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Wave-number-independent theology in a sphere

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Geofisica. — *Wave-number-independent rheology in a sphere.* Nota (*) del Socio MICHELE CAPUTO.

ABSTRACT. — We determine the displacement vector and the relaxation caused by time varying surface tractions on a spherical Earth model whose anelastic properties are described by a class of stress strain relations representing a wide variety of rheologies including the Maxwell model and the type which approximates polycrystalline halite and granite. It is seen that this class of rheologies is almost wave number independent, and that once a solution is found for a boundary condition whose time variation does not depend on the wave number, then an approximate solution is found for any other boundary condition with the same geometry.

We then study the particular case of the Burger solid and see that its relaxation is described by three different exponentials, one of which is rigorously wave number independent, whereas the other two are weakly wave number dependent.

We also study the general solution of the equations of elasticity for a spherical Earth model with assigned surface tractions when the stress-strain relations are defined using derivatives of fractional order and find a subclass of models having the same relaxation time; we see that the relaxation time is not sufficient to describe the rheology of a medium but that the relaxation of the medium needs a much more detailed description in the time domain. This rheology causes a splitting of the free modes in a set of very close lines in the frequency domain.

We discuss the effect of successive glacial loads on the Earth for this subclass of models and find it is possible to still see today the effect of more than one glaciation and that the effect of the last glaciation may be masked by the quasi-fossil effect of the previous ones.

A relation between migration of isotherms and rheology is also considered.

We show that the reciprocity theorem of Betti is valid also with the stress strain relations of the generalized Maxwell models.

We finally find a general solution of the equations which govern the deformation of an elastic sphere with Maxwell rheology.

KEY WORDS: Rheology, wave number, isostasy, glaciations, migration of isotherms, reciprocity, Apennines.

RIASSUNTO. — *Reologie quasi-indipendenti dal numero d'onda.* Si trovano il vettore spostamento ed il rilassamento causati da forze variabili nel tempo agenti in una sfera le cui proprietà reologiche sono descritte da una vasta classe di modelli che generalizzano in vari modi quello di Maxwell. Si verifica che queste reologie sono quasi indipendenti dal numero d'onda e che quando si trova la soluzione di un problema per una specifica funzione temporale dei vincoli, una soluzione approssimante è immediatamente trovata per qualsiasi altra funzione temporale e per la stessa distribuzione geometrica dei vincoli.

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Si fanno anche alcune applicazioni a problemi della geofisica.

Si studia il caso particolare del solido di Burger e si vede che il suo rilassamento è descritto da tre esponenziali uno dei quali è rigorosamente indipendente dal numero d'onda, mentre gli altri due ne sono debolmente dipendenti.

Si studia anche la soluzione generale delle equazioni dell'elasticità per una sfera con assegnate condizioni al contorno nel caso che le relazioni fra sforzo e deformazione contengano derivate di ordine reale e si trova una classe di modelli che hanno lo stesso tempo di rilassamento. Si vede che il tempo di rilassamento non è sufficiente per descrivere una reologia e che questa richiede una descrizione più dettagliata. Si vede inoltre che in questo tipo di reologia si ha una moltiplicazione di tutte le righe spettrali a formare bande attorno alle righe del modello elastico perfetto.

Si trova anche una soluzione generale delle equazioni che governano le deformazioni di una sfera con reologia regolata dal modello di Maxwell.

Si dimostra anche la validità del teorema di reciprocità di Betti nel caso delle reologie di Maxwell generalizzate.

Si discutono infine una relazione fra migrazione di isoterme e reologia nonché l'effetto di successivi periodi glaciali.

Introduction

Because of the too strict limits of applicability, the perfect elastic medium and the perfect fluid exist only in theoretical physics. In reality we have media which, at constant temperature and pressure, are somehow elastic or fluid and have elastic or anelastic properties depending on the frequency and history of the phenomena to be studied. The discipline concerned with the related problems is called rheology and is extremely important in many branches of science including Geophysics. Moreover, due to the practical difficulty of preparing samples of exactly equal physical quality it is also very difficult to verify to a satisfactory accuracy some generally accepted theoretical principles such as the reciprocity principle which (e.g. Caputo 1986) is still waiting experimental verification also for the linear field. The reciprocity principle is not to be confused with the reciprocity theorem which will be proved in Appendix 3 for the elastic media whose stress strain relations are defined by means of derivatives of fractional order and for more general rheologies.

In particular, rheology is important in the study of the deformation and related stresses of the Earth due to the various fields of forces acting on it which may arise from the thermal field generating convection within the Earth, from gravity acting on mountains and on their isostatic compensation and in general on density anomalies, and from the elastic energy released by earthquakes in form of decaying elastic waves.

The determination of the rheology of the Earth has the highest priority for the solution of all geodynamic problems with planetary scale and of many other theoretical and applied geophysical problems.

Today the rheology of the Earth and planets, considered from the most optimistic point of view, may, at best, be defined mysterious. In practice most work done uses rheologies which are defined by one parameter, which is often transduced in the so called relaxation time, or the time required to reduce to e^{-1} the stress generated in the anelastic continuum by constant exterior or inside forces.

Because of the lack of direct information, in the past, the rheology of the Earth was often discussed by assuming a Maxwell or a Voigt model for the stress strain relation, or a standard linear solid. However the analysis (Caputo, 1983; 1986) of recent laboratory data obtained by various authors (Heard, 1972; Carter and Heard, 1970; Heard and Raleigh, 1972; Carter and Hansen, 1983) indicates that none of the previously mentioned stress-strain relations fits the rheology of Yule marble, granite and polycrystalline halite at various temperatures and confining pressures in the range of linear elasticity. Therefore, other forms of stress strain relations had to be found to represent the rheology of the above mentioned materials (e.g. Caputo, 1983, 1984 *a*; 1986).

Recently a few other rheologies, more complete from the point of view of the number of parameters defining them, have been considered; the models of Bingam or Burger (Caputo, 1966; Barbarella, 1973; Sabadini *et al.*, 1985; Caputo, 1986; Caldwell and Turcotte, 1979) or that with derivatives of real order in the stress strain relations (Caputo, 1966; 1967; 1984 *a*) have been introduced in Geophysics. However, little effort has been made to approach the problem with a good rationale from the physical and mathematical points of view.

The solution of the elastodynamic problem for given boundary conditions and for the most general type of time varying source function is readily determined when the Green function is known, that is when the solution is known for that assigned boundary condition and for a source with frequency independent spectral content.

However, there are cases when the problem may be simplified in the sense that the solution for a given boundary condition may be used to obtain the solution for many other boundary conditions, as we shall study here.

THE GENERALIZED STRESS–STRAIN RELATIONS

Caputo (1984 *a*; 1986) generalized the Maxwell viscoelastic stress-strain relations by assuming

$$l * \dot{\tau}_{ij} + \mu (\tau_{ij} - \delta_{ij}\, \tau_{rr} / 3) = 2\, \mu\, l * \dot{\varepsilon}_{ij} + \delta_{ij}\, \lambda\, l * \dot{\varepsilon}_{rr} \tag{1}$$

where λ and μ are the elastic parameters, $l(t)$ is the material function and the $*$ as usual indicated convolution; here the operator l is applied to both stress τ_{ij} and strain ε_{ij}.

We may also generalize the Voigt viscoelastic stress-strain relations by assuming

$$\tau_{ij} = \lambda\, \delta_{ij}\, \varepsilon_{rr} + 2\, \mu\, \varepsilon_{ij} + 2\, g * (\dot{\varepsilon}_{ij} - \delta_{ij}\, \dot{\varepsilon}_{rr} / 3) \tag{2}$$

where $g\,(t)$ is the material function.

Combining the Maxwell and Voigt models we obtain the generalized expression of the stress-strain relation of the Standard linear solid

$$h^* \dot{\tau}_{ij} + \mu (\tau_{ij} - \delta_{ij} \tau_{rr}) = \lambda \delta_{ij} h^* \dot{\varepsilon}_{rr} + 2 \mu h^* \dot{\varepsilon}_{ij} + + f_1^* (-2 \delta_{ij} \ddot{\varepsilon}_{rr} / 3 + 2 \ddot{\varepsilon}_{ij}) \tag{3}$$

Assuming that $h(t)$ and $f_1(t)$ are different from zero, by taking the Laplace Transform (LT) of (3) we find

$$T_{ij} H p + \mu (T_{ij} - \delta_{ij} T_{,rr} / 3) = \delta_{ij} H (\lambda - 2 F p / 3) E_{rr} p + + 2 H (\mu + F p) E_{ij} p \tag{4}$$

where T_{ij} and E_{ij} are the Laplace Transform (LT) of the stress τ_{ij} and strain ε_{ij} respectively, $H(p)$ and $F_1(p)$ are the LT of the functions $h(t)$ and $f(t)$ representing the rheology of the medium and $F = F_1/H$.

It has been shown (Caputo, 1983, 1986) that the rheology of polycrystalline halite, in the range of linear elasticity, may be tentatively fitted assuming

$$H = \mu [(\overline{\alpha a} + \overline{\beta b}) p + \overline{\alpha\beta} (\bar{a} + \bar{b})] / \{[3\bar{C}\mu - 3 (\lambda + \mu) (\overline{\alpha a} + \overline{\beta b}) / /(3\lambda + 2\mu)] p^2 + [3\bar{C}\mu (\bar{\alpha} + \bar{\beta}) - 3\overline{\alpha\beta} (\bar{a} + \bar{b}) (\lambda + \mu)/(3\lambda + 2\mu)] p + + 3\bar{C}\mu\overline{\alpha\beta}] \; ; \; F = 0 \tag{5}$$

where the coefficients $\bar{\alpha}$, $\bar{\beta}$, $\bar{a}$, $\bar{b}$, are determined experimentally with constant strain rate $\dot{C}$ experiments (see the Appendix of Caputo 1986) or with creep or relaxation experiments.

