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Equazioni differenziali ordinarie. — *Connected branches of asymptotically equivalent solutions to non-linear eigenvalue problems.* Nota di ALLAN L. EDELSON (*) e MARIA PATRIZIA PERA (**), presentata (***) dal Corrisp. R. CONTI.

ABSTRACT. — We prove an existence theorem for connected branches of solutions to nonlinear operator equations in Banach spaces. This abstract result is applied to the asymptotically equivalent solutions to nonlinear ordinary differential equations.

KEY WORDS: Nonlinear eigenvalue problems; fixed point index; asymptotic equivalence of solutions.

RIASSUNTO. — *Rami connessi di soluzioni asintoticamente equivalenti per problemi agli autovalori non lineari.* Si studia l'esistenza di connessi globali di soluzioni per problemi agli autovalori non lineari in spazi di Banach e si prova, per una classe di equazioni differenziali ordinarie, l'esistenza di rami di soluzioni asintoticamente equivalenti a polinomi.

1. INTRODUCTION

This article is concerned with the topological structure of the asymptotically equivalent solutions of

$$x^{(n)} + xf(t, x) = 0, \tag{1.1}$$

where $f: [0, \infty) \times \mathbf{R} \to \mathbf{R}$ is a continuous function satisfying the conditions

(i) $f(t, x) > 0$ for $x \neq 0$;

(ii) f is non-decreasing in x (in which case (1.1) is said to be superlinear);

or

(iii) f is non-increasing in x (in which case (1.1) is said to be sublinear).

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The considerable interest in equation (1.1) has generally focused on oscillation criteria and conditions necessary or sufficient for asymptotic equivalence to the linear unperturbed equation $x^{(n)} = 0$. Our interest here will focus on the topological structure of the solutions to the parametrized equation

$$x^{(n)} + \lambda x f(t, x) = 0, \qquad \lambda \geq 0, \tag{1.2}$$

of prescribed asymptotic type. Precisely, we will prove a general result (Theorem 1) showing the existence of connected branches of solutions of non-linear eigenvalue problems $u = \mathrm{T}(\lambda, u)$, for $\lambda \in [0, \infty)$, u in a convex subset of a Banach space E and T a compact map. This result, together with a modification of known existence theorems for solutions to (1.1), will be applied to show the existence of connected branches of solutions of (1.2) having prescribed asymptotic behaviour. As a consequence of our requirement that all solutions in a branch be defined on a common interval of existence, we are required to prove existence on $[0, \infty)$, rather than on $[\tau, \infty)$, for sufficiently large τ. We note that Theorem 2, proved herein for the case of equation (1.1), can be generalized to the larger class of equations of the form

$$\mathrm{L}_n[x] + x f(t, x) = 0$$

where L_n is an n^{th} order disconjugate linear differential operator.

An early result on the asymptotic behaviour of solutions of (1.1) was a theorem of Hardy, who proved for the equation $x^{(4)} = p(t)\, x$, $p(t)$ positive and continuous, that if x is a solution asymptotic to t and satisfying $x > 0$, $x' > 0$, $x'' > 0$, $x^{(3)} < 0$, then x' is asymptotic to 1.

Asymptotic here is in the sense of

DEFINITION 1.1. *Let x, y be continuous functions with y non-zero. Then x is asymptotic to y if* $0 < \lim\limits_{t \to \infty} |x(t)/y(t)| < \infty$.

The most comprehensive result on asymptotic equivalence for solutions of (1.1) is due to Kitamura and Kusano [7]. We state here a weaker version of their result, which is sufficient for our purposes.

THEOREM A (Kitamura and Kusano). *Equation* (1.1), *either sublinear or superlinear, has a solution asymptotic to* t^m, $0 \leq m \leq n-1$, *if*

$$\int^{\infty} t^{n-1} f(t, ct^m)\, \mathrm{d}t < \infty \tag{1.3}$$

for every constant $c > 0$.

A stronger definition of asymptotic equivalence is given by

Definition 1.2. *Let x, y be continuous functions. Then x is strongly asymptotic to y if* $\lim_{t \to \infty} |x(t) - y(t)| = 0$.

For strong asymptotic equivalence there is the following [4]:

Theorem B (Edelson and Schuur). *Equation* (1.1), *either sublinear or superlinear, has a solution strongly asymptotic to t^m, $0 \leq m \leq n-1$, if*

$$\int^{\infty} t^{n+m-1} f(t, ct^m)\, dt < \infty \tag{1.4}$$

for every constant $c > 0$.

