Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Giorgio Novati

On the analysis of elastic layers by a Fourier series, Green's function approach

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **81** (1987), n.3, p. 293–304.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1987_8_81_3_293_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1987.

Atti Acc. Lincei Rend. fis.
(8), LXXXI (1987), pp. 293-304

Meccanica dei solidi. — *On the analysis of elastic layers by a Fourier series, Green's function approach* (*). Nota di GIORGIO NOVATI (**), presentata (***) dal Corrisp. G. MAIER.

ABSTRACT. — The plane strain elastic analysis of a homogeneous and isotropic layer of constant thickness, is formulated using Fourier series expansions in the direction parallel to the layer and suitable Green's functions in the transversal direction. For each frequency the unknown distributions of the Fourier coefficients relevant to the symmetric or skew-symmetric problems are governed by one-dimensional equations which can be solved exactly. The proposed method is used to critically discuss the "transfer" of static and kinematic quantities through the layer.

KEY WORDS: Elastic layers; Green's functions; Transfer matrices.

RIASSUNTO. — *Sull'analisi di strati elastici con serie di Fourier e funzioni di Green.* Il problema dell'analisi elastica in deformazioni piane di uno strato omogeneo ed isotropo di spessore costante è formulato adottando sviluppi in serie di Fourier lungo la direzione dello strato e opportune funzioni di Green nella direzione trasversale. Le equazioni, disaccoppiate per le diverse frequenze, che governano la distribuzione dei coefficienti di Fourier relativi al problema simmetrico o antisimmetrico lungo l'altezza dello strato, sono risolte senza approssimazioni. Il metodo proposto viene utilizzato per discutere criticamente la nozione di «trasferimento» di quantità statiche e cinematiche attraverso lo strato.

1. INTRODUCTION

The computationally effective analysis of an elastic layer, despite the geometrical simplicity, has attracted considerable attention in applied mechanics literature, even in recent times. An assemblage of such layers, assumed as individually homogeneous, represents an idealization often used for layered systems, such as stratified soils, in various structural or geotechnical engineering situations.

In the case of plane and parallel layers, methods resting on integral transforms [6] or Fourier series [2] prove to be much more efficient than conventional "domain" methods (e.g., by finite elements or finite differences); discretized

(*) This work is part of a research project supported by M.P.I.
(**) Department of Structural Engineering, Technical University (Politecnico) of Milan, Piazza Leonardo Da Vinci, 32 - 20133 Milan, Italy.
(***) Nella seduta del 13 dicembre 1986.

boundary integral equation methods, or "boundary elements", appear to be well suited to generate "ad hoc" solution procedures for several classes of layered systems [5].

This paper concerns a single homogeneous and isotropic layer bounded by parallel surfaces, subjected to arbitrary boundary conditions in plane strain. A solution technique is developed which makes use of series expansions along the horizontal direction and of Green's functions across the layer thickness. Once the "ad hoc" "fundamental solutions" or Green's functions are generated, the procedure implies the same sequence of steps typical of the "direct" version of the boundary element method as applied to two or three-dimensional problems [1].

The proposed method reduces the analysis of a layer to a set of small-size, one-dimensional problems, defined across the thickness in terms of Fourier coefficients and relevant (in pairs) to the various frequencies. They are solved exactly, first in terms of boundary unknowns and subsequently, if needed, for unknown values at internal points. Besides, the layer "transfer matrices" pertaining to this approach are generated in explicit form; an investigation on their conditioning provides insight into the transfer notion itself and supplements the study carried out in [4] on the transfer matrix obtainable by boundary elements for a layer. The extension of the present approach to layered systems is currently under way and will be presented elsewhere together with numerical applications.

2. Problem formulation

Let us refer the cross-section of a homogeneous isotropic layer in plane strain to a Cartesian reference frame with axes x_1 and x_2 parallel and normal to the layer, respectively.

We will conceive the layer as laterally unbounded and acted on by spatially periodic external actions. Boundary conditions on the lower and upper surface are of a single type (either essential or natural) in each of the two directions.

By taking the spatial period 2 L along axis x_1 large enough, an isolated loading can be represented with any degree of accuracy. Since the response of the system will also exhibit periodicity (with the same period 2 L), any vector function of the coordinates, say $\boldsymbol{v}(\boldsymbol{x}) = \{v_1(x_1, x_2),\ v_2(x_1, x_2)\}^t$ where t denotes transposition, can be expressed by a Fourier series:

$$\boldsymbol{v}(\boldsymbol{x}) = \sum_{0}^{\infty}{}_n\, \boldsymbol{v}_n^c(x_2) \cos \beta_n x_1 + \sum_{1}^{\infty}{}_n\, \boldsymbol{v}_n^s(x_2) \sin \beta_n x_1 \tag{1}$$

