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Analisi matematica. — *Non-smooth variational bifurcation* (*).
Nota di MARCO DEGIOVANNI (**) e ANTONIO MARINO (***), presentata (****) dal Socio E. DE GIORGI.

ABSTRACT. — We consider bifurcation problems associated with some lower semicontinuous functionals that do not satisfy the usual regularity assumptions. For such functionals it is possible to define a generalized " Hessian form " and to show that certain eigenvalues of this one are bifurcation values.

The results are applied to a bifurcation problem for elliptic variational inequalities.

KEY WORDS: Variational bifurcation; non-smooth analysis; variational inequalities.

RIASSUNTO. — *Sulla biforcazione nel caso variazionale.* Vengono considerati i problemi di biforcazione associati a funzionali semicontinui che non verificano le consuete ipotesi di regolarità. Per tali funzionali si può definire una «forma hessiana» generalizzata e mostrare che certi autovalori di tale forma sono valori di biforcazione.

I risultati così ottenuti vengono applicati ad un problema di biforcazione per disequazioni variazionali di tipo ellittico.

INTRODUCTION

Several questions of mathematical analysis can be reduced, as known, to the classical bifurcation problem.

In this one the problem

$$\text{(P)} \qquad A(u) = \lambda u$$

is considered, where A is an operator which acts on a certain functional space X, namely $A : D \to X$, where D is a suitable subset of X, and λ is a real parameter.

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Moreover one knows a branch of "trivial" solutions to (P): for sake of simplicity we consider the case

$$0 \in D \quad , \quad A(0) = 0$$

so that the pair $(\lambda, 0)$ satisfies (P) for any λ in $\mathbf{R}$.

Of course it is interesting to know whether there are other "branches" of solutions. In particular, a local analysis can be carried out on the possible points $(\lambda_0, 0)$ which emanate such other branches. The values λ_0 corresponding to such points are said to be of bifurcation for the problem (P).

In the classical case, D is an open subset of the Hilbert space X and A satisfies suitable regularity assumptions. It is quite obvious that, if λ_0 is of bifurcation, then $A'(0) - \lambda_0 I$ is not invertible, that is (under compactness assumptions)

$$(P_0) \qquad \exists\, v \neq 0 : \quad A'(0)v = \lambda_0 v .$$

The classical bifurcation problem consists in examining under which conditions the converse is true. Classical results can be found in [19].

Important studies deal with the properties of the branches of bifurcation (see, for instance, [24]).

A particular case, which is very significant, occurs when the operator A is the gradient of a suitable functional defined in D. Under differentiability assumptions on the functional, many authors have treated this case (see, for instance, [2, 3, 4, 17, 19, 21, 23, 25]).

On the other hand it is well known that the usual regularity hypotheses are not verified in many problems of analysis.

Several studies have been made, with various aims, in order to face these situations (see, for instance, [5, 8, 18]).

The researches developed in [10, 12, 13, 15] have been above all carried out in order to treat problems of non-linear analysis in which obstacle conditions appear (see, for instance, [6, 7, 22]).

In this paper we deal with a bifurcation problem with obstacle condition.

To introduce it, let us consider a bounded open set Ω in $\mathbf{R}^n$, a function $g : \Omega \times \mathbf{R} \to \mathbf{R}$, two functions $\phi_1, \phi_2 : \Omega \to \overline{\mathbf{R}}$ with $\phi_1 \leq \phi_2$ and $\rho > 0$.

The problem consists in studying the pairs (λ, u) such that

$$(VP) \quad \begin{cases} \lambda \in \mathbf{R}, \quad u \in K \cap S_\rho , \\ \displaystyle\int_\Omega (DuD(v-u) + g(x, u)(v-u))\, dx \geq \int_\Omega \lambda u (v-u)\, dx \ \forall\, v \in K \end{cases}$$

where

$$K = \{v \in H_0^1(\Omega) : \phi_1 \leq v \leq \phi_2\} ,$$

$$S_\rho = \{v \in L^2(\Omega) : \int_\Omega v^2\, dx = \rho^2\} .$$

The solutions u of problem (VP) can be regarded as "points which are critical from below" (see definition (1.1)) for the functional

$$f(u) = (1/2) \int_{\Omega} | \mathrm{D}u |^2 \, \mathrm{d}x + \int_{\Omega} \int_0^u g(x, s) \, \mathrm{d}s \, \mathrm{d}x$$

on the "constraint" $\mathrm{K} \cap \mathrm{S}_\rho$.

