Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Gabriella Kuhn, Paolo M. Soardi

On the number of representations of an element in a polygonal Cayley graph

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **81** (1987), n.4, p. 331–336.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1987_8_81_4_331_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1987.

RENDICONTI

DELLE SEDUTE

DELLA ACCADEMIA NAZIONALE DEI LINCEI

Classe di Scienze fisiche, matematiche e naturali

Ottobre - Dicembre 1987

SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Atti Acc. Lincei Rend. fis.
(8), LXXXI (1987), pp. 331-336

Teorie combinatorie. — *On the number of representations of an element in a polygonal Cayley graph.* Nota di GABRIELLA KUHN e PAOLO M. SOARDI, presentata (*) dal Socio E. MARCHIONNA.

ABSTRACT. — We compute explicitly the number of paths of given length joining two vertices of the Cayley graph of the free product of cyclic groups of order k.

KEY WORDS: Number of paths; Random walks; Cayley graphs; Free products; Fibonacci numbers.

RIASSUNTO. — *Sul numero di rappresentazioni di un elemento in un grafo di Cayley poligonale.* Si calcola esplicitamente il numero di cammini di lunghezza data che uniscono due vertici del grafo di Cayley del prodotto libero di gruppi ciclici di ordine k.

1. INTRODUCTION

Let G denote the finitely presented group

$$G = \langle a_1, \ldots, a_k \mid a_1^{N+1} = \ldots = a_k^{N+1} = e \rangle \tag{1}$$

where e is the identity, $N \geq 1$ and $K \geq 2$.

(*) Nella seduta del 19 giugno 1987.

Consider the juxtaposed product

$$a_{j_1} a_{j_2} \ldots a_{j_n} \tag{2}$$

of n (not necessarily distinct) symbols a_j $(n \geq 1)$. In this paper we study the following problem: given any $w \in G$ and any positive integer n, find the number of representations of w of the form (2). We will denote by $\chi(n, w)$ such a number and will set, for definiteness, $\chi(0, w) = 0$. Note that, if we denote by Γ the Cayley graph of G with respect to the generators $a_1, \ldots, a_k$, then $\chi(n, w)$ is the number of paths of n edges from e to w.

The analogous problem was studied by P. Gerl [G] when G is the free product of two copies of Z (the group of the relative integers).

It is not hard to show that this case corresponds to the case where G is as in (1) with $N = 1$ and $k = 4$.

The problem formulated above may be reduced to the problem of computing the n-step transition probabilities of the simple random walk on Γ. In this framework, the methods of [C-S], [W], [C] and [K] give information on the asymptotic behaviour of those probabilities, but do not allow an explicit computation. The aim of this paper is to produce such a computation by elementary methods.

2. The number of representation

Every element $w \in G$ different from the identity can be uniquely represented as a reduced word of the form

$$w = a_{j_1}^{r_1} a_{j_2}^{r_2} \ldots a_{j_s}^{r_s} \tag{3}$$

where $j_i \neq j_{i+1}$, $1 \leq j_i \leq k$, $1 \leq r_i \leq N$, $i = 1 \ldots s$.

Of course, to every reduced word correspond in general many irreduced words. For every $w \in G$, $w \neq e$, the length of w, (the so called step length) is the number

$$l(w) = r_1 + r_2 + \ldots + r_s \tag{4}$$

where $r_1, \ldots, r_s$ are the numbers appearing in the representation (3) of w. We also set $l(e) = 0$. A moment's reflection shows that $\chi(n, w_1) = \chi(n, w_2)$ whenever $l(w_1) = l(w_2)$.

Thus we set

$$X(n, l(w)) = \chi(n, w).$$

For every $r \geq 0$ let E_r denote the set of all elements of G having length r and let x_r denote the characteristic function of E_r.

The following relations are easily verified

$$\chi_1 * \chi_r = \chi_{r+1} \qquad \text{if} \quad 0 \leq r < N, \tag{5}$$

$$\chi_1 * \chi_r = \chi_{r+1} + k\chi_0 \qquad \text{if} \quad r = N, \tag{6}$$

$$\chi_1 * \chi_r = \chi_{r+1} + (k-1)\chi_{r-N} \qquad \text{if} \quad r > N. \tag{7}$$

Here * denotes the convolution on G. We can now prove our main result.