Fitting equation (4) in the one-dimensional ($\varepsilon_{22} = \varepsilon_{33} = 0$) stress strain relation obtained from (4) with $F = 0$ gives

$$T_{11} = E_{11} [\mu (\lambda + 2\mu/3) + (\lambda + 2\mu) Hp] / (\mu + Hp) \tag{5'}$$

which may be used to retrieve $H(p)$ using ε_{11} and τ_{11} observed in rigorously one-dimensional laboratory experiments.

Constant strain rate laboratory experiment data on Yule marble reported by Heard and Raleigh (1972) and creep laboratory experiment data on westerly granite reported Hansen and Carter (1983) analysed by Caputo (1986) indicate that the stress-strain relation defined by (5) represents satisfactorily also the rheology of these rocks.

It must be noted that all the above mentioned laboratory experiments have been carried to strains up of the order of 0.01, that is in the non-linear range. The data points in the linear range are too scarce to allow reliable results. Other experiments with more data points in the linear range are now available but the results of the analysis of the data are in progress and not yet available.

The deformation and the stress field of the Earth due to surface loads and to density anomalies in its interior, in the case of perfect elasticity has been studied by many authors; for recent work on this subject one may see the Journal of Geophysical Research (Vol. 85, B, 11, Nov. 1980); for a brief review one may see Caputo *et al.*, 1985, and Caputo 1984 *b*.

Few studied the case of an anelastic Earth. However, we believe that it is of great interest to know the relaxation of the stress and the creep of the deformation in the case when the rheology is represented by (5). Specifically we shall study here the deformation and relaxation of the stress field with particular attention to the dependence of the relaxation on the wave number of the perturbation and we shall see that a wide class of rheologies represented by (4) is almost independent of the wavenumber.

THE DEFORMATION OF THE ELASTIC SPHERE

The deformation of an elastic layered Earth model caused by surface and buoyant forces when the inertial terms may be neglected, as is the case in many geological problems, may be computed from the analytic solution of Caputo (1984 *a*), which is valid also in case of absence of body forces (Caputo 1961), and is expressed in form of uniformly convergent series

$$(6)\qquad {}_is_1 = \sum_{0}^{\infty}{}_n \sum_{0}^{2n}{}_k\, {}_iU_{nk}\, Y_{nk}$$

$${}_is_2 = \sum_{1}^{\infty}{}_n \sum_{0}^{2n}{}_k \left[{}_iV_{nk}(r)\, \frac{\partial Y_{nk}}{\partial \theta} + {}_iW_{nk}(r)\, \frac{\partial Y_{nk}}{\sin\theta\, \partial \psi} \right]$$

$${}_is_3 = \sum_{1}^{\infty}{}_n \sum_{0}^{2n}{}_k \left[{}_iV_{nk}(r)\, \frac{\partial Y_{nk}}{\sin\theta\, \partial \psi} - {}_iW_{nk}\, \frac{\partial Y_{nk}}{\partial \theta} \right]$$

$$(7)\qquad Y_{nk} = \begin{cases} \left[\dfrac{2n+1}{4\pi}\right]^{1/2} P_n(\cos\theta)\,, & k = 0 \\[2ex] \left[\dfrac{2n+1}{2\pi}\,\dfrac{(n-k)\,!}{(n+k)\,!}\right]^{1/2} P_n^k(\cos\theta)\, \sin^k\theta\, \cos k\psi\,, & k = 1, 2, \ldots, n \\[2ex] \left[\dfrac{2n+1}{2\pi}\,\dfrac{(k-n)\,!}{(k+n)\,!}\right]^{1/2} P_n^{k-n}(\cos\theta)\, \sin^{k-n}\theta\, \sin(k-n)\psi\,, & k = n+1, \ldots, 2n \end{cases}$$

where r, θ, ψ are spherical coordinates centred with the spheres defining the Earth model, i defines the i-th layer of the model ($i = 0$ is the central one),

λ_i and μ_i are the elastic parameters, ${}_is_1$, ${}_is_2$, ${}_is_3$ are the components of the displacement along the r, θ, ψ axes. This result has been obtained by Caputo (1961) who used a method introduced by Picone (1936) and Fichera (1949).

Y_{nk} are spherical harmonics and

$$\begin{aligned} {}_iU_{nk}(r) &= {}_iA_{nk}^{(1)} r^{n+1} + {}_iA_{nk}^{(2)} r^{-n} + {}_iA_{nk}^{(3)} r^{n-1} + {}_iA_{nk}^{(4)} r^{-n-2} \\ {}_iV_{nk}(r) &= {}_iB_{nk}^{(1)} r^{n+1} + {}_iB_{nk}^{(2)} r^{-n} + {}_iB_{nk}^{(3)} r^{n-1} + {}_iB_{nk}^{(4)} r^{-n-2} \\ {}_iW_{nk}(r) &= {}_iC_{nk}^{(1)} r^{n} + {}_iC_{nk}^{(2)} r^{-n-2} \end{aligned} \tag{8}$$

$$\begin{aligned} {}_iB_{nk}^{(1)} &= [(n+3)\gamma_1 + n + 5]\,[(n\gamma_1 + n - 2)(n+1)]^{-1}\,{}_iA_{nk}^{(1)} \\ {}_iB_{nk}^{(2)} &= [(2-n)\gamma_1 + 4 - n]\,[n((n+1)\gamma_1 + n + 3)]^{-1}\,{}_iA_{nk}^{(2)} \\ {}_iB_{nk}^{(3)} &= {}_iA_{nk}^{(3)}/n\,,\quad {}_iB_{nk}^{(4)} = -\,{}_iA_{nk}^{(4)}(n+1)\,,\quad \gamma_1 = \lambda/\mu \end{aligned} \tag{9}$$

Solutions may be written when body forces are acting on the shells; they are given in Appendix E.

To study the relation between the creep and the wave numbers (Caputo, 1984 *b*) it is sufficient to consider the case of a sphere of a radius r_0 $(i = 0)$ and a surface load defined by

$$\tau_{11} = \sum_0^{\infty}{}_n \bar{D}_n P_n(\cos\theta) \tag{10}$$

After applying the boundary conditions at $r = r_0$ we find that the solution, in case (10) is applied is

$$\begin{aligned} A_{n0}^{(1)} &= -\frac{\bar{D}_n r_0^{-n}(n+1)[n\lambda + (n-2)\mu]}{2\mu[(2n^2+4n+3)\lambda + 2(n^2+n+1)\mu]} \\ A_{n0}^{(3)} &= -\frac{n\bar{D}_n r_0^{-n+2}[(n^2+2n)\lambda + (n^2+2n-1)\mu]}{2\mu(1-n)[(2n^2+4n+3)\lambda + 2(n^2+n+1)\mu]} \end{aligned} \tag{11}$$

Obviously, due to the linearity of the algebra, the solution with body forces or other types of boundary conditions, can be easily obtained. Here we will discuss in some detail only the case of absence of body forces and boundary conditions given by (10), and mostly consider the cases in which the perturbations are simulated with high wave numbers, as it is generally realistic, and gravity plays a negligible role (Slichter and Caputo, 1961). The terms with $n = 1$ have been added for sake of completeness, in fact they represent a translation.

THE CREEP IN POLAR SPHERICAL COORDINATES

In spherical polar coordinates when the rheology is defined by (5) the LT of the displacement is obtained as a uniformly convergent series by substituting

$$
\begin{aligned}
&\lambda \rightarrow [\mu(\lambda + 2\mu/3) + \lambda p \mathrm{H} - (2/3) p^2 \mathrm{H} \mathrm{F}] \,/\, (\mu + p \mathrm{H}) \\
&\mu \rightarrow p \mathrm{H}(\mu + p \mathrm{F}) \,/\, (\mu + p \mathrm{H})
\end{aligned}
\tag{12}
$$

in the LT of (6) as shown by Caputo (1966, 1985).