The minimization of such functional and the existence of a solution to (VP) have been considered for instance in [1, 26].

In [6, 7] the multiplicity of solutions to (VP) is studied by means of the associated evolution variational inequality and an adaptation of Ljusternik-Schnirelmann techniques.

In this paper the corresponding bifurcation problem is treated: one has to individuate a "tangent" problem corresponding to (P_0) and establish the connections between the "eigenvalues" of this one and the "branches" of solutions of (VP).

The same topic has been the object of the lecture [14].

§ 1. NON-SMOOTH BIFURCATION

Throughout this section H will denote a real Hilbert space whose scalar product and norm are denoted by $(\cdot \mid \cdot)$ and $\| \cdot \|$ respectively.

(1.1) DEFINITION. *Let* W *be an open subset of* H *and* $f : \mathrm{W} \to \mathbf{R} \cup \{+\infty\}$ *a function. Set* $\mathrm{D}(f) = \{u \in \mathrm{W} : f(u) < +\infty\}$. *Let* $u \in \mathrm{D}(f)$. *The function f is said to be sub-differentiable at u if there exists* α *in* H *such that*

$$\liminf_{v \to u} \frac{f(v) - f(u) - (\alpha \mid v - u)}{\| v - u \|} \geq 0 .$$

For every u in $\mathrm{D}(f)$ *we denote by* $\partial^- f(u)$ *the (possibly empty) set of such* α*'s and we set* $\mathrm{D}(\partial^- f) = \{u \in \mathrm{D}(f) : \partial^- f(u) \neq \emptyset\}$. *Since* $\partial^- f(u)$ *is convex and closed, for every u in* $\mathrm{D}(\partial^- f)$ *we can denote by* $\mathrm{grad}^- f(u)$ *the element of* $\partial^- f(u)$ *of minimal norm.*

Finally, a point u in $\mathrm{D}(f)$ *is said to be critical from below if* $0 \in \partial^- f(u)$. *A real number c is said to be a critical value if there exists u in* $\mathrm{D}(f)$ *such that* $0 \in \partial^- f(u)$ *and* $f(u) = c$.

Now we consider a function $f : \mathrm{H} \to \mathbf{R} \cup \{+\infty\}$ such that

$$f(0) = 0 \quad , \quad 0 \in \partial^- f(0) . \tag{1.2}$$

Our purpose is to study the set of the pairs (λ, u) in $\mathbf{R} \times D(\partial^- f)$ such that

$$\lambda u \in \partial^- f(u). \tag{1.3}$$

Of course for every λ in $\mathbf{R}$ the pair $(\lambda, 0)$ satisfies (1.3).

(1.4) DEFINITION. *A real number λ is said to be of bifurcation for $\partial^- f$ if there exists a sequence $((\lambda_h, u_h))$ in $\mathbf{R} \times D(\partial^- f)$ such that*

$$\forall h \in \mathbf{N} : \lambda_h u_h \in \partial^- f(u_h), \qquad u_h \neq 0;$$

$$\lim_h (\lambda_h, u_h) = (\lambda, 0) \text{ in } \mathbf{R} \times H.$$

In order to give a characterization of the values λ of bifurcation, we make the following further assumptions on f:

(1.5) the function f is lower semi-continuous and there exists a continuous function $q : H \to \mathbf{R}$ such that

$$f(v) \geq f(u) + (\alpha \mid v - u) - q(u) \mid v - u \mid^2$$

whenever $v \in H$, $u \in D(\partial^- f)$, $\alpha \in \partial^- f(u)$;

(1.6) there exists $\bar{\rho} > 0$ such that, if $(u_h)_h$ is a sequence in H with $0 < \mid u_h \mid \leq \bar{\rho}$, $\sup_h f(u_h)/(\mid u_h \mid^2) < +\infty$, then $(u_h/\mid u_h \mid)_h$ possesses a convergent subsequence;

(1.7) there exists a function $f_0 : H \to \mathbf{R} \cup \{+\infty\}$ such that for every sequence $(\rho_h)_h$ in $]0, 1]$ with $\lim_h \rho_h = 0$ we have

$$f_0 = \Gamma^-(H) \lim_h f_{\rho_h}$$

where $f_\rho(u) = f(\rho u)/(\rho^2)$ (see [11] for the definition of Γ-limit).