where $\boldsymbol{v}_n^c \equiv \{v_{1n}^c, v_{2n}^c\}^t$, $\boldsymbol{v}_n^s \equiv \{v_{1n}^s, v_{2n}^s\}^t$ and $\beta_n \equiv n\pi/\mathrm{L}$ is the frequency of the n-th cosine and sine harmonic terms. Of course any loading can be split into a symmetric and a skew-symmetric part. We choose to refer physical symmetry to the x_2 axis. Vector quantities $\boldsymbol{v}$ (such as displacements, tractions on horizontal surfaces or body forces) pertaining to the symmetric problem are expressed in terms of Fourier coefficients solely of type $v_{1n}^{\,s}$, $v_{2n}^{\,c}$; those of the skew-symmetric one in terms of $v_{1n}^{\,c}$, $v_{2n}^{\,s}$ only.

For brevity, only the symmetric problem will be worked out here in detail; the skew-symmetric case can be dealt with in an analogous way. By virtue of the series expansions the analysis of a symmetrically loaded layer, extending from $x_2 = a$ to $x_2 = b$ (with $h = b - a > 0$), reduces to a sequence of uncoupled one-dimensional problems in the different frequencies β_n $(n = 0, 1, 2\ldots)$. Each problem is formulated in terms of the unknown displacement coefficient $u_{1n}^{\,s}(x_2)$ and $u_{2n}^{\,c}(x_2)$, as follows:

$$\mathrm{A}_n\,(u_{1n}^{\,s}) + \mathrm{B}_n\,(u_{2n}^{\,c}) + \frac{1}{\mathrm{G}}\,b_{1n}^{\,s} = 0 \tag{2 a}$$

$$\overline{\mathrm{A}}_n\,(u_{2n}^{\,c}) + \overline{\mathrm{B}}_n\,(u_{1n}^{\,s}) + \frac{1}{\mathrm{G}}\,b_{2n}^{\,c} = 0 \tag{2 b}$$

having set:

$$\mathrm{A}_n(\cdot) \equiv \frac{\mathrm{d}^2}{\mathrm{d}x_2^2}(\cdot) - g\beta_n^2(\cdot)\,,\ \overline{\mathrm{A}}_n(\cdot) \equiv g\,\frac{\mathrm{d}^2}{\mathrm{d}x_2^2}(\cdot) - \beta_n^2(\cdot) \tag{3 a}$$

$$\mathrm{B}_n(\cdot) \equiv -g'\,\beta_n\,\frac{\mathrm{d}}{\mathrm{d}x_2}(\cdot)\,,\quad \overline{\mathrm{B}}_n(\cdot) \equiv -\mathrm{B}_n(\cdot) \tag{3 b}$$

$$g \equiv 2\,(1-\nu)/(1-2\,\nu)\,,\qquad g' \equiv 1/(1-2\,\nu)\,. \tag{3 c}$$

The relevant boundary conditions are represented by an appropriate combination of traction and/or displacement coefficient values at the lower and the upper surfaces.

The expressions for the Fourier coefficients representing the amplitudes of the traction distributions on horizontal surfaces, are obtained through the differentiation of the displacement field and the enforcement of Hooke's law; they read:

$$p_{1n}^{\,s} = \mathrm{G}\left(\frac{\mathrm{d}}{\mathrm{d}x_2}\,u_{1n}^{\,s} - \beta_n\,u_{2n}^{\,c}\right) \tag{4 a}$$

$$p_{2n}^{\,c} = \mathrm{G}\left(g\,\frac{\mathrm{d}}{\mathrm{d}x_2}\,u_{2n}^{\,c} + 2\,g'\,\nu\,\beta_n\,u_{1n}^{\,s}\right) \tag{4 b}$$

Eqs (2 *a*, *b*) can be obtained formally by the following two-step procedure:

(i) write Navier's equations using half-range series representation for the displacements;

(ii) multiply these equations by $\sin \beta_n x$ $(n = 1 \ldots)$ and by $\cos \beta_n x$ $(n = 0, 1 \ldots)$ and integrate them over the interval $-L \leqq x \leqq +L$ making use of the orthogonality property of trigonometric functions.

Note that the application of the same step (ii) to the original boundary conditions leads to their expressions in terms of Fourier coefficients. In the subsequent sections, quantities such as (u_{1n}^{s}, u_{2n}^{c}) will be often referred to as " displacements " for conciseness; and analogously for the other Fourier coefficients.