In the case of the boundary conditions (10) we obtain the following expressions for the LT of the Green function for the displacement vector of the sphere of radius r_0 ($\bar{\mathrm{D}}_n = \mathrm{D}_n \,\delta(t)$) with generalized rheology of Maxwell type.

$$
\mathrm{A} = \mu\left(\lambda + \frac{2}{3}\mu\right)
$$

$$
\begin{aligned}
\mathrm{S}_1 = \sum_0^\infty{}_n \Bigg\{ & -\frac{n(n+1)\left[\mathrm{A} + p\mathrm{H}\left(\lambda + \dfrac{n-2}{n}\mu\right)\right]}{(2n^2+4n+3)\left[\mathrm{A} + p\mathrm{H}\left(\lambda + \dfrac{2n^2+2n+2}{2n^2+4n+3}\mu\right)\right]}\left(\frac{r}{r_0}\right)^2 + \\
& - \frac{n^2(n+2)\left[\mathrm{A} + p\mathrm{H}\left(\lambda + \dfrac{n(n^2+n-1)}{n^2(n+2)}\mu\right)\right](1-\delta_{1n})}{(n-1)(2n^2+4n+3)\left[\mathrm{A} + p\mathrm{H}\left(\lambda + \dfrac{2n^2+2n+2}{2n^2+4n+3}\mu\right)\right]} \Bigg\} \cdot \\
& \left(\frac{r}{r_0}\right)^{n-2} \frac{r\,\mathrm{D}_n(\mu + p\mathrm{H})}{2\mu\, p\mathrm{H}}\, \mathrm{Y}_{n0}
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
\mathrm{S}_2 = \sum_1^\infty{}_n \Bigg\{ & -\frac{(n+3)\left[\mathrm{A} + p\mathrm{H}\left(\lambda + \dfrac{n+5}{n+3}\mu\right)\right]}{(2n^2+4n+3)\left[\mathrm{A} + p\mathrm{H}\left(\lambda + \dfrac{2n^2+2n+2}{2n^2+4n+3}\mu\right)\right]}\left(\frac{r}{r_0}\right)^2 + \\
& - \frac{n(n+2)\left[\mathrm{A} + p\mathrm{H}\left(\lambda + \dfrac{n^2+n-1}{n(n+2)}\mu\right)\right](1-\delta_{1n})}{(n-1)(2n^2+4n+3)\left[\mathrm{A} + p\mathrm{H}\left(\lambda + \dfrac{2n^2+2n+2}{2n^2+4n+3}\mu\right)\right]} \Bigg\} \cdot \\
& \left(\frac{r}{r_0}\right)^{n-2} \frac{r\,\mathrm{D}_n(\mu + p\mathrm{H})}{2\mu\, p\mathrm{H}}\, \frac{\partial \mathrm{Y}_{n0}}{\partial \theta}
\end{aligned}
$$

From analysis of polycristalline halite and granite Caputo (1985, 1986) found that $h(t)$ is of the type (5)

$$h(t) = a_1 e^{-\alpha_1 t} + b_1 e^{-\beta_1 t}$$

$$\mathrm{H}(p) = \frac{(a_1 + b_1)p + \alpha_1 b_1 + \beta_1 a_1}{p^2 + (\alpha_1 + \beta_1)p + \alpha_1 \beta_1} = \frac{a_0 p + b_0}{p^2 + \mathrm{d}_0 p + u_0} \tag{14}$$

Therefore substituting (14) in (13) we find that S_1 and S_2 are

$$\mathrm{S}_1 = \sum_0^\infty {}_n \left[-\mathrm{D}_n \frac{r^{n+1}}{r_0^n} \frac{\sum_0^4 {}_i \mathrm{C}_{1i} p^i}{p(p+\mathrm{F})(p^2 + \mathrm{M}p + \mathrm{N})} + \right. \tag{15}$$

$$\left. -\mathrm{D}_n \frac{r^{n-1}}{r_0^{n-2}} \frac{n}{n-1} \frac{\sum_0^4 {}_i \mathrm{C}_{3i} p^i (1 - \delta_{1n})}{p(p+\mathrm{F})(p^2 + \mathrm{M}p + \mathrm{N})} \right] \Psi_{n0}$$

$$\mathrm{S}_2 = \sum_1^\infty {}_n \left[-\mathrm{D}_n \frac{r^{n+1}}{r_0^n} \frac{\sum_0^4 {}_i \mathrm{E}_{1i} p^i}{n p(p+\mathrm{F})(p^2 + \mathrm{M}p + \mathrm{N})} + \right.$$

$$\left. -\mathrm{D}_n \frac{r^{n-1}}{r_0^{n-2}} \frac{(1 - \delta_{ln})}{n-1} \frac{\sum_0^4 {}_i \mathrm{E}_{3i} p^i}{p(p+\mathrm{F})(p^2 + \mathrm{M}p + \mathrm{N})} \right] \frac{\delta \Psi_{n0}}{\delta \theta} .$$

the definition of C_{1i}, C_{3i}, E_{1i}, E_{3i} is given in Appendix A.

The LT^{-1} of (15) is easily found in closed form by expanding it in partial fractions. Note that F here is different than in (4) and is defined below; M and N are given in (17).

We must note that one of the poles is $p_1 = -(a_1\beta_1 + b_1\alpha_1)/(a_1 + b_1 =) - b_0/a_0 = -\mathrm{F}$, with $\alpha_1 > 0$, $\beta_1 > 0$, $a_1 > 0$, $b_1 > 0$, which depends exclusively on the analytic expression of $h(t)$. The other two poles depend on n, however, as we shall see for n large the exponents are almost independent of n, the poles are:

$$p_2 = (\mathrm{M}/2)(-1 + [1 - 4\,\mathrm{N}/\mathrm{M}^2]^{1/2}) \, , \; p_3 = (\mathrm{M}/2)(-1 + \tag{16}$$

$$- [1 - 4\,\mathrm{N}/\mathrm{M}^2]^{1/2})$$

where

$$\text{(17)} \qquad \mathrm{M} = \frac{\mu\left(\lambda + \frac{2}{3}\mu\right)\mathrm{d}_0 + b_0\left\{\lambda + 2\,\frac{n^2+n+1}{2\,n^2+4\,n+3}\,\mu\right\}}{\mu\left(\lambda + \frac{2}{3}\mu\right) + a_0\left\{\lambda + 2\,\frac{n^2+n+1}{2\,n^2+4\,n+3}\,\mu\right\}}$$

$$\mathrm{N} = \frac{u_0\mu\left(\lambda + \frac{2}{3}\mu\right)}{\mu\left(\lambda + \frac{2}{3}\mu\right) + a_0\left\{\lambda + 2\,\frac{n^2+n+1}{2\,n^2+4\,n+3}\,\mu\right\}}$$

We obtain H from the first of formulas (10) of Caputo (1986) with its $\mathrm{N} = 0$; E_{11} is given by the LT of the creep (9) of Caputo (1986), obtained with the theoretically applied pressure σ_1 and confining pressure σ_3. We then introduce the values of a, b, α, β obtained from the experimental creep curves presented in the paper of Caputo (1986). We find

$$\mathrm{H} = -\,[2\,(\alpha a + \beta b)\,p + \alpha\beta\,(a + b)]\,(\sigma_1 - \sigma_3)\,\mu/3\,[p\,((\alpha a + \beta b)\,p +$$
$$+\,\alpha\beta\,(a + b))\,(\sigma_1 - (\sigma_1 + 2\,\sigma_3)\,\lambda/(3\,\lambda + 2\,\mu)) - 2\,\mu\,(p + \alpha)\,(p + \beta)]$$

which gives

$$a_0 = -\,(\alpha a + \beta b)\,2\,(\sigma_1 - \sigma_3)\,\mu/3\,\mathrm{Z}\ ; \quad b_0 = -\,\alpha\beta\,(a_0 + b_0)\,2\,(\sigma_1 - \sigma_3)\,\mu/3\,\mathrm{Z}$$

$$\mathrm{d}_0 = [\alpha\beta\,(a + b)\,(\sigma_1 - (\sigma_1 + 2\,\sigma_3)\,\lambda/(3\,\lambda + 2\,\mu)) - 2\,(\alpha + \beta)\,\mu]/\mathrm{Z}$$

$$u_0 = -\,2\,\mu\alpha\beta/\mathrm{Z} \quad ; \quad \mathrm{Z} = -\,2\,\mu + (\alpha a + \beta b)\,(\sigma_1 - (\sigma_1 + 2\,\sigma_3)\,\lambda/(3\,\lambda + 2\,\mu))$$

where a, b, α, β are the coefficients of (6) of Caputo (1986) obtained from the values of the parameters A, B, D, J of the creep curves presented in Caputo (1986).

M and N depend on n through the term $(2\,n^2 + 2\,n + 2)/(2\,n^2 + 4\,n + 3)$ which is a monotonous function increasing from $2/3$ for $n = 0$ to 1 for $n = \infty$ as shown by the triangles in fig. 1. For most rheological problems of geophysics it is allowed to assume that the variation of the function $(n^2 + n + 1)/(n^2 + 2\,n + 1.5)$ is negligible and assume it $2/3$. M and N then reduce to

$$\text{(18)} \qquad \mathrm{M} = (\mu \mathrm{d}_0 + b_0)/(\mu + a_0); \quad \mathrm{N} = u_0\mu/(\mu + a_0)\,.$$

For polycrystalline halite it is $a_0 \ll \mu$, $b_0 \ll \mu\mathrm{d}_0$ (Caputo, 1983) then $\mathrm{M} = \alpha_1 + \beta_1$ and $\mathrm{N} = \alpha_1\beta_1$, which give $p_2 = -\,\beta_1$, $p_3 = -\,\alpha_1$ and also the exponentials $e^{p_2 t}$, $e^{p_3 t}$ may be factored out of their summations.