In these hypotheses it is readily proved that

$$f_0(0) = 0 \quad , \quad 0 \in \partial^- f_0(0);$$

$$\forall s > 0, \forall u \in H : f_0(su) = s^2 f_0(u).$$

(1.8) DEFINITION. *A real number λ is said to be an eigenvalue of $\partial^- f_0$ if there exists w in $D(\partial^- f_0)$ such that*

$$w \neq 0 \quad , \quad \lambda w \in \partial^- f_0(w).$$

(1.9) THEOREM. *Under assumptions* (1.2), (1.5), (1.6), (1.7), *if* λ *is of bifurcation for* $\partial^- f$, *then* λ *is an eigenvalue of* $\partial^- f_0$.

The converse is, in general, not true, as the following example shows.

(1.10) *Example.* Let $H = \mathbf{R}^2$, $f(x, y) = x^2 + 2y^2 - x(x^2 + y^2) \exp(-x^2 - y^2)$ if $x \geq 0$ and $y \geq 0$, $f(x, y) = +\infty$ elsewhere.

Then, assumptions (1.2), (1.5), (1.6), (1.7) are satisfied and we have $f_0(x, y) = x^2 + 2y^2$ if $x \geq 0$ and $y \geq 0$, $f_0(x, y) = +\infty$ elsewhere.

On the other hand it is easy to check that $\lambda = 4$ is an eigenvalue of $\partial^- f_0$, but it is not of bifurcation for $\partial^- f$ (the reason is that $\lambda = 4$ is not "topologically essential" in a sense that will be demonstrated later).

We have to make a further assumption which allows us to state the converse of (1.9). For this purpose define $\tilde{f}_0 : H \to \mathbf{R} \cup \{+\infty\}$ by means of $\tilde{f}_0(u) = f_0(u)$ if $|u| = 1$, $\tilde{f}_0(u) = +\infty$ if $|u| \neq 1$.

(1.11) PROPOSITION. *Let* $\lambda \in \mathbf{R}$. *Then the following facts are equivalent*:

a) λ *is an eigenvalue of* $\partial^- f_0$;

b) $(\lambda/2)$ *is a critical value of* $\tilde{f}_0$.

Now we can state our bifurcation theorem, which concerns only the eigenvalues which are "topologically essential".

If $c \in \mathbf{R}$, set $\tilde{f}_0^c = \{u \in H : \tilde{f}_0(u) \leq c\}$.

(1.12) THEOREM. *Suppose that* (1.2), (1.5), (1.6), (1.7) *hold. Let* λ *be an eigenvalue of* $\partial^- f_0$ *such that for some* $\varepsilon > 0$ $\tilde{f}_0^{(\lambda/2)-\varepsilon}$ *is not a weak deformation retract of* $\tilde{f}_0^{(\lambda/2)+\varepsilon}$ *in* $\tilde{f}_0^{(\lambda/2)+\varepsilon}$ *and* $(\lambda/2)$ *is the unique critical value of* $\tilde{f}_0$ *in* $[(\lambda/2) - \varepsilon, (\lambda/2) + \varepsilon]$.

Then λ *is of bifurcation for* $\partial^- f$.

More precisely, there exists $\rho_0 > 0$ *and two sets* $\{\lambda_\rho : 0 < \rho \leq \rho_0\} \subset \mathbf{R}$ *and* $\{u_\rho : 0 < \rho \leq \rho_0\} \subset D(\partial^- f)$ *such that*

$$\forall \rho : \lambda_\rho u_\rho \in \partial^- f(u_\rho) \quad , \quad |u_\rho| = \rho \, ; \; \lim_{\rho \to 0} \lambda_\rho = \lambda .$$

(1.13) COROLLARY. *Suppose that* (1.2), (1.5), (1.6), (1.7) *hold and that* $f_0(u) < +\infty$ *for some* $u \neq 0$.