3. Fundamental solutions

Let us formulate for $j = 1, 2$, the following two problems associated to the frequency β_n (with $n \geqq 1$):

$$\mathrm{A}_n (u_{1n}^{s}) + \mathrm{B}_n (u_{2n}^{c}) + \frac{1}{\mathrm{G}} \delta_{1j} \Delta (x_2 - \bar{x}_2) = 0 \tag{5 a}$$

$$\bar{\mathrm{A}}_n (u_{1n}^{s}) + \bar{\mathrm{B}}_n (u_{2n}^{c}) + \frac{1}{\mathrm{G}} \delta_{2j} \Delta (x_2 - \bar{x}_2) = 0 \tag{5 b}$$

$$\lim_{|x_2| \to \infty} u_{1n}^{s} = 0 \qquad \lim_{|x_2| \to \infty} u_{2n}^{c} = 0 \tag{5 c}$$

where δ_{ij} is the Kronecker delta and $\Delta (\eta - \bar{\eta})$ is the one-dimensional delta function with the property:

$$\int_a^b f(\eta)\, \Delta (\eta - \bar{\eta})\, \mathrm{d}\eta = \begin{cases} f(\bar{\eta}), & \text{for } a < \bar{\eta} < b \\ 0, & \text{otherwise} \end{cases} \tag{6}$$

The pair of functions which are solution to the j-th of the two problems (5) will be denoted henceforth by $(\overset{*}{u}{}_{1n}^{js}, \overset{*}{u}{}_{2n}^{jc})$; they play the role of " fundamental solution " in the procedure developed in the next section. Their physical interpretation is straightforward: for $j = 1, 2$, $\overset{*}{u}{}_{1n}^{js} (\bar{x}_2, x_2) \sin \beta_n x_1$ and $\overset{*}{u}{}_{2n}^{jc} (\bar{x}_2, x_2) \cos \beta_n x_1$ represent the displacement field induced in an infinite stratum by a loading which consists of forces, directed as axis x_j, which are continuously distributed along the line $x_2 = \bar{x}_2$ according to the n-th harmonic of unit amplitude: sine harmonic if $j = 1$, cosine if $j = 2$.

The generation of the j-th fundamental solution $(j = 1, 2)$ has been carried out as follows. Its representation on each of the two disjointed intervals $-\infty < x_2 < \bar{x}_2$ and $\bar{x}_2 < x_2 < +\infty$ is given by the general solution of the homogeneous version of the system (5 a, b). The application of the boundary conditions (5 c) and of the appropriate matching conditions at $x_2 = \bar{x}_2$ to such two separate representations, leads to the sought expression for $(\overset{*}{u}{}_{1n}^{js}, \overset{*}{u}{}_{2n}^{jc})$ on the whole x_2 axis. Namely, by defining $\mathrm{d} \equiv x_2 - \bar{x}_2$, the two fundamental solutions and the associated tractions turn out to be, for $n \geqq 1$:

$$\left\{\begin{array}{c} \overset{*}{u}{}^{1s}_{1n} \\ \hdashline \overset{*}{u}{}^{1c}_{2n} \end{array}\right\} = \frac{1}{8\,\mathrm{G}\,(1-\nu)}\, e^{-\beta_n |\mathrm{d}|} \left\{\begin{array}{c} (3-4\,\nu)/\beta_n - |\,\mathrm{d}\,| \\ \hdashline -\,\mathrm{d} \end{array}\right\} \tag{7 a}$$

$$\left\{\begin{array}{c} \overset{*}{p}{}^{1s}_{1n} \\ \hdashline \overset{*}{p}{}^{1c}_{2n} \end{array}\right\} = \frac{-1}{4\,(1-\nu)}\, e^{-\beta_n |\mathrm{d}|} \left\{\begin{array}{c} 2\,(1-\nu)\,\mathrm{d}/|\,\mathrm{d}\,| - \beta_n \mathrm{d} \\ \hdashline (1-2\,\nu) - \beta_n\,|\,\mathrm{d}\,| \end{array}\right\} \tag{7 b}$$

$$\left\{\begin{array}{c} \overset{*}{u}{}^{2s}_{1n} \\ \hdashline \overset{*}{u}{}^{2c}_{2n} \end{array}\right\} = \frac{1}{8\,\mathrm{G}\,(1-\nu)}\, e^{-\beta_n |\mathrm{d}|} \left\{\begin{array}{c} +\,\mathrm{d} \\ \hdashline (3-4\,\nu)/\beta_n + |\,\mathrm{d}\,| \end{array}\right\} \tag{8 a}$$

$$\left\{\begin{array}{c} \overset{*}{p}{}^{2s}_{1n} \\ \hdashline \overset{*}{p}{}^{2c}_{2n} \end{array}\right\} = \frac{-1}{4\,(1-\nu)}\, e^{-\beta_n |\mathrm{d}|} \left\{\begin{array}{c} (1-2\,\nu) + \beta_n\,|\,\mathrm{d}\,| \\ \hdashline 2\,(1-\nu)\,\mathrm{d}/|\,\mathrm{d}\,| + \beta_n\,\mathrm{d} \end{array}\right\} \tag{8 b}$$