The result we obtain apply to both asymptotic equivalence and strong asymptotic equivalence.

2. Connected branches of solutions

Our study of (1.1) will begin with the abstract eigenvalue problem

$$u = T(\lambda, u)$$

where T is a continuous and compact map, i.e. T sends bounded sets into relatively compact sets. In this context, we can establish the following result, whose proof is obtained by combining a classical point set topology result with a fixed point argument:

Theorem 1. *Let E be a Banach space, Q a closed convex subset of E, U an open (relative to $[0, \infty) \times Q$) subset of $[0, \infty) \times Q$. Let $T : \overline{U} \to Q$ be a continuous and compact map. Assume $U \cap (\{0\} \times Q) \neq \varnothing$ and $T(O, u) = u_0$ for all u, with $(0, u_0) \in U$. Then the equation $u = T(\lambda, u)$ has a connected branch C of solutions $(\lambda, u) \in \overline{U}$, emanating from $(0, u_0)$ and satisfying at least one of the following conditions:*

(i) C *is unbounded*;

(ii) C *intersects* ∂U (*the boundary of* U *relative to* $[0, \infty) \times Q$).

Proof. Let S denote the solution set $S = \{(\lambda, u) \in \overline{U} : u = T(\lambda, u)\}$. Let us show first that

(iii) For any bounded open subset W of U containing $(0, u_0)$ we have $S \cap \partial W \neq \varnothing$.

Denote by W_λ the slice of W at λ, i.e. $W_\lambda = \{u \in Q : (\lambda, u) \in W\}$. Let $T_\lambda : W_\lambda \to Q$ be the map $T_\lambda(u) = T(\lambda, u)$. Since W is open in $[0, \infty) \times Q$, we have that W_λ is open in Q. Moreover, W_λ is an open absolute neighbourhood retract (ANR), since it is an open subset of a convex set. Therefore, if we assume by contradiction that $S \cap \partial W = \varnothing$ then the fixed point index $i(T_\lambda, W_\lambda, Q)$ is well-defined for each λ (for a detailed exposition of the index theory for arbitrary ANR's see e.g. [2], [6]).

Take $\bar{\lambda} > 0$ such that $W \subset [0, \bar{\lambda}) \times Q$. By the general homotopy invariance of the fixed point index, $i(T_0, W_0, Q) = i(T_{\bar{\lambda}}, W_{\bar{\lambda}}, Q)$. But $W_{\bar{\lambda}} = \varnothing$, so that $i(T_{\bar{\lambda}}, W_{\bar{\lambda}}, Q) = 0$. On the other hand, since the map T_0 is the constant map $T_0(u) = u_0$ with u_0 belonging to W_0, from the normalization property of the index, it follows $i(T_0, W_0, Q) = 1$, which gives a contradiction. This proves (iii).

Now denote by C the component of $(0, u_0)$ in S. A standard point set topology argument based on Whyburn's lemma [10] (see also [8], Chapter 5) applies showing that C satisfies either i) or ii).

Q.E.D

It should be noted that in Theorem 1 the hypothesis $T(0, u_0) = u_0$ can be replaced by the following more general assumptions:

(h_1) $T(0, u) \neq u$ for any $u \in \partial U_0$, where U_0 denotes the slice of U at 0;

(h_2) the fixed point index of $T(0, \cdot)$ in U_0, $i(T(0, \cdot), U_0)$, is defined and non-zero.

With these assumptions the conclusion of the theorem remains the same except that the branch C emanates from the bounded subset of U $\{(\lambda, u) \in \bar{U}: T(0, u) = u\}$. The proof is a standard modification of the previous one.

The idea of using degree methods in proving the existence of global branches of solutions to non-linear eigenvalue problems in Banach spaces goes back to P.H. Rabinowitz [9]. For maps acting between positive cones of ordered Banach spaces, the analogue of Rabinowitz's result has been obtained by E.N. Dancer [3], and H. Amann [1]. Continuation methods in locally convex spaces and applications to ordinary differential equations in non-compact intervals have been recently developed by M. Furi and the second author in [5].

Let us now pass to consider the non-linear eigenvalue problem

$$x^{(n)} + \lambda x f(t, x) = 0, \qquad \lambda \geq 0. \tag{2.1}$$

A solution of (2.1) is a pair (λ, x) with $x \in C^n([0, \infty))$. Recalling Theorem A, we seek to prove the existence of an unbounded, connected branch of solutions of (2.1), with x asymptotically equivalent to t^m, $1 \leq m \leq n-1$, as $t \to \infty$, and therefore we will work in the Banach space

$$E_m = \{x \in C([0, \infty)) : -\infty < \lim_{t \to \infty} (x(t)/(1 + t^m)) < \infty\}$$

with norm $\| x \|_m = \sup \{| x(t) |/(1 + t^m): 0 \leq t < \infty\}$.