Then $\tilde{f}_0 \not\equiv +\infty$, *there exists* $(\lambda/2) := \min \tilde{f}_0$ *and* λ *is of bifurcation for* $\partial^- f$. *More precisely, all the thesis of theorem* (1.12) *holds.*

Bifurcation theorems for the "first eigenvalue" were already proved in [26, 27].

A significant case, in which theorem (1.12) is true for all the eigenvalues of $\partial^- f_0$, occurs when f_0 is a "quadratic form".

(1.14) THEOREM. *Suppose that* (1.2), (1.5), (1.6), (1.7) *hold. Suppose moreover that* $D(f_0)$ *is a linear subspace of* H *and that* f_0 *is a quadratic form* (*in the usual sense*) *on every finite dimensional subspace of* $D(f_0)$. *Let* λ *be an eigenvalue of* $\partial^- f_0$.

Then λ *is of bifurcation for* $\partial^- f$. *More precisely, all the thesis of theorem* (1.12) *holds.*

Under the assumptions of theorem (1.14) it is easy to prove that for every λ in $\mathbf{R}$ the set $E_\lambda = \{u \in D(\partial^- f_0) : \lambda u \in \partial^- f_0(u)\}$ is a linear subspace of H of finite dimension. A real number λ is said to be a simple eigenvalue of $\partial^- f_0$ if E_λ has dimension one.

(1.15) THEOREM. *Under the assumptions of theorem* (1.14) *let* λ *be a simple eigenvalue of* $\partial^- f_0$.

Then λ *gives rise to two branches of bifurcation for* $\partial^- f$, *that is there exists* $\rho_0 > 0$ *and*

$$\{\lambda_\rho^{(i)} : 0 < \rho \leq \rho_0 \,,\ i = 1\,,2\} \subset \mathbf{R}\,,$$

$$\{u_\rho^{(i)} : 0 < \rho \leq \rho_0 \,, i = 1\,,2\} \subset D(\partial^- f)$$

such that

$$\forall \rho\,, i : \lambda_\rho^{(i)}\, u_\rho^{(i)} \in \partial^- f(u_\rho^{(i)})\,,\ |\, u_\rho^{(i)} \,| = \rho\,, u_\rho^{(1)} \neq u_\rho^{(2)};\ \lim_{\rho \to 0} \lambda_\rho = \lambda\,.$$

§ 2. BIFURCATION FOR SOME VARIATIONAL INEQUALITIES

In this section we expose an application of the results of the previous section to a bifurcation problem for a nonlinear elliptic variational inequality.

Let Ω be a bounded open subset of $\mathbf{R}^n$, $g : \overline{\Omega} \times \mathbf{R} \to \mathbf{R}$ a function of class C^1 such that $g(x\,,0) = 0$, $\phi_1 : \Omega \to [-\infty\,,0]$ an upper semi-continuous function and $\phi_2 : \Omega \to [0\,,+\infty]$ a lower semi-continuous function.

For $i = 1\,,2$ set $F_i = \{x \in \Omega : \phi_i(x) = 0\}$ and denote by K the closure in $H_0^1(\Omega)$ of the set

$$\{u \in C_0^\infty(\Omega) : \forall x \in \Omega : \phi_1(x) \leq u(x) \leq \phi_2(x)\}$$

and by K_0 the closure in $H_0^1(\Omega)$ of the set

$$\{u \in C_0^\infty(\Omega) : \forall x \in F_1 : u(x) \geq 0\ ;\ \forall x \in F_2 : u(x) \leq 0\}\,.$$

The sets K and K_0 can also be directly defined by means of the notion of capacity (see, for instance, [9]).

We want to study the pairs (λ, u) in $\mathbf{R} \times \mathrm{K}$ such that

$$(2.1) \qquad u \in \mathrm{L}^\infty(\Omega), \int_\Omega (DuD(v-u) + g(x, u)(v-u))\,dx \geq$$

$$\geq \int_\Omega \lambda u (v-u)\,dx \ \forall v \in \mathrm{K}.$$

Because of the assumptions we have made, for every λ in $\mathbf{R}$ the pair $(\lambda, 0)$ satisfies (2.1).