THEOREM. *Suppose* G *is the group* (1) *and let* $X(n, r)$ *denote the number of different representations* (2) *of any element of step-length* r $(r \geq 0, n \geq 0)$. *If* $n \equiv r \pmod{(N+1)}$, *then*

$$X(n, r) = (N + 1 - kN) \sum_{j=0}^{b} k^j (k-1)^{b-j} \binom{a+b-j-1}{a-1} + N(k-1)^{b+1} \binom{a+b}{a}$$

where $n = a + b$ *and* $r = a - bN$. *If* $n \not\equiv r \pmod{(N+1)}$, *then* $X(n, r) = 0$.

If $k = 2$ *the above expression simplifies to*

$$X(n, r) = (1 - N)\left(2^{a+b} - 2^{a-1} + 1 - \binom{a+b+1}{a}\right) + \binom{a+b}{a}.$$

Proof. Let x_1^{*n} denote the n — *th* convolution power of x_1. Then, by (5) – (7) and the definition of $X(n, r)$, we necessarily have

$$\chi_1^{*n} = \sum_{r=0}^{n} X(n, r)\, \chi_r. \tag{8}$$

From (8) and equations (5) – (7) we obtain the recursive equations

$$\begin{aligned} X(n+1, 0) &= kX(n, N) \\ X(n+1, r) &= X(n, r-1) + (k-1)X(n, r+N). \end{aligned} \tag{9}$$

Note that $X(n, r) = 0$ if $r > n$ or $n \not\equiv r \pmod{(N+1)}$. Therefore we may restrict X to the domain $D_1 = \{(n, r) \in Z^2 : 0 \leq r \leq n, n \equiv r \pmod{(N+1)}\}$.

Setting $n = a + b$ and $r = a - bN$, D_1 is mapped onto the domain $D = \{(a, b) \in Z^2 : 0 \leq b \leq a/N\}$. We let

$$x(a, b) = X(n(a, b), r(a, b)),$$

so that equations (9) become

$$(10)\qquad \begin{aligned} x(bN, b) &= kx(bN, b-1) \\ x(a+1, b+1) &= x(a, b+1) + (k-1) \times (a+1, b), \end{aligned}$$

for $(a, b) \in D$. We also have the conditions

$$(11)\qquad x(0, 0) = 0 \quad , \quad x(a, 0) = 1 \quad \text{if } a > 1,$$

since $x(a, 0) = X(a, a) = 1$ for $a > 1$.

Let $f(a, z)$ denote the generating function $\sum_{b=0}^{\infty} x(a, b) z^b$. From the second equation of (10) and from (11) we get

$$(12)\qquad f(a, z) = C(z)(1 - (k-1)z)^{-a},$$

where $a \geq 1$ and $C(z) = \sum_{j=0}^{\infty} c_j z^j$ has to be determined (cfr. [J]).

From (12) and the definition of $f(a, z)$ we obtain

$$(13)\qquad x(a, b) = \sum_{j=0}^{b} c_j (k-1)^{b-j} \binom{a+b-j-1}{a-1}.$$

Taking the first equation of (10) into account, we get by (13)

$$(14)\qquad \sum_{j=0}^{b} (c_j - kc_{j-1})(k-1)^{b-j} \binom{b(N+1)-j-1}{b-j} = 0,$$

with the convention $c_{-1} = 0$.

Setting $d_j = (c_j - kc_{j-1})(k-1)^{-j}$, equation (15) becomes

$$(14')\qquad \sum_{j=0}^{b} d_j \binom{b(N+1)-j-1}{b-j} = 0.$$

The solution of (14') is given by $d_0 = c_0 = 1$, $d_j = -N$ for $j \geq 1$, whence

$$(15)\qquad c_j = (1 + N(1-k))k^j + N(k-1)^{j+1}, \; j \geq 1.$$

From (13) and (15) we obtain the first part of the theorem.

If $k = 2$, the expression for $x(a, b)$ can be simplified as indicated in the statement of the theorem, by repeated use of summation by parts and elementary identities for binomial coefficients.

3. RELATION WITH RANDOM WALKS AND CONCLUDING REMARKS

For every element $x \in G$ let δ_x denote the unit mass at x. Let μ denote the probability measure on G

$$\mu = k^{-1}(\delta_{a_1} + \delta_{a_2} + \ldots \delta_{a_k}) = k^{-1}\chi_1$$

(as χ_1 can obviously be identified with a measure).