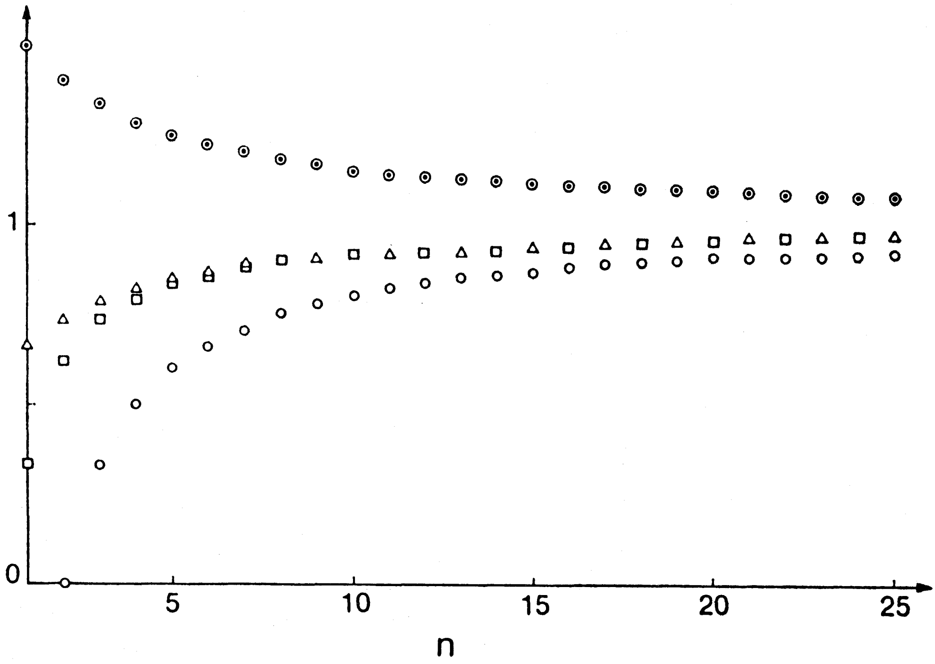


Fig. 1. – Wave number dependence of the relaxation time of the generalized Maxwell solid with the material function $h(t)$ as in (13); circled dots represent $(n+5)/(n+3)$, squares represent $(n^2+n-1)/(n^2+2n)$, triangles represent $2(n^2+n+1)/(2n^2+$ $+4n+3$, circles represent $(n-2/n$.

Note that in this case $p_1 = p_2 a_1/(a_1+b_1) + p_3 b_1/(a_1+b_1)$ and therefore, since the experimental results show that $a_1 \sim b_1$ and $\alpha_1 \sim \beta_1$, it follows that $p_2 \sim p_3$ and $p_1 \sim (p_3+p_2)/2$.

The inversion of the LT (15) gives

$$
\begin{aligned}
(19)\qquad s_1 = \sum_0^\infty{}_n \Bigg[& -D_n \frac{r^{n+1}}{2\mu r_0^n} \left\{ \frac{(a_0+\mu)(f_1+a_0 f_2)}{a_0(f_3+a_0 f_4)}\,\delta(t) + \frac{f_1}{f_3}\frac{\mu u_0}{b_0} + \right. \\
& + \frac{F_{13}p_1 - F_{14}p_1 + F_{15}}{p_1^2 - p_1(p_2+p_3) + p_2 p_3} e^{p_1 t} + \frac{F_{13}p_2^2 - F_{14}p_2 + F_{15}}{p_2^2 - p_2(p_1+p_2) + p_1 p_3} e^{p_2 t} + \\
& \left. + \frac{F_{13}p_3^2 - F_{14}p_3 + F_{15}}{p_3 - p_3(p_1+p_2) + p_1 p_2} e^{p_3 t} \right\} - D_n \frac{r^{n-1}(1-\delta_{1n})\,n}{2\mu r_0^{n-2}(n-1)} \left\{ \frac{(a_0+\mu)(f_5+a_0 f_6)}{a_0(f_3+a_0 f_4)}\,\delta(t) + \right. \\
& + \frac{f_5}{f_3}\frac{\mu u_0}{b_0} + \frac{F_{33}p_1^2 - F_{34}p_1 + F_{35}}{p_1^2 - p_1(p_2+p_3) + p_2 p_3} e^{p_1 t} + \\
& \left. + \frac{F_{33}p_2^2 - F_{34}p_2 + F_{35}}{p_2^2 - p_2(p_1+p_3) + p_1 p_3} e^{p_2 t} + \frac{F_{33}p_3^2 - F_{34}p_2 + F_{35}}{p_3^2 - p_3(p_1+p_2) + p_1 p_2} e^{p_3 t} \right\} \Bigg] Y_{n0}
\end{aligned}
$$

$$(20) \qquad s_2 = \sum_1^\infty {}_n \left[- D_n \frac{r^{n+1}}{2\mu n r^n} \left\{ \frac{(a_0+\mu)(f_7+a_0 f_8)}{a_0(f_3+a_0 f_4)} \delta(t) + \frac{f_7}{f_3} \frac{\mu u_0}{b_0} + \right.\right.$$

$$+ \frac{R_{13} p_1^2 - R_{14} p_1 + R_{15}}{p_1^2 - p_1(p_2+p_3) + p_2 p_3} e^{p_1 t} + \frac{R_{13} p_2^2 - R_{14} p_2 + R_{15}}{p^2 - p_2(p_1+p_3) + p_1 p_3} e^{p_2 t} +$$

$$+ \frac{R_{13} p - R_{14} p_3 + R_{15}}{p_3^2 - p_3(p_1+p_2) + p_1 p_2} e^{p\ t} \Big\} - \frac{D_n r^{n-1}(1-\delta_{1n})}{2\mu(n-1) r_0^{-2}} \left\{ \frac{(a_0+\mu)(f_9+a_0 f_{10})}{\partial_0(f_3+a_0 f_4)} \delta(t) + \right.$$

$$+ \frac{f_9}{f_9} \frac{\mu u_0}{b_0} + \frac{R_{33} p_1^{\cdot} - R_{34} p_1 + R_{35}}{p_1^2 - p_1(p_2+p_3) + p_2 p_3} e^{p_1 t} +$$

$$\left.\left. + \frac{R_{33} p_2^2 - R_{34} p_2 + R_{35}}{p_2^2 - p_2(p_1+p_3) + p_1 p_2} e^{p_2 t} + \frac{R_{33} p_3^2 - R_{34} p_3 + R_{35}}{p^{\ 2} - p_3(p_1+p_2) + p_1 p_2} e^{p_3 t} \right\} \right] \frac{\partial Y_{n0}}{\delta\theta}$$

the values of C_{ij}, F_{ij}, R_{ij}, E_{ij}, f_i are given in the Appendix A. It is worth noting that these coefficients depend weakly on n through ratios of polynomials f_i which are of the same order in n. However one may verify that the Fourier Coordinates (FC) of s_1 are weakly dependent on n but the FC of s_2 are inversely proportional to n. The factors p_i in the exponents of the exponentials are weakly dependent on n.

Since a and b have the same order of magnitude and α and β are also of the same order of magnitude, the values of p_i are of the order of α, (or β) and the transient half life would be only α^{-1} sec, (or β^{-1} sec) depending on the confining pressure and the temperature. This implies that the transient would last but a short time, then creep would take place for an indefinite time at a vanishing rate resulting in a residual finite strain.

An analysis of (13) shows that the results obtained with the form (14) for $h(t)$ apply also to the more general case when $h(t)$ is such that

$$(21) \qquad \lim_{t\to\infty} h(t) = 0\,, \int_0^\infty h(t)\, dt = \text{finite}\,.$$

An inspection of (13) shows that for large values of n the terms containing p are reduced to $(\mu + pH)/\mu\, pH$ which may be factored out of the summation. Then the LT^{-1} of (13), in case $H(p)$ is given by (14), contains the time only in the function

$$(22) \qquad -\left(\frac{1}{a_0} + \frac{1}{\mu}\right) \delta(t) + \frac{u_0}{b_0} + ((d_0 - b_0/a_0 - a_0 u_0/b_0)/a_0)\, e^{-b_0 t/a_0}$$

which is factored out of the summations. It is thus clear that this type of rheology for large values of n is almost wave-number-independent, is dominated by the relaxation time a_0/b_0 and that, when the geometry of the boundary condition does not change in time the displacement and stress field, in the time domain, are described by convolving (22) with the function $\psi_0(t)$ describing the boundary condition in the time domain.

We shall discuss in Appendix D the case of the Maxwell rheology which is a special case of (13) obtained with $h(t) = \gamma\delta(t)$, $\mathrm{H}(p) = \gamma$.

An application to a geophysical problem

To this point no values have been specified for the FC of the load D_n and the results obtained are quite general.

We may now consider the case of geologic interest of the reaction of the Earth to the load of polar ice-caps which have been accumulated for some time. For this purpose we may use the formulas of the previous paragraphs assuming that there is axial symmetry and symmetry with respect to the equator. The normal section of the two antipodal caps can then be approximated with a limited number of Legendre polynomials obtained by truncating the series

$$\tau_{11} = \left\{ \sqrt{2}\,(1 - \cos\omega) + \sum_1^\infty{}_n \left[\mathrm{P}_{2n-1}(\cos\omega) - \right.\right.$$
$$\left.\left. - \mathrm{P}_{2n+1}(\cos\omega)\right] \mathrm{Y}_{2n,0}(\cos\theta)\left(\frac{2}{4n+1}\right)^{1/2}\right\}\psi_0 = f(\theta)\,\psi_0(t) \tag{23}$$

where $\psi_0(t)$ represents the time history of the load, $f(\theta)$ represents the following box-like function which is different from zero on polar caps of radius ω

$$f(\theta) = \begin{array}{l} \mathrm{D}\,, 0 < \theta < \omega\,, \quad \pi - \omega < \theta < \pi \\ \\ 0\,, \omega < \theta < \pi - \omega\,. \end{array} \tag{24}$$

The symmetry with respect to the equator has been assumed in order to have the system in equilibrium; it is also assumed that the values of ω are such that the loads on the two regions where $f(\theta) \neq 0$, which are symmetric with respect to the equator, have a negligible interaction.