(2.2) DEFINITION. *A real number λ is said to be of bifurcation for $-\Delta u + g(x, u)$ with respect to* K, *if there exists a sequence $((\lambda_h, u_h))_h$ in $\mathbf{R} \times \mathrm{K}$ such that*

$\forall h \in \mathbf{N} : (\lambda_h, u_h)$ *satisfies* (2.1), $u_h \neq 0$; $\lim_h \lambda_h = \lambda$; $\lim_h u_h = 0$ in $\mathrm{L}^\infty(\Omega)$.

(2.3) DEFINITION. *A real number λ is said to be an eigenvalue of $-\Delta u + g'_u(x, 0)\, u$ with respect to K_0, if there exists u in K_0 such that*

$$u \neq 0, \int_\Omega (DuD(v-u) + g'_u(x, 0)\, u(v-u))\,dx \geq$$

$$\geq \int_\Omega \lambda u(v-u)\,dx \ \forall v \in \mathrm{K}_0.$$

(2.4) THEOREM. *Let $\lambda \in \mathbf{R}$. If λ is of bifurcation for $-\Delta u + g(x, u)$ with respect to* K, *then λ is an eigenvalue of $-\Delta u + g'_u(x, 0)\, u$ with respect to K_0.*

To formulate the converse, we have to consider the functional $\tilde{f}_0 : \mathrm{L}^2(\Omega) \to \mathbf{R} \cup \{+\infty\}$ defined by

$$\tilde{f}_0(u) = \int_\Omega (1/2)(|Du|^2 + g'_u(x, 0)\, u^2)\,dx$$

if $u \in \mathrm{K}_0$ and $\int_\Omega u^2\,dx = 1$; $\tilde{f}_0(u) = +\infty$ elsewhere.

(2.5) THEOREM. *Let λ be an eigenvalue of $-\Delta u + g'_u(x, 0)\, u$ with respect to K_0. Suppose that for some $\varepsilon > 0$ $\tilde{f}_0^{(\lambda/2)-\varepsilon}$ is not a weak deformation retract of $\tilde{f}_0^{(\lambda/2)+\varepsilon}$ in $\tilde{f}_0^{(\lambda/2)+\varepsilon}$ and that $(\lambda/2)$ is the unique critical value of $\tilde{f}_0$ in $[(\lambda/2)-\varepsilon, (\lambda/2)+\varepsilon]$.*

Then λ *is of bifurcation for* $-\Delta u + g(x, u)$ *with respect to* K.

More precisely, there exists $\rho_0 > 0$ *and* $\{\lambda_\rho : 0 < \rho \leq \rho_0\} \subset \mathbf{R}$, $\{u_\rho : 0 < \rho \leq \rho_0\} \subset \mathrm{K}$ *such that* $\forall \rho : (\lambda_\rho, u_\rho)$ *satisfies* (2.1), $\int_\Omega u_\rho^2 \, dx = \rho^2$; $\lim_{\rho \to 0} \lambda_\rho = \lambda$; $\lim_{\rho \to 0} u_\rho = 0$ *in* $\mathrm{L}^\infty(\Omega)$ *and in* $\mathrm{H}_0^1(\Omega)$.

A particular case of theorem (2.5) is the bifurcation theorem for the "first eigenvalue" (see also [26, 27]).

(2.6) Corollary. *Suppose we are not in the trivial case* $\phi_1 = \phi_2 = 0$. *Then there exists* $\lambda := 2 \min \tilde{f}_0$, λ *is an eigenvalue of* $-\Delta u + g'_u(x, 0) u$ *with respect to* K_0 (*the "first eigenvalue"*) *and* λ *is of bifurcation for* $-\Delta u + g(x, u)$ *with respect to* K. *More precisely, all the thesis of theorem* (2.5) *holds.*

Now we consider the particular case in which $\mathrm{F}_1 = \mathrm{F}_2$ and we set $\Omega' = \Omega \setminus \mathrm{F}_1 = \Omega \setminus \mathrm{F}_2$.

