Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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Dynamical systems with Newtonian type potentials

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **81** (1987), n.3, p. 271–277.

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<http://www.bdim.eu/item?id=RLINA_1987_8_81_3_271_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1987.

Atti Acc. Lincei Rend. fis.
(8), LXXXI (1987), pp. 271–278

Analisi matematica. — *Dynamical systems with Newtonian type potentials* (*). Nota di MARCO DEGIOVANNI (**), FABIO GIANNONI (***) e ANTONIO MARINO (****), presentata (*****) dal Socio E. DE GIORGI.

ABSTRACT. — We study the existence of regular periodic solutions to some dynamical systems whose potential energy is negative, has only a singular point and goes to zero at infinity. We give sufficient conditions to the existence of periodic solutions of assigned period which do not meet the singularity.

KEY WORDS: Dynamical systems; Newtonian potential; Periodic solutions.

RIASSUNTO. — *Sistemi dinamici con potenziali di tipo newtoniano.* Si studia l'esistenza di soluzioni period che regolari per certi sistemi dinamici con energia potenziale negativa avente un solo punto singolare e infinitesima all'infinito. Vengono date condizioni sufficienti per l'esistenza di soluzioni periodiche di assegnato periodo che non passano mai per la singolarità.

INTRODUCTION

In this paper we consider the problem of finding periodic solutions of a conservative dynamical system

$$\text{(P)} \qquad \ddot{q} + \operatorname{grad} V(q) = 0 .$$

It is well known that the solutions can be found among the critical points of the functional

$$f(q) = (1/2)\int_0^T |\dot{q}|^2 \, dt - \int_0^T V(q)\, dt ,$$

$$q \in H^1(0, T; \mathbf{R}^n) \quad , \quad q(0) = q(T) ,$$

(*) Lavoro eseguito nell'ambito di un progetto nazionale di ricerca finanziato dal Ministero della Pubblica Istruzione (40%-1985).

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(*****) Nella seduta del 24 aprile 1987.

where V is the potential energy and T is the period.

Many important studies have been done, when V is regular (see, for instance, the survey paper [10] and references therein).

The case of potentials of Newtonian type (namely $V(x) = -1/|x|$) has led to the study of (P) under the assumption that V goes to ∞ in the points of a certain "singular" set.

A first case, studied for instance in [4, 6, 9], concerns potentials satisfying the so called "strong force" hypothesis, which is verified, for example, by $V(x) = -1/|x|^\alpha$ if $\alpha \geq 2$, but not when $\alpha < 2$. Indeed, under strong force assumption we have that $f(q) = +\infty$ whenever q is a curve which intersects the singular set. As a consequence, the sublevels $f^c = \{q : f(q) \leq c\}$ ($c \in \mathbf{R}$) are complete (for instance in H^1-metric) and do not contain any curve which intersects the singular set: by means of this fact one can overcome the difficulty involved in the presence of the singularities.

Other cases, concerning singular potentials, deal with situations in which $f(q) = -\infty$ whenever q intersects the singular set (see, for instance, [5]). A critical point q, which of course does not intersect the singular set, can be found by means of a careful evaluation of the homology groups of the sublevels.

In [3] a case is treated, in which V goes to $+\infty$ on the singular set, but $f(q)$ may be finite even if q intersects such a set. Nevertheless a solution is found which does not meet the singularities because of the assumptions on V and grad V. A similar case is treated in [1] under convexity assumptions, finding solutions of minimal period and extending the result to more general Hamiltonian systems.

A case in which V goes to $+\infty$ in certain points (with the assumptions of [3]) and goes to $-\infty$ in other points (with strong force hypothesis) is considered in [8].

A treatment of several problems of singular potentials (under strong force assumption) as well as other problems which exhibit a lack of compactness can be found in [2].

We wish to study a case in which V goes to $-\infty$ at a certain singular point, under assumptions which include the case $V(x) = -1/|x|^\alpha$ with $\alpha \geq 1$.