4. SOLUTION BY GREEN'S FUNCTION METHOD

Eqs. $(2\,a\,,b)$ are recast into the following weighted residual statement:

$$\int_a^b \left\{ w \left[\mathrm{A}_n(u^s_{1n}) + \mathrm{B}_n(u^c_{2n}) + \frac{1}{\mathrm{G}}\, b^s_{1n} \right] + z \left[\bar{\mathrm{A}}_n(u^c_{2n}) + \right.\right.$$
$$\left.\left. + \bar{\mathrm{B}}_n(u^s_{1n}) + \frac{1}{\mathrm{G}}\, b^c_{2n} \right] \right\} \mathrm{d}x_2 = 0 \tag{9}$$

where $w\,(x_2)$ and $z\,(x_2)$ are two arbitrary weighting functions. Let us now process eq. (9) integrating by parts twice; since the operators $\mathrm{A}_n(\cdot)$ and $\bar{\mathrm{A}}_n(\cdot)$ are self-adjoint and $\bar{\mathrm{B}}_n(\cdot)$ is the adjoint of $\mathrm{B}_n(\cdot)$, the above statement is transformed into the following "inverse" one:

$$\int_a^b \{u^s_{1n}\,[\mathrm{A}_n(w) + \mathrm{B}_n(z)] + u^c_{2n}\,[\bar{\mathrm{A}}_n(z) + \bar{\mathrm{B}}_n(w)]\}\,\mathrm{d}x_2 +$$
$$+ \frac{1}{\mathrm{G}} \int_a^b (w\,b^s_{1n} + z\,b^c_{2n})\,\mathrm{d}x_2 + \mathrm{b.t.} = 0 \tag{10 a}$$

with boundary terms (b.t.) given by:

$$\text{b.t.} = \left[w \frac{1}{G} p_{1n}^{s} - \left(\frac{dw}{dx_2} - \beta_n z \right) u_{1n}^{s} + z \frac{1}{G} p_{2n}^{c} + \right.$$
$$\left. - \left(g \frac{dz}{dx_2} + 2 g' \nu \beta_n w \right) u_{2n}^{c} \right]_{x_2 = a}^{x_2 = b} \tag{10 b}$$

This relation holds for any function pair $[w(x_2), z(x_2)]$. If we interpret these weighting functions as Fourier coefficients of a displacement field

$$\tilde{u}_{1n}(\boldsymbol{x}) = w(x_2) \sin \beta_n x_1 \quad , \quad \tilde{u}_{2n}(\boldsymbol{x}) = z(x_2) \cos \beta_n x_1$$

relevant to another elastic solution (denoted by a tilde) with the same frequency β_n, eq. (10) becomes the counterpart of Betti's theorem. In fact, taking into account eqs. $(2\,a\,,b)$ and $(4\,a\,,b)$, written for the corresponding quantites marked by the tilde, eq. (10) can be expressed, in a concise matrix form, as:

$$\int_a^b \boldsymbol{u}_n^t \tilde{\boldsymbol{b}}_n \, dx_2 + \left[\boldsymbol{u}_n^t \tilde{\boldsymbol{p}}_n \right]_{x_2 = a}^{x_2 = b} = \int_a^b \tilde{\boldsymbol{u}}_n^t \boldsymbol{b}_n \, dx_2 + \left[\tilde{\boldsymbol{u}}_n^t \boldsymbol{p}_n \right]_{x_2 = a}^{x_2 = b} \tag{11}$$

where: $\boldsymbol{u}_n \equiv \{u_{1n}^{s}, u_{2n}^{c}\}^t$, $\tilde{\boldsymbol{u}}_n \equiv \{\tilde{u}_{1n}^{s}, \tilde{u}_{2n}^{c}\}^t$ and similarly for the body force and traction vectors.

From the reciprocity relation (11) one can derive two Somigliana-type scalar equations for the unknown displacement Fourier coefficients by choosing as fictitious elastic state either of the two associated to the "fundamental solutions" defined in the previous section. In fact, let us replace the quantities marked by the tilde with the corresponding starred ones relevant to the j-th fundamental solution ($j = 1\,,2$). Using the sifting property of the delta function, eq. (6), we obtain, in vector notation:

$$\boldsymbol{u}_n(\bar{x}_2) = \left[\overset{*}{\boldsymbol{u}}_n(\bar{x}_2, x_2) \, \boldsymbol{p}_n(x_2) - \overset{*}{\boldsymbol{p}}_n(\bar{x}_2, x_2) \, \boldsymbol{u}_n(x_2) \right]_{x_2 = a}^{x_2 = b} +$$
$$+ \int_a^b \overset{*}{\boldsymbol{u}}_n(\bar{x}_2, x_2) \, \boldsymbol{b}_n(x_2) \, dx_2 \tag{12}$$

having set:

$$\overset{*}{\boldsymbol{u}}_n(\bar{x}_2, x_2) \equiv \begin{bmatrix} \overset{*}{u}{}_{1n}^{1s} & \overset{*}{u}{}_{2n}^{1c} \\ \overset{*}{u}{}_{1n}^{2s} & \overset{*}{u}{}_{2n}^{2c} \end{bmatrix} \quad ; \quad \overset{*}{\boldsymbol{p}}_n(\bar{x}_2, x_2) \equiv \begin{bmatrix} \overset{*}{p}{}_{1n}^{1s} & \overset{*}{p}{}_{2n}^{1c} \\ \overset{*}{p}{}_{1n}^{2s} & \overset{*}{p}{}_{2n}^{2c} \end{bmatrix} \tag{13}$$