In such a case the eigenvalues of $-\Delta u + g'(x, 0) u$ with respect to K_0 coincide with the classical eigenvalues of $-\Delta u + g'_u(x, 0) u$ in the open set Ω'.

(2.7) Theorem. *Suppose that* $\mathrm{F}_1 = \mathrm{F}_2$ *and let* λ *be an eigenvalue of* $-\Delta u + g'_u(x, 0) u$ *in the open set* Ω'.

Then λ *is of bifurcation for* $-\Delta u + g(x, u)$ *with respect to* K. *More precisely, all the thesis of Theorem* (2.5) *holds.*

(2.8) Theorem. *Suppose that* $\mathrm{F}_1 = \mathrm{F}_2$ *and let* λ *be a simple eigenvalue of* $-\Delta u + g'_u(x, 0) u$ *in the open set* Ω'.

Then λ *gives rise to two branches of bifurcation for* $-\Delta u + g(x, u)$ *with respect to* K, *namely there exists* $\rho_0 > 0$ *and* $\{\lambda_\rho^{(i)} : 0 < \rho \leq \rho_0, i = 1, 2\} \subset \mathbf{R}$, $\{u_\rho^{(i)} : 0 < \rho \leq \rho_0, i = 1, 2\} \subset \mathrm{K}$ *such that*

$\forall \rho, i : (\lambda_\rho^{(i)}, u_\rho^{(i)})$ *satisfies* (2.1), $\int_\Omega |u_\rho^{(i)}|^2 \, dx = \rho^2$, $u_\rho^{(1)} \neq u_\rho^{(2)}$; $\lim_{\rho \to 0} \lambda_\rho^{(i)} = \lambda$; $\lim_{\rho \to 0} u_\rho^{(i)} = 0$ *in* $\mathrm{L}^\infty(\Omega)$ *and in* $\mathrm{H}_0^1(\Omega)$.

§ 3. Some open problems

In this section we wish to formulate some open problems which seem to be very natural.

Theorems (1.14) and (2.7) ensure that there exists a branch of bifurcation. One can ask whether or not, in some instances, there are many branches of bifurcation. Theorems (1.15) and (2.8) provide an answer only for simple eigenvalues.

However, in the regular case there are several known results (see, for instance, [4, 17, 21, 25]).

For example, one may conjecture that there are two branches of bifurcation when f_0 is even and $2\,m$ branches when f is even, where m is the multiplicity, in some sense, of the eigenvalue.

Another interesting question is the study of the " semiquadratic " function f_0, namely of the " eigenvalues of $\partial^- f_0$ ".

In the concrete case, this is the study of the points that are critical from below for the functional

$$\tilde{f}_0(u) = \begin{cases} (1/2)\displaystyle\int_\Omega (|Du|^2 + a_0 u^2)\,dx & \text{if } u \in K_0 \cap S_1 \\ +\infty & \text{elsewhere,} \end{cases}$$

$$K_0 = \{u \in H_0^1(\Omega) : u \geq 0 \text{ in } F_1,\ u \leq 0 \text{ in } F_2\},$$

$$S_1 = \{u \in L^2(\Omega) : \int_\Omega u^2\,dx = 1\},$$

where a_0 is an assigned function in $L^\infty(\Omega)$ and F_1, F_2 are two assigned closed subsets of Ω.

Of course, if $F_1 = F_2$ the existence of infinitely many critical points is trivial.

One can look for other conditions which ensure that there are infinitely many critical points for $\tilde{f}_0$ or that the critical points of $\tilde{f}_0$ are " topologically essential " in the sense of Theorem (2.5).

For instance, the existence of infinitely many critical points is proved in [20], if Ω is an interval in $\mathbf{R}$, F_1 is a point and $F_2 = \varnothing$.

Finally, variational inequalities of second order have been considered in § 2, as well as in the previous studies [6, 7].

It would be interesting to known whether all these results (eigenvalues with obstacle and related evolution equation, bifurcation with obstacle) are still true for von Karman's equation describing thin elastic plates, which is of fourth order. In the case without obstacle, there are results in [2, 3, 23], whereas in the case with obstacle, the study of the first eigenvalue and related bifurcation is carried out in [16, 26].

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