The solution for a load constant in the time domain is obtained by convolving (19) and (20) with 1 and contains the same exponentials appearing in (19) and (20); the δ-like term and the constant are substituted with the constant and linear terms included in the following formulas

$$s_1 = \sum_0^\infty{}_n - \mathrm{D}_n \left\{ \frac{r^{n+1}}{2\,\mu\, r_0} \left[f_1\, u_0\,(\mu \mathrm{d}_0 + b_0) + \mu u_0\,(f_1\,\mathrm{d}_0 + f_2\, b_0) + \right.\right. \tag{25}$$
$$\left. - (a_0\, u_0 f_3 + b_0\,(\mathrm{d}_0 f_3 + b_0 f_4))\,\frac{\mu u_0 f_1}{b_0 f_3}\right] \frac{1}{b_0\, u_0 f_3} + \frac{r^{n-1}}{2\,\mu r_0^{n-2}}\,\frac{n}{n-1}\left[f_5\, u_0\,(\mu \mathrm{d}_0 + \right.$$

$$+ b_0) + \mu u_0 (f_5 \, d_0 + f_6 \, b_0) - (a_0 \, u_0 f_3 + b_0 (d_0 f_3 + b_0 f_4)) \frac{\mu u_0 f_5}{b_0 f_3} \Big] \frac{1}{b_0 \, u_0 f_3} \Big\} +$$

$$- D_n \left\{ \frac{r^{n+1}}{2 \, \mu r_0^n} \frac{\mu u_0 f_1}{b_0 f_3} + \frac{r^{n-1}}{2 \, \mu \, r_0^{n-2}} \frac{n}{n-1} \frac{\mu u_0 f_5}{b_0 f_3} \right\} t + \ldots$$

$$(26) \qquad s_2 = \sum_1^\infty {}_n - D_n \left\{ \frac{r^{n+1}}{2 \, \mu n \, r_0^n} \Big[f_7 \, u_0 (\mu d_0 + b_0) + \mu u_0 (f_9 \, d_0 + f_{10} \, b_0) + \right.$$

$$- (a_0 \, u_0 f_3 + b_0 (d_0 f_3 + b_0 f_4)) \frac{\mu u_0 f_9}{b_0 f_3} \Big] \frac{1}{b_0 \, u_0 f_3} + \frac{r^{n+1}}{2 \, \mu r_0^{n-2} (n-1)} \Big[f_9 \, u_0 (\mu d_0 + b_0) +$$

$$\left. + \mu u_0 (f_9 \, d_0 + b_0 f_{10}) - (a_0 \, u_0 f_3 + b_0 (d_0 f_3 + b_0 f_{10})) \frac{\mu u_0 f_9}{b_0 f_3} \Big] \frac{1}{b_0 \, u_0 f_3} \right\} +$$

$$- \frac{D_n \, \mu u_0}{2 \, \mu \, b_0 f_3} \left\{ \frac{r^{n+1}}{r_0^n} f_7 + \frac{r^{n-1}}{r_0^{n-2}} f_9 \right\} t + \ldots$$

The constant term represents the instantaneous reaction, the linear term gives the creep. Convolving (19) and (20) with a Box-life function of duration T we obtain the response of the sphere to a load acting for the time T, as in first approximation, an ice age polar cap would do. The response of the system, is similar to that of the uniaxial laboratory experiment illustrated by Caputo (1986) and shown in figure 2 (*a* and *b*).

The mechanisms which produce the variation of height of mountains are many; among the most important we may quote: *a*) tectonic forces, *b*) erosion, *c*) sinking of the crust in the mantle to generate isostatic adjustment and *d*) the migration of isotherms of the mantle into the portion of the crust forming the roots of the mountains, which changes the physical properties of the crust and causes flow of its lowest layer.

Rheology is indirectly involved in *a*) and *b*) and directly involved in *c*) and *d*).

Concerning *c*) and *d*), in case the isostatic adjustment has taken place as seems true in many cases, the sinking of the mountain range is due to the changes in rheological properties of the lowest parts of the roots of the mountain range, caused by the migration of the isotherm. This layer of rock with new temperature and new rheological properties, due to its buoyancy, would eventually migrate towards the surface by moving along the sides of the mountain roots. The rate of removal would depend on the rheological properties of the crust at the new temperature, and on the distance covered by this material in order reach the border of the roots of the mountain range; however the layer, with the new temperature and the new rheological properties, would eventually cease to support the mountain range and would make room for a new isostatic adjustment which would occur at the rate of the migration of the isotherms of the mantle into the crust. This rate of migration may be tentatively estimated from an analogous phenomenon occurring in the sinking slabs.

One considers a slab of 100 km thickness sinking in the mantle with a velocity of 7 cm/year considering also that when the slab reaches the depth of 700 km it loses its rigidity; we may therefore assume that at that depth the isotherm which causes a relevant loss of rigidity has reached the centre portion of the slab; we thus tentatively obtain a rate of migration of that isotherm into the slab of about 5 mm/year.

The temperature gradient between the upper mantle and the lower crust is lower than that between the mantle and the slab at greater depth and we may therefore assume that the rate of 5 mm/year estimated above is an upper limit for the rate of change to the rheological properties correspondig to the new temperature.

The lighter and viscous material newly formed, will flow along the surface separating the roots of the mountain ranges from the lower strata; the rate of flow will depend on its viscosity and its elimination will depend on the slope and the lateral extension of the roots; a balance between the volume of rocks being formed at the new temperature and the volume of outgoing material will eventually be reached.

At steady state, if we neglect the creep of the crust, due to the almost perfectly elastic response of the crust relative to the mantle, the rate of migration of the isotherm gives the rate of change of isostatic support and we may infer that the phenomenon of change of height of the mountains associated with *c*) and *d*) is dominated by the rate of migration of the isotherm in the mantle.

The lack of isostatic adjustment generates an additional uncompensated load and the surface displacement varies in time as in fig. 2 (*d*) for $t < t_0$.

The Maxwellian viscosity of the material below the crust should be extremely large in order to explain the low rate of height decrease observed in the places where there is lack of isostasy as in the Apennines (Caputo *et al.*, 1984) which exhibit a rate of height decrease of about 1 mm/year (Arca e Beretta, 1985).

In this case one would be inclined to seek other rheologies than the generally accepted Maxwell one studied in detail in Appendix D; examples of other rheologies are studied in the next section. In fact the rate of height decrease in the case of the Apennines would require too large mantle viscosities as seen in Appendix D or the presence of active tectonic forces to support the mountain range.

In this particular case however the lack of isostasy under the mountain range, associated with the large negative gravity anomaly to the East of the axis of the range, which may reveal the presence of a buoyant mass anomaly, indicates the possibility of a torque acting on the crust (Caputo *et al.*, 1984, 1985) which in turn may generate a rotation and compensate for the subsidence.

In the isostatically compensated mountain ranges, besides the mechanism of height decrease due to the migration of the isotherm, the other important one concerns the crust which is compressed between the load of the mountains and that of the isostatic buoyant masses. This would be the case of the Alps. A simple one-dimensional model of a 35 km thick slab squeezed by opposite forces of the order of those generated by the load of the Alps, shows

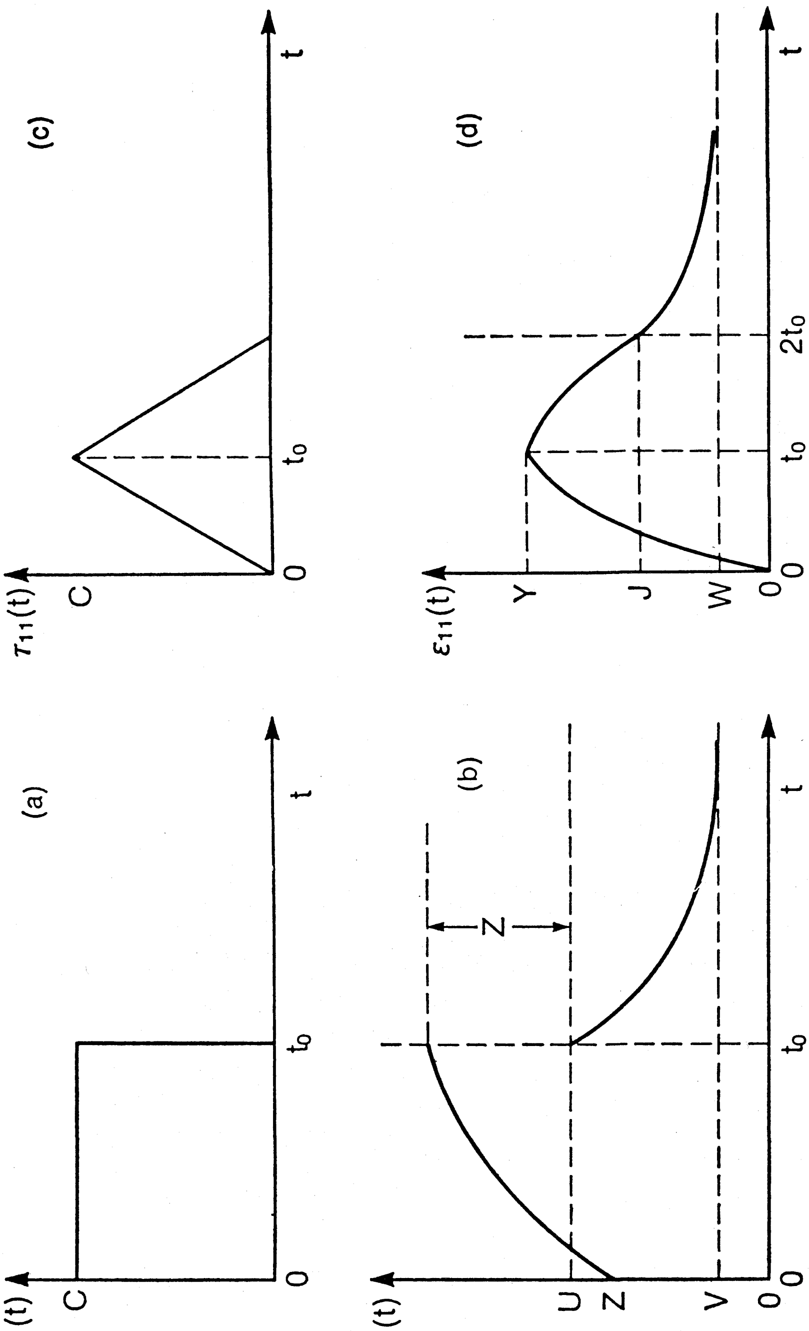


Fig. 2. – Relaxation of a medium whose material function $h(t)$ is as in (14); (*b*) and (*d*) relaxation caused by pressure on polar caps described in (*a*) and (*c*) respectively.

that the observed height decrease of about 1 mm/year is compatible with a Maxwellian viscosity of about $5 \cdot 10^{23}$.