Eq. (12) gives the displacements at $\bar{x}_2$ (with $a < \bar{x}_2 < b$) as a function of all the boundary quantities (displacements and tractions on both the bounding horizontal surfaces) and of the body force distribution within the layer. In the sequel vanishing body forces will be assumed for simplicity; however their presence could be easily accounted for in what follows by adding extra-terms, similar to the integral in eq. (12), which would depend on data only.

Let us now rewrite eq. (12) by taking the load point $\bar{x}_2$ to either end of the interval (a, b). This limiting process is straightforward in the present one-dimensional context (unlike the corresponding one in the boundary element method for two and three-dimensional elasticity). The two vector equations thus obtained involve only boundary quantities and can be expressed simultaneously as:

$$(14)\qquad \left[\begin{array}{c:c} \mathbf{I} - \overset{*}{\boldsymbol{p}}_n(a_+, a) & \overset{*}{\boldsymbol{p}}_n(a, b) \\ \hdashline -\overset{*}{\boldsymbol{p}}_n(b, a) & \mathbf{I} + \overset{*}{\boldsymbol{p}}_n(b_-, b) \end{array}\right] \left\{\begin{array}{c} \boldsymbol{u}_n(a) \\ \hdashline \boldsymbol{u}_n(b) \end{array}\right\} = \left[\begin{array}{c:c} -\overset{*}{\boldsymbol{u}}_n(a, a) & \overset{*}{\boldsymbol{u}}_n(a, b) \\ \hdashline -\overset{*}{\boldsymbol{u}}_n(b, a) & \overset{*}{\boldsymbol{u}}_n(b, b) \end{array}\right] \left\{\begin{array}{c} \boldsymbol{p}_n(a) \\ \hdashline \boldsymbol{p}_n(b) \end{array}\right\}$$

having set, for $\varepsilon > 0$:

$$(15)\qquad \overset{*}{\boldsymbol{p}}_n(a_+, a) \equiv \lim_{\varepsilon \to 0} \overset{*}{\boldsymbol{p}}_n(a + \varepsilon, a) \quad , \quad \overset{*}{\boldsymbol{p}}_n(b_-, b) \equiv \lim_{\varepsilon \to 0} \overset{*}{\boldsymbol{p}}_n(b - \varepsilon, b) .$$

As for the problem associated to the constant terms in the expansions adopted, the analogous relations is easily obtained by simple physical arguments and reads:

$$(16)\qquad \begin{bmatrix} -1 & +1 \\ -1 & +1 \end{bmatrix} \begin{Bmatrix} u_{20}(a) \\ u_{20}(b) \end{Bmatrix} = \begin{bmatrix} +\dfrac{h}{\mathrm{G}g} & 0 \\ 0 & +\dfrac{h}{\mathrm{G}g} \end{bmatrix} \begin{Bmatrix} p_{20}(a) \\ p_{20}(b) \end{Bmatrix}$$

However, the constant coefficients could be also treated in the same fashion as the previous ones: the introduction of a suitable fundamental solution ($\overset{*}{u}_{20} = -\,|\,\mathrm{d}\,|/2\,\mathrm{G}g$) would lead to a Somigliana-type scalar equation from which eqs. (16) could be derived assuming in turn $x_2 = a_+$, $x_2 = b_-$ (with $b_{20} = 0$).

For each frequency β_n, $n = 1 \ldots,$ eq. (14) provides four scalar constraints among the eight Fourier coefficients defined on the two surfaces of the layer; for $n = 0$ eq. (16) represents two constraints for the four corresponding coefficients. The boundary conditions of any well-posed (symmetric) problem in the frequency β_n are such that for $n \geq 1$ four (for $n = 0$ two) of its Fourier coefficients are assigned. Therefore, the above equations govern the elastic response of the layer at its boundary.