Our principal result is the following theorem.

THEOREM 2. *Consider equation* (2.1) *either sublinear or superlinear, and assume*

(i) $f(t, x) > 0$ for $x \neq 0$;

(ii) $\int^{\infty} t^{n-1} f(t, ct^m)\, dt < \infty, \ \forall c > 0$.

Then (2.1) *has un unbounded, connected branch* C *of solutions pairs* (λ, x), $\lambda > 0$, *such that*

(i) x *is asymptotic to* t^m, *as* $t \to \infty$;

(ii) C *emanates from* $(0, 1 + t^m)$.

In particular, the projection of C *on the x-space is a connected branch of solutions* $x(t)$, *asymptotic to* t^m.

Proof. We will give the proof for (2.1) superlinear, the sublinear casc being similar. Define a closed, convex subset Q_m of E_m by

$$Q_m = \{x \in E_m : \lim_{t \to \infty} (x(t)/(1 + t^m)) = 1\}$$

and for $x \in Q_m$, $\lambda \geq 0$, define

$$T(\lambda, x)(t) = 1 + t^m + \lambda/((m-1)!\,(n-m-1)!) \int_0^t (t-s)^{m-1} \int_s^{\infty} (s-\sigma)^{n-m-1} x(\sigma) f(\sigma, x(\sigma))\, d\sigma\, ds .$$

Then solutions in E_m of the operator equation

$$T(\lambda, x)(t) = x(t) \tag{2.2}$$

also satisfy (2.1). We will show that T is a continuous and compact mapping of $[0, \infty) \times Q_m$ into Q_m, and then apply Theorem 1, with $U = [0, \infty) \times Q_m$.

(i) $T([0, \infty) \times Q_m) \subset Q_m$. By L'Hospital's rule

$$\lim_{t \to \infty} (1 + t^m)^{-1} T(\lambda, x)(t) =$$

$$= 1 + \lim_{t \to \infty} (\lambda/((m-1)!\,(n-m-1)!)) (1 + t^m)^{-1} \int_0^t (t-s)^{m-1} \int_s^{\infty} (s-\sigma)^{n-m-1} x(\sigma) f(\sigma, x(\sigma))\, d\sigma\, ds = 1 + \lim_{t \to \infty} \lambda/(m!\,(n-m-1)!)$$

$$\int_t^{\infty} (t-\sigma)^{n-m-1} x(\sigma) f(\sigma, x(\sigma))\, d\sigma .$$

Since $x(t)$ is asymptotic to t^m and f is non-decreasing in x, we see that

$$\left| \int_t^\infty (t-\sigma)^{n-m-1} x(\sigma) f(\sigma, x(\sigma)) \, d\sigma \right| \leq \int_t^\infty \sigma^{n-m-1} (k\sigma^m) f(\sigma, k\sigma^m) \, d\sigma \leq$$

$$\leq k \int_t^\infty \sigma^{n-1} f(\sigma, k\sigma^m) \, d\sigma \text{ for some constant } k > 0 .$$

This last integral has limit 0 as $t \to \infty$, which shows that

$$\lim_{t\to\infty} (1 + t^m)^{-1} T(\lambda, x)(t) = 1 .$$

(ii) T is continuous. Continuity in λ is obvious. Let $\{x_k\}_{k\in N}$ be a sequence in Q_m with $\| x_k - x \|_m \to 0$. Then $| T(\lambda, x_k)(t) - T(\lambda, x)(t) | (1 + t^m)^{-1} \leq (\lambda/((m-1)!(n-m-1)!)) (1 + t^m)^{-1} \int_0^t (t-s)^{m-1} \int_s^\infty (\sigma - s)^{n-m-1}$

$$| x_k(\sigma) f(\sigma, x_k(\sigma)) - x(\sigma) f(\sigma, x(\sigma)) | \, d\sigma \, ds .$$

Let $R_k(\sigma) = | x_k(\sigma) f(\sigma, x_k(\sigma)) - x(\sigma) f(\sigma, x(\sigma)) |$ so that

$$0 \leq R_k(\sigma) \leq | x_k(\sigma) | | f(\sigma, x_k(\sigma)) - f(\sigma, x(\sigma)) | + f(\sigma, x(\sigma)) | x_k(\sigma) - x(\sigma) | .$$