Let us consider a sequence $Y_1, Y_2, \ldots, Y_n, \ldots$ of G-valued independent identically distributed random variables with common distribution μ. The random variables

$$W_n = Y_0 Y_1 Y_2 \ldots Y_n \qquad (Y_0 = e \text{ identically})$$

define a random walk on G (with initial point e) whose transition probabilities are $P(W_{n+1} = y \mid W_n = x) = \mu(x^{-1}y)$. Let $P^{(n)}_{e,w} = P(W_n = w)$.

Then, by independence, $P^{(n)}_{e,w} = \mu^{*n}(w) = k^{-n}\chi_1^{*n}(w)$. It follows that if $l(w) = r$, $P^{(n)}_{e,w} = k^{-n} X(n, r)$. An application of Theorem 5 in Woess' paper [W] gives the asymptotic estimate for $X(n, r)$ as $n \to \infty$:

$$X(n, r) \simeq k^n \rho^{-n} n^{-3/2} \qquad \text{if} \quad n \equiv r \pmod{(N+1)}$$

$$X(n, r) = 0 \qquad \text{otherwise.}$$

Here C is a constant depending on r, N and k only and ρ is the convergence norm of the random walk.

The numbers $k^{-n} X(n, r)$ represent the probability that at the time n the random walk is at a given element of length r. Another natural problem is the computation of the probability that at the time n the random walk is at distance r from the initial point e (the distance between two points x and y being simply $l(y^{-1}x)$). Clearly, if q_r denotes the cardinality of E_r, one has

$$P(l(W_n) = r) = q_r k^{-n} X(n, r).$$

Hence the problem reduces to computing q_r. It turns out that the q_r's are sums of generalized Fibonacci numbers. We say that the numbers f_r ($r = 0, 1, \ldots$) are (k, N)-Fibonacci numbers if they satisfy the recursive relation

$$f_r = (k-1)(f_{r-1} + f_{r-2} + \ldots f_{r-N}) \qquad \text{if} \quad r > N \tag{16}$$

with the initial conditions: $f_r = 0$ if $0 \leq r < N$, $f_N = 1$ (cfr. [Ka]).

It is easy, passing to the l^1 norms in (5)–(7) and then using the method of the generating function, to verify that $q_r = f_r + f_{r+1} + \ldots + f_{r+N}$ for

every r. Arguing as in [M] one can conclude that the numbers q_r have the expression

$$(17) \qquad q_r = \sum_{s=0}^{N} \sum_{j=1}^{N} \alpha_j^{r+s-N} (p'(\alpha_j))^{-1}, \qquad \text{if } r > N,$$

$$q_r = k^r \qquad , \quad \text{if } r \leq N.$$

Where $\alpha_1, \alpha_2, \ldots, \alpha_N$ are the N distinct roots of the polynomial $p(z) = z^N - (k-1)(1 + z + z^2 + \ldots + z^{N-1})$. If $N = 1$, equations (17) give the well known expression for the cardinality of the r — th corona in a homogeneous tree of degree k. In the case $k = 2$ we have, by the definition of generalized Fibonacci numbers, $q_r = f_{r+N+1}$. The (2, N)-Fibonacci numbers were studied in detail by Miles [M].

References

[C] D. Cartwright (1986) - *Some remarks about random walks on free products of discrete groups*, preprint, Sydney 1986.

[C-S] D. Cartwright and P.M. Soardi (1986) - *Random walks on free products, quotients and amalgams*, « Nagoja Math. J. », *102*, 163-180.

[G] P. Gerl (1971) - *Ueber die Anzahl der Darstellungen von Worten*, « Monat. Math. », *75*, 205-214.

[J] C. Jordan, *The Calculus of finite differences*, New York.

[K] G. Kuhn (1986) - *Some more examples of Plancherel measures*, preprint, Milano.

[Ka] D. Kalman (1982) - *Generalized Fibonacci numbers by matrix methods*, « The Fibonacci Quart. », *20*, 73-76.

[M] E.P. Miles (1960) - *Generalized Fibonacci numbers and associated matrices*, « Amer. Math. Monthly », *67*, 745-752.

[W] W. Woess - *Nearest neighbour random walks on free products of discrete groups*, to appear in « Boll. Un. Mat. Soc. ».