Once the boundary unknowns have been determined, displacement coefficients within the layer can be worked out using the Somigliana identity, eq. (12), for each frequency separately. The associated stress fied can be obtained through Hooke's law and the strain-displacement relationship:

$$\left\{\begin{array}{c}\sigma_{11n}(\boldsymbol{x})\\ \hdashline \sigma_{22n}(\boldsymbol{x})\end{array}\right\} = \left\{\begin{array}{c}\sigma_{11n}^{c}(x_2)\\ \hdashline \sigma_{22n}^{c}(x_2)\end{array}\right\} \cos \beta_n x_1 , \tag{17 a}$$

$$\sigma_{12n}(\boldsymbol{x}) = \sigma_{12n}^{s}(x_2) \sin \beta_n x_1 \tag{17 b}$$

where:

$$\sigma_{11n}^{c} = 2\,\mathrm{G}g' \left[(1-\nu)\, \beta_n u_{1n}^{s} + \nu \frac{\mathrm{d}}{\mathrm{d}x_2} u_{2n}^{c} \right] \tag{18 a}$$

$$\sigma_{22n}^{c} = 2\,\mathrm{G}g' \left[\nu\, \beta_n u_{1n}^{s} + (1-\nu) \frac{\mathrm{d}}{\mathrm{d}x_2} u_{2n}^{c} \right] \tag{18 b}$$

$$\sigma_{12n}^{s} = \mathrm{G} \left[\frac{\mathrm{d}}{\mathrm{d}x_2} u_{1n}^{s} - u_{2n}^{c} \right] \tag{18 c}$$

Each of the above stress Fourier coefficients can be given an expression similar to Somigliana equation in terms of boundary quantities only; this is done by differentiating eq. (12) with respect to $\bar{x}_2$ to obtain $\mathrm{d}/\mathrm{d}x_2\,(u_{1n}^{s})$ and $\mathrm{d}/\mathrm{d}x_2\,(u_{2n}^{c})$ and substituting in eqs. (18) their expressions together with those for u_{1n}^{s} and u_{2n}^{c}.

5. Transfer matrix

With reference to the symmetric problem in the n-th frequency β_n and to either of the layer surfaces, let us introduce the non-dimensional "state vector" $\boldsymbol{v}_n \equiv \{(1/h)\,\boldsymbol{u}_n^t,\ (1/\mathrm{G})\,\boldsymbol{p}_n^t\}$ and define as layer «transfer matrix» $\mathbf{T}_n$ the matrix which transforms $\boldsymbol{v}_n(a)$ into $\boldsymbol{v}_n(b)$. This gives rise to a transfer relation "from bottom to top". In order to generate such a matrix for $n \geq 1$, eq. (14) is rearranged into the form:

$$\mathbf{A}_n \boldsymbol{v}_n(a) = \mathbf{B}_n \boldsymbol{v}_n(b) \tag{19}$$

having set:

$$\mathbf{A}_n \equiv \left[\begin{array}{c:c} h\,[\mathbf{I} - \overset{*}{\boldsymbol{p}}_n(a_+ , a)] & \mathrm{G}\,\overset{*}{\boldsymbol{u}}_n(a , a) \\ \hdashline h\,\overset{*}{\boldsymbol{p}}_n(b , a) & -\mathrm{G}\,\overset{*}{\boldsymbol{u}}_n(b , a) \end{array}\right] , \tag{20 a}$$

$$\mathbf{B}_n \equiv \left[\begin{array}{c:c} -h\,\overset{*}{\boldsymbol{p}}_n(a , b) & \mathrm{G}\,\overset{*}{\boldsymbol{u}}_n(a , b) \\ \hdashline h\,[\mathbf{I} + \overset{*}{\boldsymbol{p}}_n(b_- , b)] & -\mathrm{G}\,\overset{*}{\boldsymbol{u}}_n(b , b) \end{array}\right] \tag{20 b}$$

Then, by inverting matrix $\mathbf{B}_n$, one obtains:

$$\boldsymbol{v}_n(b) = [(\mathbf{B}_n)^{-1}\mathbf{A}_n]\,\boldsymbol{v}_n(a) \equiv \mathbf{T}_n\,\boldsymbol{v}_n(a)\,. \tag{21}$$

The transfer relationship can be re-written in a partitioned form corresponding to the subdivision of the state variables into nondimensional displacements and tractions:

$$\left\{\begin{array}{c} \boldsymbol{u}_n(b)/h \\ \hline \boldsymbol{p}_n(b)/\mathrm{G} \end{array}\right\} = \left[\begin{array}{c|c} \mathbf{T}_n^{uu} & \mathbf{T}_n^{up} \\ \hline \mathbf{T}_n^{pu} & \mathbf{T}_n^{pp} \end{array}\right] \left\{\begin{array}{c} \boldsymbol{u}_n(a)/h \\ \hline \boldsymbol{p}_n(a)/\mathrm{G} \end{array}\right\} \tag{22}$$