We have

$$(2.3) \qquad (\lambda/((m-1)!(n-m-1)!)) (1 + t^m)^{-1} \int_0^t (t-s)^{m-1} \int_s^\infty (\sigma - s)^{n-m-1}$$

$$R_k(\sigma) \, d\sigma \, ds \leq \lambda (1 + t^m)^{-1} \int_0^t (t-s)^{m-1} \int_s^\infty (\sigma - s)^{n-m-1} | x_k(\sigma) |$$

$$| f(\sigma, x_k(\sigma)) - f(\sigma, x(\sigma)) | \, d\sigma \, ds + \lambda (1 + t^m)^{-1} \int_0^t (t-s)^{m-1} \int_s^\infty (\sigma - s)^{n-m-1}$$

$$f(\sigma, x(\sigma)) | x_k(\sigma) - x(\sigma) | \, d\sigma \, ds .$$

We let I_1 and I_2 denote respectively the first and second integrals on the right hand side of (2.3). Assume $\lambda > 0$ and fix $\varepsilon > 0$. For $\delta > 0$, there exists N_δ such that $k > N_\delta$ implies $| x_k(\sigma) - x(\sigma) | < \delta (1 + \sigma^m)$, for $0 \leq \sigma < \infty$. Recall that Q_m is closed, so x belongs to Q_m. Since $\| x \|_m < \infty$, there exists $\mu > 0$ such that $0 \leq | x(\sigma) | \leq \mu (1 + \sigma^m)$, for $0 \leq \sigma <$

$< \infty$, and by the superlinearity property it follows that $0 \leq f(\sigma, x(\sigma)) \leq$ $\leq f(\sigma, \mu(1+\sigma^m))$, for $0 \leq \sigma < \infty$.

To estimate I_2:

$$I_2 \leq \lambda\delta(1+t^m)^{-1}\int_0^t (t-s)^{m-1}\int^{\infty}(\sigma-s)^{n-m-1}(1+\sigma^m)$$

$$f(\sigma, \mu(1+\sigma^m))\, d\sigma\, ds \leq \lambda\delta K_\mu$$

where $K_\mu = \int_0^\infty \sigma^{n-m-1}(1+\sigma^m) f(\sigma, \mu(1+\sigma^m))\, d\sigma < \infty$. Therefore $I_2 < \varepsilon/2$ for $\delta < \varepsilon/2\lambda K_\mu$, $k > N_\delta$.

To estimate I_1:
for every $T_0 \in (0, \infty)$ we have

$$I_1 \leq \lambda \int_0^{T_0} \sigma^{n-m-1} |x_k(\sigma)| |f(\sigma, x_k(\sigma)) - f(\sigma, x(\sigma))| d\sigma + \tag{2.4}$$

$$+ \lambda \int_{T_0}^{\infty} \sigma^{n-m-1} |x_k(\sigma)| |f(\sigma, x_k(\sigma)) - f(\sigma, x(\sigma))| d\sigma .$$

With N_δ and μ as before, we have

$$0 \leq |x_k(\sigma)| \leq \mu(1+\sigma^m), \ |x_k(\sigma) - x(\sigma)| < \delta(1+\sigma^m), \text{ for } 0 \leq \sigma < \infty.$$

Therefore

$$0 \leq |x_k(\sigma)| \leq (\mu+\delta)(1+\sigma^m). \text{ We also have } |f(\sigma, x_k(\sigma)) -$$

$$- f(\sigma, x(\sigma))| \leq 2 f(\sigma, (\mu+\delta)(1+\sigma^m)).$$

Choose T_0 so large that

$$2\lambda(\mu+\delta)\int_{T_0}^{\infty} \sigma^{n-m-1}(1+\sigma^m) f(\sigma, (\mu+\delta)(1+\sigma^m))\, d\sigma < \varepsilon/4 ,$$

and the second integral in (2.4) will be less than $\varepsilon/4$. Now, on the compact interval $[0, T_0]$, x_k converges uniformly to x, and f is uniformly continuous.

Hence there exists M_δ such that, for $k > M_\delta$,

$$|f(\sigma, x_k(\sigma)) - f(\sigma, x(\sigma))| < \varepsilon \left\{4\lambda \int_0^{T_0} \sigma^{n-m-1}(\mu + \delta)(1 + \sigma^m)\, d\sigma\right\}^{-1},$$

so that for $k > \max\{N_\delta, M_\delta\}$, the first integral in (2.4) will be less than $\varepsilon/4$, and this completes the proof.