Therefore, it may happen that $f(q) \in \mathbf{R}$ even if q intersects the singular point. Nevertheless we want to find a periodic solution of (P) which does not meet such a point.

In order to explain the heart of the question, we have considered both the singular potential (Theorems (1.3) and (1.8)) both the symmetric singular potential (Theorem (2.1)) under assumptions which include potentials of Newtonian type, but are also rather simple: for instance, the singular set is reduced to a point and the symmetry is the antipodal one.

§ 1. SOME SINGULAR POTENTIALS

We wish to consider a class of potentials V like $-b/|x|^{\alpha}$ with $\alpha \geq 1$, $b > 0$.

Let us introduce the following notations:

$$\theta_0(\alpha) = \inf \{(1/2) \int_0^1 \dot{\gamma}^2 \, dt + \int_0^1 (1/\gamma^{\alpha}) \, dt : \gamma \in H_0^1(0, 1; \mathbf{R}), \gamma \geq 0\}, \tag{1.1}$$

$$\theta_1(\alpha) = \min \{2\pi^2 R^2 + (1/R^{\alpha}) : R > 0\},$$

$$\phi(\alpha) = (\theta_0(\alpha)/\theta_1(\alpha))^{(\alpha+2)/2}.$$

(1.2) *Remark.*

In this way we have defined a real extended map $\phi : [1, +\infty[\to [1, +\infty]$ such that $\phi(1) = 1$, $\phi(\alpha) > 1$ if $\alpha > 1$, $\phi(\alpha) = +\infty$ iff $\alpha \geq 2$.

(1.3) THEOREM.

Let $V \in C^1(\mathbf{R}^n \setminus \{0\}; \mathbf{R})$ *be such that*

i) $\lim_{|x| \to \infty} \operatorname{grad} V(x) = 0$, $\lim_{|x| \to \infty} V(x) = 0$;

ii) $\exists a, r > 0$, $\alpha \geq 1$ *such that*

$$a/|x|^{\alpha} \leq -V(x) \leq \phi(\alpha) \, a/|x|^{\alpha}, \; \forall x \in B(0, r) \setminus \{0\}.$$

Then for every $T > 0$ *there exists a* T*-periodic solution* q *of* (P) *such that* $q(t) \neq 0$ *for any* t.

To prove this result, we look for a critical point of the functional $f : H \to \mathbf{R} \cup \{+\infty\}$ defined by

$$f(q) = (1/2) \int_0^1 |\dot{q}|^2 \, dt - T^2 \int_0^1 V(q) \, dt$$

where $H = \{q \in H^1(0, 1; \mathbf{R}^n) : q(0) = q(1)\}$.

The proof is obtained by a combination of the following lemmas with the usual deformation techniques based on the study of the associated evolution equation.

(1.4) LEMMA.

Set $X_0 = \{q \in H : \exists t$ *such that* $q(t) = 0\}$. *Then, under assumptions of Theorem* (1.3), *there exist a subset* $\overline{X}$ *of* H *and* $\sigma > 0$ *such that*

i) $\overline{X} \cap X_0 = \emptyset$, $\overline{X} \cap f^\sigma \neq \emptyset$, $f^\sigma \cap X_0 = \emptyset$;

ii) $\sup \{f(q) : q \in \overline{X}\} \leq \inf \{f(q) : q \in X_0\}$;

iii) *there does not exist a continuous deformation* $H_s : \overline{X} \to H \setminus X_0$, $0 \leq s \leq 1$, *such that* $H_0(u) = u$, $H_s(\overline{X} \cap f^\sigma) \subset f^\sigma$, $H_1(\overline{X})$ *is contained in the set of constant trajectories in* $\mathbf{R}^n \setminus \{0\}$.