The expressions of the four submatrices of $\mathbf{T}_n$ have been worked out explicitly through some lengthy algebraic manipulations of the submatrices of $\mathbf{A}_n$ and $\mathbf{B}_n$. These submatrices turn out to depend on the parameter $(\beta_n h)$ and on the Poisson's ratio ν according to the following formulas:

$$\mathbf{T}_n^{uu} = \frac{1}{2(1-\nu)} \begin{bmatrix} [2(1-\nu)\,\mathrm{C}_n + \beta_n h\,\mathrm{S}_n] & [(1-2\nu)\,\mathrm{S}_n + \beta_n h\,\mathrm{C}_n] \\ [(1-2\nu)\,\mathrm{S}_n - \beta_n h\,\mathrm{C}_n] & [2(1-\nu)\,\mathrm{C}_n - \beta_n h\,\mathrm{S}_n] \end{bmatrix} \tag{23 a}$$

$$\mathbf{T}_n^{up} = \frac{1}{4(1-\nu)}\,\frac{1}{\beta_n h} \begin{bmatrix} [(3-4\nu)\,\mathrm{S}_n + \beta_n h\,\mathrm{C}_n] & [\beta_n h\,\mathrm{S}_n] \\ [-\beta_n h\,\mathrm{S}_n] & [(3-4\nu)\,\mathrm{S}_n - \beta_n h\,\mathrm{C}] \end{bmatrix} \tag{23 b}$$

$$\mathbf{T}_n^{pu} = \frac{1}{(1-\nu)}\,\beta_n h \begin{bmatrix} [\mathrm{S}_n + \beta_n h\,\mathrm{C}_n] & [\beta_n h\,\mathrm{S}_n] \\ [-\beta_n h\,\mathrm{S}_n] & [\mathrm{S}_n - \beta_n h\,\mathrm{C}_n] \end{bmatrix} \tag{23 c}$$

$$\mathbf{T}_n^{pp} = \frac{1}{2(1-\nu)} \begin{bmatrix} [2(1-\nu)\,\mathrm{C}_n + \beta_n h\,\mathrm{S}_n] & [-(1-2\nu)\,\mathrm{S}_n + \beta_n h\,\mathrm{C}_n] \\ [-(1-2\nu)\,\mathrm{S}_n - \beta_n h\,\mathrm{C}_n] & [2(1-\nu)\,\mathrm{C}_n - \beta_n h\,\mathrm{S}_n] \end{bmatrix} \tag{23 d}$$

having set: $\mathrm{S}_n \equiv \mathrm{S}h\,(\beta_n h)$, $\mathrm{C}_n \equiv \mathrm{C}h\,(\beta_n h)$.

The corresponding matrix $\mathbf{T}_0$ is easily obtained by rearranging eq. (16); its four elements, listed by rows, read: 1, $1/g$, 0, 1.

At this stage it is instructive to compare the above transfer matrices to those dealt with in [4]. The latter were generated through the application of the standard boundary element method to a finite elastic layer, endowed with fictitious side boundaries and with equally discretized lower and upper boundaries; of course in that boundary element context the " state variables " at the horizontal surfaces were represented by the displacement and traction components at the nodes. Some concepts related to matrix conditioning, extensively discussed in [3], will be used in what follows.

To start with, one notices that some properties established for the transfer matrices in [4], hold true for the matrices $\mathbf{T}_n$ of the present context as well (as can be proved following the same path of reasoning); for $n \geqq 1$, they read:

$$\det \mathbf{T}_n = 1 ; \qquad \mu_{n,5-i} = (\mu_{n,i})^{-1} , \qquad i = 1 \ldots 4 \tag{24 a, b}$$

where $\mu_{n,1} \geqq \ldots \geqq \mu_{n,4} > 0$ are the singular values of matrix $\mathbf{T}_n$; for $n = 0$, $\det \mathrm{T}_0 = \mu_{0,1} = \mu_{0,2} = 1$.

Besides, in [4] it was pointed out that the transfer concept in elasticity is closely related to the solution of an ill-posed boundary value problem; hence one expects that this aspect be reflected also in the present algebraized version of the transfer problem represented, for each frequency $\beta_n \, (n \geqq 1)$, by eq. (22).

That this is indeed the case, emerges from the scrutiny of the condition number, cond $\mathbf{T}_n$, of the transfer matrix $\mathbf{T}_n$, namely from the following two circumstances:

(i) numerical tests, illustrated further on, show that cond $\mathbf{T}_n$ becomes higher and higher at increasing values of the parameter $(\beta_n h)$;

(ii) with reference to $\boldsymbol{v}_n(a) = \mathbf{T}_n^{-1} \boldsymbol{v}_n(b)$, conceived as an equation in $\boldsymbol{v}_n(b)$, cond $\mathbf{T}_n$ bounds the ratio of the relative uncertainty of the solution state vector $\boldsymbol{v}_n(b)$ to that of the given state vector $\boldsymbol{v}_n(a)$:

$$\frac{\| \delta \boldsymbol{v}_n(b) \|}{\| \delta \boldsymbol{v}_n(a) \|} \, \frac{(\| \boldsymbol{v}_n(b) \|)^{-1}}{(\| \boldsymbol{v}_n(a) \|)^{-1}} \leq \mathrm{cond}\, \mathbf{T}_n = \| \mathbf{T}_n^{-1} \| \; \| \mathbf{T}_n \| \tag{25}$$

Here $\| \cdot \|$ denotes the euclidean vector or matrix norm.