The other possible cause is the migration of the isotherms of the mantle and the flow of material along the surface limiting the roots of the Alps which would imply a present rate of migration of the isotherm of the order of 1 mm/yr.

An example of new rheology

After the rheology of polycrystalline halite, which is of classic type in the sense that the relaxation time, related to exponentials, is descriptive of the rheology, it is of interest to study a more general rheology in which the relaxation time is not indicative of the behaviour of the continuum after the time to reduce the stress to $1/e$ has elapsed. This type of rheology is described assuming the following stress-strain relation where appear the derivatives of fractional order

$$\frac{\partial^z f}{\partial t^z} = \int_0^t \frac{(t-\tau)^{-z}}{\Gamma(1-z)} f'(\tau)\, d\tau\,.$$

$$\mathrm{LT}\left[\frac{\partial^z f}{\partial t^z}\right] = p^{z-1} f(0) + p^z\, \mathrm{LT}\,[f] \tag{27}$$

where $0 < z < 1$ is a real number, which corresponds to the assumption that in formulas (3) and (4) we set $f_1 = 0$ and

$$h = \eta t^{-z}/\Gamma(1-z) \qquad \mathrm{H} = \eta p^{z-1}\,. \tag{28}$$

In this case formulas (13) are defined and the determination of s_1 and s_2 is reduced to find the LT^{-1} of a function of the following type

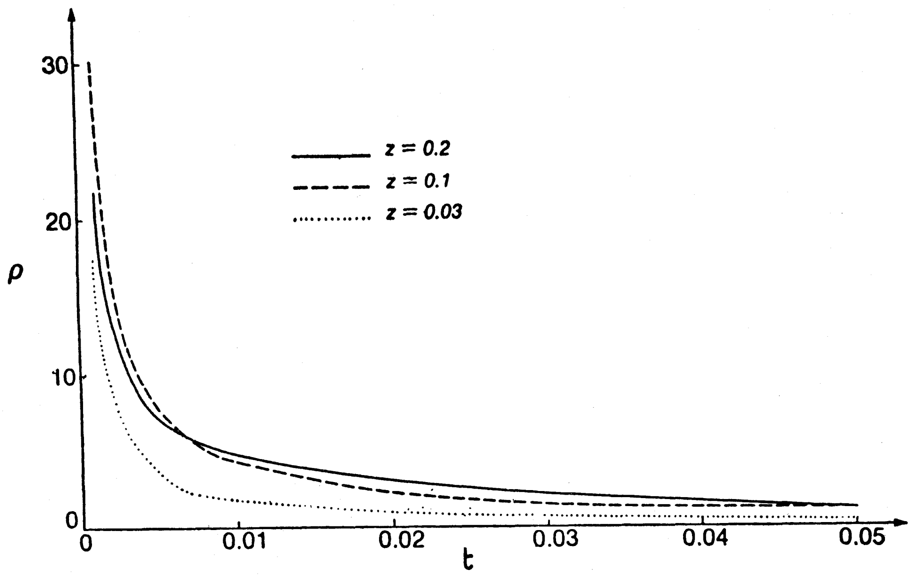
$$[1/\mu + 1/\eta p^z]\,[\mathrm{U} + \eta \mathrm{B} p^z]/[\mathrm{U} + \eta\, \mathrm{C} p^z] \tag{29}$$

where U, B, C are functions of n with very limited variation; the LT^{-1} of (29) is computed in the Appendix B. It is

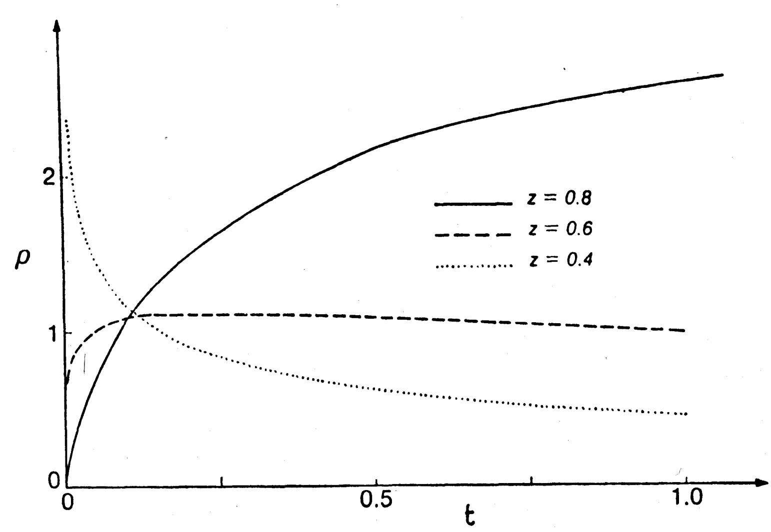
$$(\mathrm{U}/\mu - \mathrm{BU}/\mu\mathrm{C} + \mathrm{B})\, \chi(t)/\eta\mathrm{C} + \mathrm{U}\rho(t)/\eta\mathrm{C}^2 + \mathrm{B}\delta(t)/\mathrm{C}\mu \tag{30}$$

$\chi(t)$ and $\rho(t)$ are shown in fig. 3 and fig. 4 for several values of z.

To obtain s_1 and s_2, in the case $\tau_{11} \neq 0$ and $\tau_{12} = 0$ on $r = r_0$, (30) must be convolved with the function which gives the time history of τ_{11}, this eliminates the singularity for $t = 0$ in (30).



a)

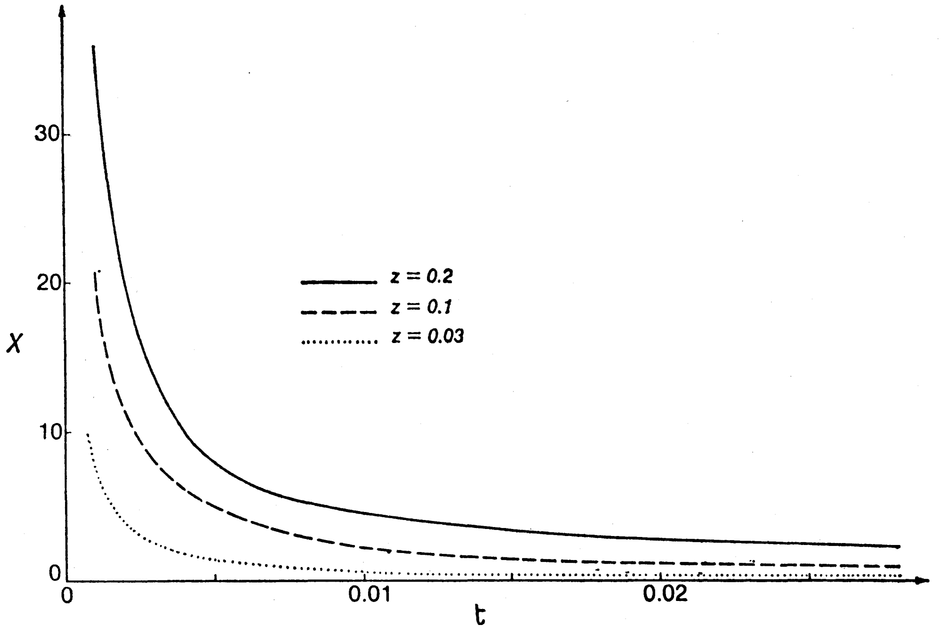


b)