(iii) T is compact. A subset $F \subset Q_m$ is relatively compact if and only if, (α) the functions in F are uniformly bounded in the norm of Q_m; (β) the family $\{x(t)(1 + t^m)^{-1} : x \in F\}$ is equicontinuous; (γ) $\forall \varepsilon > 0$, $\exists \tau > 0$ such that $t > \tau$ implies $|x(t)(1 + t^m)^{-1} - 1| < \varepsilon$ for all $x \in F$.

Let W be a bounded subset of Q_m. Thus there exists an $M > 0$ such that $|x(t)|(1 + t^m)^{-1} \leq M$, for $0 \leq t < \infty$, and $\lambda \leq M$, for any $(\lambda, x) \in W$.

Define $K_M = \int_0^\infty \sigma^{n-m-1}(1 + \sigma^m) f(\sigma, M(1 + \sigma^m))\, d\sigma$.

(α) $$|T(\lambda, x)(t)| \leq 1 + t^m + M^2/((m-1)!\,(n-m-1)!) \int_0^t (t-s)^{m-1} \int_0^\infty \sigma^{n-m-1}(1 + \sigma^m) f(\sigma, M(1 + \sigma^m))\, d\sigma\, ds \leq 1 + t^m + M^2 K_M \int_0^t (t-s)^{m-1}\, ds,$$

and, therefore, $|T(\lambda, x)(t)|(1 + t^m)^{-1} \leq 1 + M^2 K_M$, so that the family of functions $\{T(\lambda, x) : (\lambda, x) \in W\}$ is uniformly bounded.

(β) To show equicontinuity it suffices to show that the family of functions $\{(d/dt)[(T(\lambda, x)(t)(1 + t^m)^{-1}] : (\lambda, x) \in W\}$ is uniformly bounded on $[0, \infty)$.

We have, for $m > 1$, $|(d/dt)[(T(\lambda, x)(t)(1 + t^m)^{-1}]| \leq$

$$\leq |(d/dt)(1 + \lambda/((m-1)!\,(n-m-1)!) \int_0^t (t-s)^{m-1}(1 + t^m)^{-1} \int_s^\infty (s-\sigma)^{n-m-1} x(\sigma) f(\sigma, x(\sigma))\, d\sigma\, ds)| \leq$$

$$\leq M^2 K_M \int_0^t |(m-1)(t-s)^{m-2}(1 + t^m)^{-1} - mt^{m-1}(t-s)^{m-1}(1 + t^m)^{-2}|\, ds \leq 2 M^2 K_M.$$

An analogous estimate is valid also for $m = 1$.

(γ) Choose $\tau_0 = \tau_0(\mathrm{M})$ so large that

$$\mathrm{M}^2/((m-1)!\,(n-m-1)!) \int_{\tau_0}^{\infty} (\sigma - \tau_0)^{n-m-1} (1 + \sigma^m)$$

$$f(\sigma, \mathrm{M}(1 + \sigma^m))\, d\sigma < \varepsilon/2 .$$

Then, for $t > \tau_0$, we have

$$|(\mathrm{T}(\lambda, x)(t)(1 + t^m)^{-1} - 1| \leq \mathrm{M}^2/((m-1)!\,(n-m-1)!)\,[\mathrm{K_M} \int_0^{\tau_0} (t-s)^{m-1}$$

$$(1 + t^m)^{-1}\, ds + \int_{\tau_0}^{t} (t-s)^{m-1} (1 + t^m)^{-1} \int_{\tau_0}^{\infty} (\sigma - \tau_0)^{n-m-1} (1 + \sigma^m)$$

$$f(\sigma, \mathrm{M}(1 + \sigma^m))\, d\sigma\, ds] \leq \mathrm{M}^2 \mathrm{K_M} \int_0^{\tau_0} (t-s)^{m-1} (1 + t^m)^{-1}\, ds +$$

$$+ \varepsilon/2 \leq \mathrm{M}^2 \mathrm{K_M}\, \tau_0 (t^{m-1}/(1 + t^m)) + \varepsilon/2 < \varepsilon \text{ for } t$$

sufficiently large.

This completes the proof that T is compact. Consequently Theorem 1 applies with $\mathrm{U} = [0, \infty) \times \mathrm{Q}$ so that the equation (2.2) has an unbounded connected branch of solutions (λ, x), $\lambda > 0$, emanating from $(0, 1 + t^m)$.

Q.E.D.

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