Roughly speaking, $\overline{X}$ is the set of the circular trajectories which are parallel to a given one and lie on a suitable ellipsoid centred in the origin.

We remark that equality in ii) holds if $V(x) = -b/|x|$. See also the computations which are made in [7] in the case $V(x) = -1/|x|$.

(1.5) LEMMA.

Under assumptions of Theorem (1.3), *for every* $\sigma > 0$ *there exists* ε *in* $]0, \sigma[$ *such that the set of constant trajectories in* $\mathbf{R}^n \setminus \{0\}$ *contains a deformation of the sublevel* f^ε *in* f^σ .

Theorem (1.3) can be generalized by substituting $1/|x|^\alpha$ with a convex function of $1/|x|$.

More precisely, let $\psi : \mathbf{R}^+ \to \mathbf{R}^+$ be a convex function such that $\psi(0) = 0$, $\lim_{s \to +\infty} \psi(s) = +\infty$. Let us prescribe the period T .

We introduce the following notations:

(1.6)
$$\theta_0(\psi, T) = \inf \{(1/2) \int_0^1 \dot{\gamma}^2 \, dt + T^2 \int_0^1 \psi(1/\gamma) \, dt : \gamma \in H_0^1(0, 1; \mathbf{R}) \quad , \quad \gamma \geq 0\};$$

$$\theta_1(\psi, T) = \min \{2\pi^2 R^2 + T^2 \psi(1/R) : R > 0\} .$$

(1.7) *Remark.*

The following facts hold:

i) $$\theta_0(\psi, T) \geq \theta_1(\psi, T) = \inf \{(1/2) \int_0^1 \dot{\gamma}^2 \, dt + T^2 \psi(\int_0^1 1/\gamma \, dt) : \gamma \in H_0^1(0, 1; \mathbf{R}) \quad , \quad \gamma \geq 0\};$$

ii) $\theta_0(\psi, .)$ and $\theta_1(\psi, .)$ are continuous;

iii) if ψ is not eventually affine as $s \to +\infty$, then $\theta_1(\psi, T) < \theta_0(\psi, T)$, $\forall T$;

iv) if $\liminf_{s\to+\infty} \psi(s)/s^2 > 0$, then $\theta_0(\psi, T) = +\infty \ \forall T$;

v) if ψ is positively homogenuous (that is $\psi(s) = c.s_\alpha$, $\alpha \geq 1$), then $\theta_0(\psi, .)$ and $\theta_1(\psi, .)$ are both positively homogenuous of some degree β.

(1.8) THEOREM.

Let $V \in C^1(\mathbf{R}^n \setminus \{0\}; \mathbf{R})$. *Suppose that*

i) $\lim_{|x|\to\infty} \operatorname{grad} V(x) = 0$, $\lim_{|x|\to\infty} V(x) = 0$;

ii) *there exists* $r > 0$, $0 < a \leq A < +\infty$, $\psi : \mathbf{R}^+ \to \mathbf{R}^+$ *convex, with* $\psi(0) = 0$, $\lim_{s\to+\infty} \psi(s) = +\infty$, $T_0 > 0$ *such that*

$$a^2\psi(1/|x|) \leq -V(x) \leq A^2\psi(1/|x|), \ \forall x \in B(0, r)\setminus\{0\},$$

$$\theta_1(\psi, TA) \leq \theta_0(\psi, Ta) \quad , \quad \forall T \in]0, T_0].$$

Then for every $T > 0$ *there exists a* T*-periodic solution* q *of* (P) *such that* $q(t) \neq 0$ *for any* t.

(1.9) *Remark.*

Under the assumptions of Theorem (1.8), if grad $V(x) \neq 0 \ \forall x \in \mathbf{R}^n \setminus B(0, r)$, it is possible to find the solution q in $B(0, r)\setminus\{0\}$.

In particular, if grad $V(x) \neq 0 \ \forall x \neq 0$, for every T there exist T-periodic solutions with arbitrarily small supremum norm.