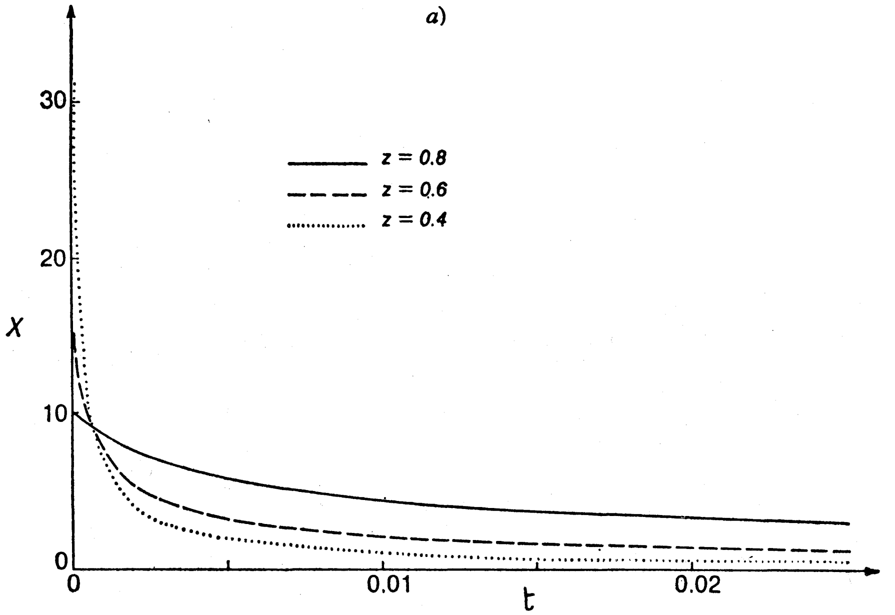
Figs. 3 *a* and 3 *b*. – The function $\rho(t)$ of the relaxation of the solid whose stress strain relation contains derivatives of real order, the time is measured in units of $(\mathrm{U}/\eta\,\mathrm{C})^{1/2}$. For $\lambda = \mu$ and $n = 10$, $\mathrm{U}/\eta\,\mathrm{C}$ is only 4% more than the asymptotic value $5\mu/6\eta$ $(n = \infty)$.

In case the rheology is defined by (28) when n is large, in s_1 and s_2 one may factor out of the summations the term

$$(\mu + \mathrm{H}p)/\mu\,\mathrm{H}p = 1/\eta p^z + 1/\mu\,. \tag{31}$$



a)



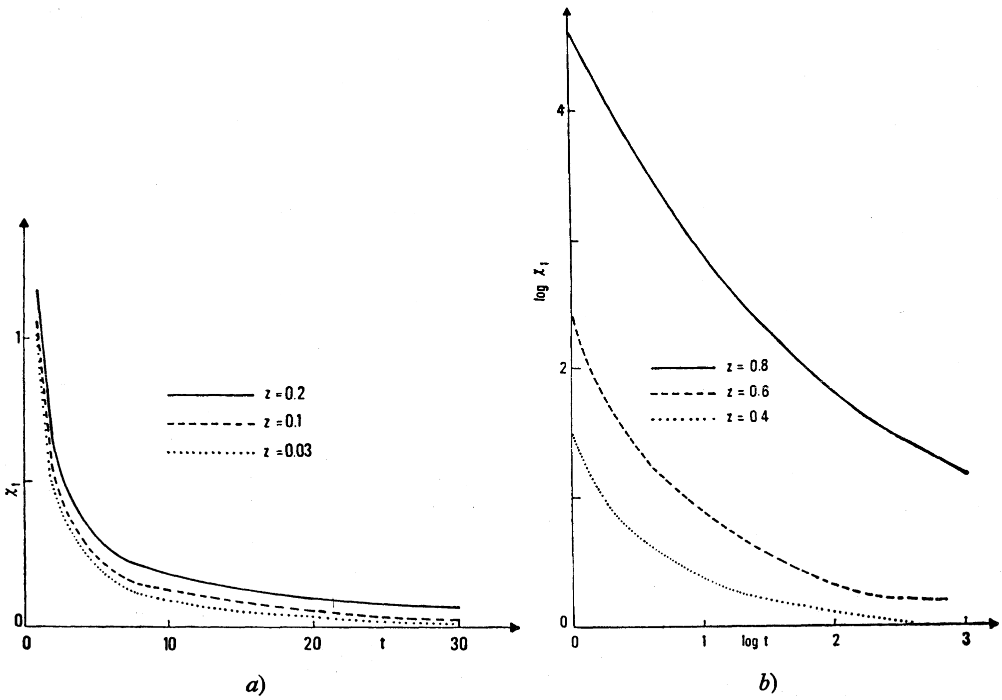
b)

Figs. 4 *a* and 4 *b*. – As figs. 3 *a* and 3 *b* for the function $\chi(t)$.

Then the time depending factor in s_1 and s_2 is

$$\chi_1 = \delta(t)/\mu + ((\sin \pi z)/\eta\pi) \int_0^\infty r^{-z} e^{-rt} \, dr = \delta(t)/\mu + t^{z-1}/\eta\Gamma(z) \tag{32}$$

shown in figs. 5 *a* and 5 *b* as $t^{z-1} \Gamma(1 - z)$.



Figs. 5 *a* and 5 *b*. - The function $\chi_1(t)$ valid for the wave number independent case of the solid whose stress strain relations contain derivatives of real order. Time is in sec.

The convolution of $\chi_1(t)$ with a Box of duration T gives

$$\chi_{1B}(t) = 1/\mu + t^z/\eta\Gamma(1+z); \qquad 0 \le t \le T \tag{33}$$

$$\chi_{1B}(t) = (t^z - (t-T)^z)/\eta\Gamma(1+z); \qquad t \ge T$$

which are shown in figs. 6 *a* and 6 *b*.

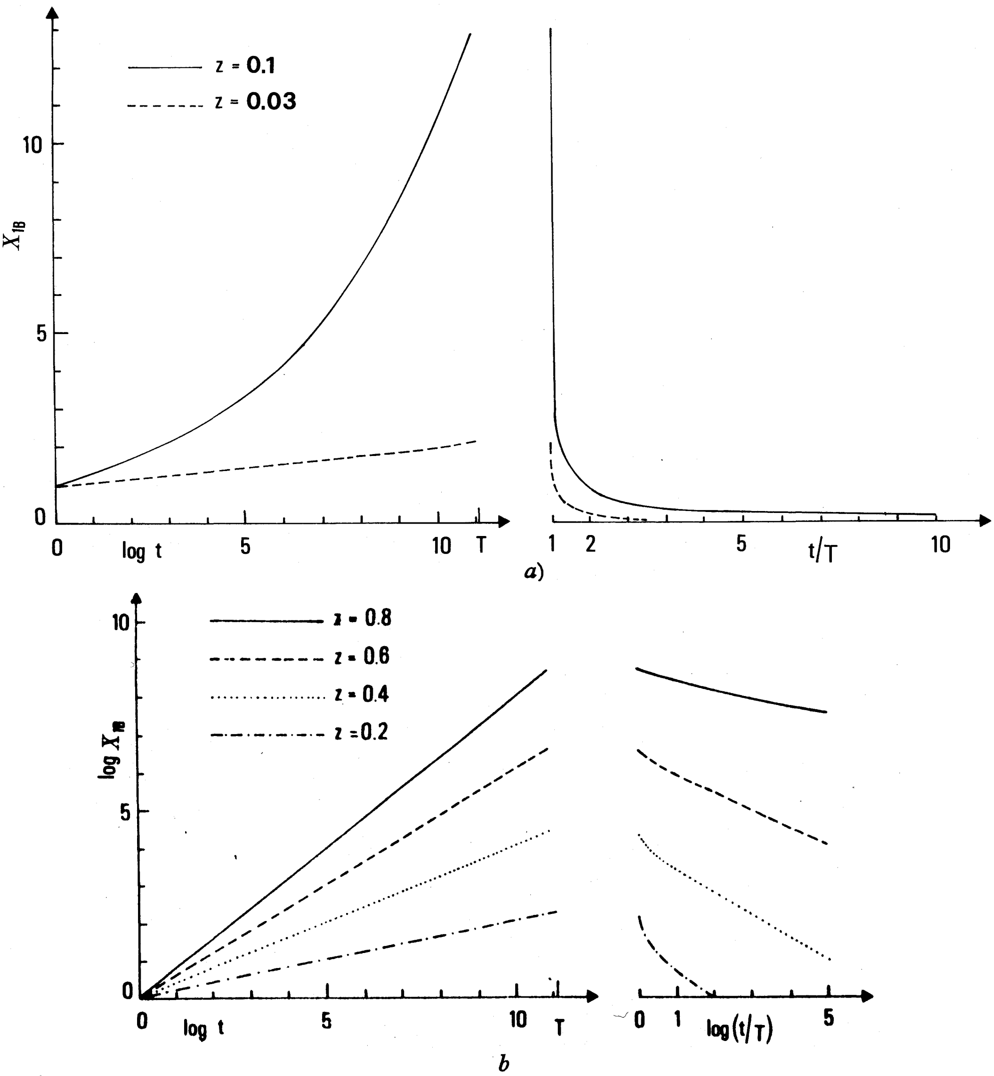
The creep rate

$$(t^{z-1} - (t-T)^{z-1})/\eta\Gamma(z)$$

is negative and asymptotically increasing to zero.

If the Earth has this type of rheology (Caputo, 1985), and $z > 0.6$, we may still be seeing now changes in the surface displacement due to the effect of previous polar caps, superimposed to the effect of the last glaciation.

We may note that for T = 2500 years (the approximate duration of a glaciation) and for relatively small values of z, the displacement, in case of a constant load applied for a time T, first occurs rather rapidly but its rate decreases also very rapidly; as one may see in fig. 6 *a* for $t < T$ and $t > T$; however for relatively large z and $t > T$ we see in fig. 6 *b* that the effect of the load is removed in a very long time after the load has ceased to act.



Figs. 6 *a* and 6 *b*. – The function χ_{1B} is the convolution of χ_1 with a box of duration T = 2500 years. Time is in sec.

