $D_t \cup D_\tau$. Since x is not a base point of Σ, the subsystem Σ_x is a pencil (Remark 2.3) and therefore, since $x \notin D_t \cup D_\tau$, no base point of Σ_x, apart from the base points of Σ, belongs to $D_t \cup D_\tau$ (if y is a base point of Σ_x, not of Σ, then $\Sigma_y = \Sigma_x$). Furthermore, almost all $D_\eta \in \Sigma_x$ are such that $G_{\eta t} = qA$, $G_{\eta\tau} = qB$, A, B not base points of Σ (nor of Σ_x) (for, one such divisor is D_θ). Therefore, through almost all $A \in D_t$ there is just one $D_\eta \in \Sigma_x$, which meets D_τ at a single point B, not a base point of Σ. By the symmetric argument, with almost all $B \in D_\tau$, there is just one corresponding point $A \in D_t$. □

Because of 4.3, if x is a generic point of X, then there exists precisely one irreducible component E_x of Γ_x projecting onto D_t and D_τ, E_x being thus an irreducible generically $1-1$ correspondence between D_t and D_τ. At this point, we could spend some time showing that $\{\Gamma_x\}$ is a 2–dimensional family of correspondences (in the same sense as in § 1 for a family of cycles) parametrized by $x \in X$ in a generically $1-1$ way. However, all we need is the following property:

Lemma 4.4. *For almost all pairs* A, $A' \in D_t$, $B, B' \in D_\tau$, *there exists some* E_x *under which* (A, B) *and* (A', B') *are pairs of corresponding points and, moreover, no point different from* B (*resp.* B') *does correspond with* A (*resp.* A').

Proof. For almost all $D_\theta \in \Sigma$, one has $G_{\theta t} = qA$, $G_{\theta\tau} = qB$, A, B not base points of Σ. Now, almost all $D_\eta \in \Sigma$ are such that $G_{\eta t} = qA'$, $G_{\eta\tau} = qB'$, $G_{\eta\theta} = qx$, A', B', x not base points of Σ. It is clear, from the proof of 4.3, that Γ_x is generically $1-1$, so that E_x is defined. The second half of the statement follows since no base point of Σ_x belongs to $D_t \cup D_\tau$. $\square$

Now, if D_t is a generic member of Σ, then we can choose some auxiliary D_τ as before, and some point $y \in X$ such that Γ_y determines an irreducible generically $1-1$ correspondence E_y between D_τ and D_t, in order to obtain a family $\{F_x = E_x \circ E_y\}$ of irreducible generically $1-1$ correspondences of D_t with itself, by just composing each E_x with this fixed E_y. Clearly, this family $\{F_x\}$ satisfies the same property as in 4.4. Furthermore, from $\{F_x\}$ we can then construct a family $\{\phi_x\}$ of birational maps of D_t with itself, which still enjoys some property coming from 4.4, as we shall see.

This is because in the diagram

$$\begin{tikzcd} & F_x \arrow[dl, "\pi"'] \arrow[dr, "\pi'"] & \\ D_t & & D_t \end{tikzcd}$$

both projections π, π' are generically $1-1$ morphisms having the same inseparability degree $q = q'$. In fact, the obvious automorphism $D_t \times D_t \leftrightarrow D_t \times D_t$, $(A, B) \leftrightarrow (B, A)$, carries F_x isomorphically onto itself and exchanges projections π and π'. Now, any morphism $C \xrightarrow{\pi} D$ between curves is obtained as a composition $C \xrightarrow{\sigma} C' \overset{\psi}{\dashrightarrow} D$, where σ is a Frobenius map of degree equal to the inseparability degree of π, and ψ is a separable rational map whose degree is then the separability degree of π. Therefore, if we denote by σ the q-Frobenius map $F_x \to F'_x$, then there exist birational maps $F'_x \overset{\psi, \psi'}{\dashrightarrow} D_t$ such that the diagram

$$
\begin{tikzcd}
& F_x \arrow[ddl, "\pi"'] \arrow[d, "\sigma"] \arrow[ddr, "\pi'"] & \\
& F'_x \arrow[dl, dashed, "\psi"'] \arrow[dr, dashed, "\psi'"] & \\
D_t \arrow[rr, dashed, "\phi_x"'] & & D_t
\end{tikzcd}
$$

is commutative, where $\phi_x = \psi' \circ \psi^{-1}$ also is a birational map.

This family $\{\phi_x\}$ enjoys the following property:

LEMMA 4.5. *For almost all* $A, A', B, B' \in D_t$, *there exists some* ϕ_x *such that* $\phi_x(A) = B$, $\phi_x(A') = B'$ *and, moreover, no point* C *different from* A (*resp.* A') *is such that* $\phi_x(C) = B$ (*resp.* B').

Proof. There are open subsets $U \subseteq D_t$, $V \subseteq D_\tau$ such that E_y induces a bijective correspondence between U and V. Therefore, with a generic set $A, A', B, B' \in D_t$, such that $A, A' \in U$, then a set $C, C' \in D_\tau$, $B, B' \in D_t$, corresponds through E_y, which is generic in the sense of 4.4. If we take E_x as in 4.4, relatively to such C, C', B, B' then, clearly, (A, B), (A', B') are pairs of corresponding points under $F_x = E_x \circ E_y$ and, moreover, no point different from B (resp. B') corresponds with A (resp. A'). Furthermore, almost all such A, A', B, B' are smooth points of D_t, so ϕ_x is automatically defined at A, A' and $\phi_x(A) = B$, $\phi_x(A') = B'$. $\square$

In particular, we see that for almost all pairs $A, B \in D_t$ there are infinitely many birational maps ϕ_x such that $\phi_x(A) = B$. This implies that D_t is a rational curve, as we claimed in 4.2. Indeed, a curve whose desingularization has genus ≥ 2 possesses finitely many birational maps at all (see [10], p. 66). On the other hand, for a curve whose desingularization is an elliptic curve, there are only finitely many birational maps taking a given point into another (see [11], p. 182). The proof of 4.2 is now achieved, hence the proof of Enriques' theorem is so too.

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