We wish to point out some particular cases in which the assumptions of Theorem (1.8) are satisfied:

a) $V(x) = -b/|x|^\alpha$ with $b > 0$, $\alpha \geq 1$, and T_0 is an arbitrary positive number (we apply remarks (1.7)i and (1.7)v with $\psi(s) = s^\alpha$, $a^2 = A^2 = b$);

b) V is obtained by a "small perturbation" of $\psi(1/|x|)$, where ψ is not eventually affine as $s \to +\infty$ (we apply Remarks (1.7) ii and (1.7) iii);

c) V is obtained by a possibly "large perturbation" of $\psi(1/|x|)$, where $\liminf_{s\to+\infty} \psi(s)/s^2 > 0$ (we apply remark (1.7 iv).

In particular, case *b*) contains also some potentials V which are not radially symmetric and such that $\lim_{|x|\to 0} V(x)|x|$ is finite.

§ 2. Some symmetric singular potentials

Another result concerning Newtonian potentials can be obtained under an eveness assumption on the potential V. Under strong force hypothesis, results for even singular potentials have been obtained in [4].

(2.1) Theorem.

Let $V \in C^1(R^n \setminus \{0\}; R)$ *be such that*

i) $V(x) = V(-x)\ \forall x \in R^n \setminus \{0\}$;

ii) $\exists \alpha \geq 1$, $\exists a > 0$:

$$a/|x|^\alpha \leq -V(x) \leq 2^\alpha \phi(\alpha)\, a/|x|^\alpha \qquad \forall x \in R^n \setminus \{0\}.$$

Then for every $T > 0$ *there exists a* T*-periodic solution* q *of* (P) *which does not cross the origin and has minimal period* T (*function* ϕ *has been defined in* (1.1)).

We remark that we obtain a solution q which is "symmetric" with respect to the origin, that is such that $q(t + T/2) = -q(t)$.

The proof is based on the following observations:

a) since V is even the set H_S of the curves which are "symmetric" with respect to the origin constitutes a natural constraint for the functional f;

b) we have

$$\lim_{\substack{\|q\| \to \infty \\ q \in H_S}} f(q) = +\infty,$$

where $\|\cdot\|$ is the H^1 norm, and therefore f has a minimum point in H_S;

c) there exists $\bar{q}$ in $H_S \setminus X_0$ such that $f(\bar{q}) \leq \inf\{f(q) : q \in X_0 \cap H_S\}$ because of hypothesis ii).

§ 3. Some open problems

We wish to suggest some open problems which seem to be interesting and could be studied.

a) First of all, do the results hold true under the only assumption that V goes to $-\infty$ at the origin and to 0 at infinity?

b) Are there suitable conditions which do not imply the eveness of V and allow to find solutions of prescribed minimal period?

c) In this paper only problems with assigned period are considered. Of course one can look also for periodic solutions of assigned energy.

d) What other symmetries, besides the antipodal one, can be considered in Theorem (2.1)?

e) If the potential V has several singularities, can one still state the existence of a periodic solution for certain periods? Is there other information on the trajectories solving (P), in this case?

f) The non-autonomous case, in which V depends also on the time, is interesting: it can represent the motion of a satellite revolving round a planet which rotates on an axis and whose equator is not perfectly circular.

g) Still about the motion of a satellite round a planet, one can also consider trajectories which are periodic with respect to a system rotating with the planet. In particular, the following two problems can be considered typical:

– the problem of the geosynchronous satellite (whose period is equal to the period of the planet);

– the problem of the geostationary satellite (which is in rest in the rotating system).

h) Concerning the planet which is not perfectly spherical, a first approximation may consist in assuming that $-\mathrm{V}(x) = a\,(1/|x| + \varepsilon h(x)/|x|^5)$, where $a > 0$, $\varepsilon > 0$ is "small" and h is an indefinite quadratic form.

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