Thus, for a fixed layer thickness h, the l.h.s. of inequality (25) suffers an unbounded growth at increasing frequency values. This behaviour is the expected counterpart of that established in [4]. In fact, in that context the condition number of the relevant transfer matrix was shown to get higher and higher at increasing values of the parameter (h/d), h being the layer thickness and d a typical boundary element length.

It is worth noting that the transfer matrix procedure cannot be relied on even for solving well-posed problems. In fact, for a layer with given tractions on its top surface and assigned zero displacements at its bottom, an accurate determination of the unknown bottom reactions through the second of eqs. (22), $\boldsymbol{p}_n(a) = (\mathbf{T}_n^{pp})^{-1} \bar{\boldsymbol{p}}_n(b)$, is jeopardized by the fact that the submatrix $\mathbf{T}_n^{pp}$ turns out to be ill-conditioned as well.

The numerical tests which substantiate the preceding remarks were performed using double precision and assuming: $\mathrm{L} = 20$, $h = 4$, $\nu = 0.3$.

The condition numbers cond $\mathbf{T}_n$ and cond $\mathbf{T}_n^{pp}$ were evaluated for increasing values of n and, hence, of the parameter $(\beta_n h)$; they are shown in Table I. Cond $\mathbf{T}_n$ increases faster than cond $\mathbf{T}_n^{pp}$; for both the trend is an unbounded

TABLE I.

n	cond $\mathbf{T}_n$	cond $\mathbf{T}_n^{pp}$
1	8.57	1.74
5	1.17×10^5	2.32×10
9	1.75×10^8	7.33×10
13	1.15×10^{11}	1.51×10^2
17	5.08×10^{13}	2.56×10^2
21	1.79×10^{16}	3.89×10^2
25	2.16×10^{18}	5.50×10^2

growth. It is worth noticing that in the frequency range considered, the generation of the transfer matrix coefficients is practically unaffected by truncation errors. This is demonstrated by the fact that eqs. (24 *a*, *b*) turn out to be substantially fulfilled within this range while they cease to be so for higher frequencies.

6. CONCLUSIONS

Only the symmetric problem has been discussed here in some detail. The skew-symmetric one, formulated in terms of $u_{1n}^c(x_2)$ and $u_{2n}^s(x_2)$, can be treated in an analogous way making use of appropriate "fundamental solutions". As for the corresponding transfer matrices, they are bound to suffer the same ill-conditioning pointed out for those relevant to the symmetric problem.

Thus for a generic loading, the proposed procedure reduces the determination of the elastic response at the layer boundary to the solution of two decoupled one-dimensional problems (the symmetric and skew-symmetric ones) for each one of the frequencies β_n; for $n \geqq 1$ ($n = 0$) each of these problems consists of a fourth (second) order linear system in the unknown Fourier coefficients at both the lower and upper surface of the layer. Subsequently, displacement and stress Fourier coefficients at internal points are easily obtained through an algebraic equation of Somigliana-type, with quadratures needed only in the presence of body forces; all the unknown fields can then be reconstructed by combining the considered harmonics. The proposed method is exact up to the representation of the data through Fourier series. Its application to the analysis of multi-layer elastic systems will be dealt with and numerically tested elsewhere.

Acknowledgement. Support from MPI is gratefully acknowledged.

Reference

[1] C.A. Brebbia, J.C.F. Telles and L.C. Wrobel (1984) - *Boundary Element Techniques*, Springer-Verlag, Berlin.

[2] Y.K. Cheung and S.C. Fan (1979) - *Analysis of Pavements and Layered Foundations by the Finite Layer Method*, « Proc. Int. Conf. on Num. Meth. in Geomech., Aachen », Vol. 3, A.A. Balkema, Rotterdam, 1129-1135.

[3] G.E. Forsythe and C.B. Moler (1967) - *Computer Solution of Linear Algebraic System*, Prentice-Hall, Englewood Cliffs, N.J.

[4] G. Maier and G. Novati (1986) - *On Boundary Element - Transfer Matrix Analysis of Layered Elastic System*, « Engineering Analysis », *3*, 208-216.

[5] G. Maier and G. Novati (1987) - *Boundary Element Elastic Analysis of Layered Soils by a Successive Stiffness Method*, « Int. J. Num. Anal. Meth. Geomechanics », *11*, 435-447.

[6] J.C. Small and J.R. Booker (1984) - *Finite Layer Analysis of Layered Elastic Materials using a Flexibility Approach*, « Int. J. Num. Meth. Engng. », *20*, 1025-1037.