On the Earth's surface we have many large mountain ranges with average elevation more than 2 km and in the ocean bottoms many abyssal plains with an average depth more than 4 km, which indicate that the Earth is not in hydrostatic equilibrium. Some of these features have not significantly changed their shape in the past millions of years. Remarkable examples are the Rocky Mountains in the northern portion of the United States fold and thrust belt whose orogen terminated about 45 M yr ago and the portion of the Appalachian Range whose orogen terminated about 225 M yr ago. These examples suggest that it is possible that in the long range the rheology of the Earth's crust may behave as a continuum of the type described here.

Since the last few glaciations occurred at intervals of about 10^4 years, it is possible that we have today in the crust a relevant residual stress due to the most recent glaciations.

In this geophysical application the gravity forces acting on the Earth have been neglected; we could have taken them rigorously into account by adding to the general solution the particular solution (given in Appendix E) for a layer added over the sphere, which would have complicated the formulae without giving more insight.

At the present stage in the Earth's evolution we are far from the asymptotic conditions not only in the tectonically active regions but also in those mountain ranges which are not tectonically active such as the Appalachain Range and the Rocky Mountains in the northern portion of the United States where a large maximum shear stress still exists (Caputo, 1984 *a*). The same applies in general to some intraplate regions which are considered stable, such as those mentioned above, where the occasional seismicity confirms the presence of a residual maximum shear stress, which is not surprising as it is needed to support the load of the mountains.

From the discussion made in the previous paragraphs we may infer that the shape of the deformation caused by a load on the surface of the Earth, as well as the consequent isostatic compensation, would not change in time due to the weak dependence of s_1 and of its time derivative on the wave-number, only the amplitude would change.

THE Q

The Q^{-1} and the phase velocity v of a medium, whose anelastic properties are represented by (3) with $f_1 = 0$, are obtained from the complex index of refraction $n(\omega)$ using the response of the medium to an impulse of stress. In the one dimensional case we have $n^2(\omega) = H_2(i\omega)/H_2(\infty)$ where $H_2(i\omega)$ is E_{11} of (5′) with $T_{11} = 1$. If H is given by (14), substituting in (5′), we see that H_2 is the ratio of two second degree polynomials and obtain

$$n^2(\omega) = (a_2\omega^2 - e_2 - ib_2\omega)/c_0(\omega^2 - u_2 - id_2\omega)$$

$$n = n_r - i\,n_i$$

$$Q^{-1} = 2\,n_i/n_r \qquad v = c/n_r$$

$$c_0 = \lim_{p\to\infty} H_2(p) = a_2$$

where c is the velocity of the wavefront.

Simple algebra gives

$$A_2 = (((a_2\omega^2 - e_2)^2 + b_2^2\omega^2)/((\omega^2 - u_2)^2 + d_2^2\omega^2)\, a_2^2)^{1/4}$$

$$B_2 = \tan^{-1}(\omega\,(u_2b_2 - d_2e_2 + \omega^2\,(d_2a_2 - b_2))/((a_2\omega^2 - e_2)\,(\omega^2 - u_2) + b_2d_2\omega^2))$$

$$n_i = A_2 \sin(B_2/2) \qquad n_r = A_2 \cos(B_2/2)\,.$$

It may be seen that when ω is large then $Q^{-1} = (d_2a_2 - b_2)/\,a_2\omega$ which is always the case when the polynomials in p, whose ratio gives H_2, have the same degree.

When $h(t) = \eta t^z/\Gamma(1-z)$, $(H = \eta p^{z-1}\,;\ N = 0)$, we obtain for the one dimensional case

$$E_{11} = T_{11}\,(\mu\,(\lambda + 2\,\mu/3) + (\lambda + 2\,\mu)\,\eta p^z)/(\mu + \eta p^z)$$

$$n^2(\omega) = (p^z + \mu/\eta)/(p^z + \mu\,(\lambda + 2\,\mu/3)/(\lambda + 2\,\mu)\eta)$$

and then

$$Q^{-1} = (\omega^z\,(\sin(\pi z/2))\,4\,\mu^2/3\,\eta\,(\lambda + 2\,\mu))/(\omega^{2z} +$$
$$+ \mu\,(2\,\lambda + 8\,\mu/3)\,\omega^z\,(\cos(\pi z/2))/(\eta\,(\lambda + 2\,\mu) + (\lambda + 2\,\mu/3)\,\mu^2/(\lambda + 2\mu)\,\eta^2)\,.$$

When μ/η is small with respect to ω^z we obtain

$$Q^{-1} = 4\,\mu^2\,\omega^{-z}\,(\sin \pi z/2)/3\,\eta\,(\lambda + 2\,\mu)\,.$$

The rheological model used in this paragraph causes a splitting of the lines of the free modes of a sphere into a set of an infinite number of very close lines limited in a narrow frequency band (Caputo, 1984) whose width is proportional the Q^{-1}. One may easily see that the rheological model represented by (14) causes also a splitting.

From the records of underground nuclear explosions Caputo (1981) found $Q = 2000$ at $2\,Hz$, which in turn gives for the P waves assuming $\lambda = \mu$,

$$2000^{-1} = 4\,(4\,\pi)^{-z}\,(\sin \pi z/2))\,\mu/9\,\eta\,. \tag{39}$$

This equation is a condition for the two unknown z and μ/η.

A second condition equation with the Q^{-1} observed for P waves at a different frequency (not yet available with sufficient accuracy) would allow to compute z and μ/η for compressional waves.

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APPENDIX A

$$C_{11} = [(\mu + a) f_1 + a f_2]/a (f_3 + a f_4)$$

$$C_{12} = [(\mu + a)(f_1 \mathrm{d} + f_2 b) + (\mu \mathrm{d} + b)(f_1 + a f_2)]/a (f_3 + a f_4)$$

$$C_{13} = [f_1 u (\mu + a) + \mu u (f_1 + f_2 a) + (\mu \mathrm{d} + b)(f_1 \mathrm{d} + f_2 b)]/a (f_3 + a f_4)$$

$$C_{14} = [f_1 u (\mu \mathrm{d} + b) + \mu u (f_1 \mathrm{d} + f_2 b)]/a (f_3 + a f_4)$$

$$C_{15} = f_1 u \mu / a (f_3 + a f_4)$$

$$f_1 = n(n+1)\left(\lambda + \frac{2}{3}\mu\right)\mu \quad , \quad f_2 = n(n+1)\lambda + (n+1)(n-2)\mu$$

$$f_3 = (2n^2 + 4n + 3)\left(\lambda + \frac{2}{3}\mu\right)\mu \quad , \quad f_4 = (2n^2 + 4n + 3)\lambda + 2(n^2 + n + 1)\mu$$

$$f_5 = n(n+2)\left(\lambda + \frac{2}{3}\mu\right)\mu \quad , \quad f_6 = n(n+2)\lambda + (n^2 + 2n + 1)\mu$$

$$F_{13} = C_{12} - \frac{(a+\mu)(f_1 + a f_2)}{a^2 (f_3 + a f_4)} [a (f_3 \mathrm{d} + f_4 b) + b (f_3 + a f_4)] - \frac{f_1}{f_3} \frac{\mu u}{b}$$

$$F_{14} = C_{13} - \frac{(a+\mu)(f_1 + a f_2)}{a^2 (f_3 + a f_4)^2} [a u f_3 + b (\mathrm{d} f_3 + a f_4)] - \frac{f_1 \mu u}{f_3 b} \left[\frac{\mathrm{d} f_3 + b f_4}{f_3 + a f_4} + b\right]$$

$$F_{15} = C_{14} - \frac{(a+\mu)(f_1 + a f_2)}{a^2 (f_3 + a f_4)} b u f_3 - \frac{f_1}{f_3} \frac{\mu u}{b} \frac{a u f_3 + b (\mathrm{d} f_3 + b f_4)}{a (f_3 + a f_4)}$$

$$C_{31} = (a + \mu)(f_5 + a f_6)/a (f_3 + a f_4)$$

$$C_{32} = [(\mu + a)(f_5 \mathrm{d} + f_6 b) + (\mu \mathrm{d} + a)(f_5 + f_6 a)]/a (f_3 + a f_4)$$

$$C_{33} = [f_5 u (\mu + a) + \mu u (f_5 + f_6 a) + (\mu \mathrm{d} + b)(f_5 + f_6 b)]/a (f_3 + a f_4)$$

$$C_{34} = [f_5 u (\mu \mathrm{d} + b) + \mu u (f_5 + f_6 b)]/a (f_3 + a f_4)$$

$$C_{35} = f_5 u \mu / a (f_3 + a f_4)$$

$$F_{33} = C_{32} - \frac{(a+\mu)(f_5 + a f_6)}{a^2 (f_3 + a f_4)^2} [a (f_3 \mathrm{d} + f_4 b) + b (f_3 + a f_4)] - \frac{f_5 u \mu}{f_3 b}$$

$$F_{34} = C_{33} - \frac{(a+\mu)(f_5+af_6)}{a^2(f_3+af_4)^2}[auf_3 + b(\mathrm{d}f_3+bf_4)] - \frac{f_5\mu u}{f_3 b}\left[\frac{\mathrm{d}f_3+bf_4}{f_3+af_4}+b\right]$$

$$F_{35} = C_{34} - \frac{(a+\mu)(f_5+af_6)}{a^2(f_3+af_4)^2}buf_3 - \frac{f_5\mu u}{f_3 b}\frac{auf_3+b(\mathrm{d}f_3+bf_4)}{a(f_3+af_4)}$$

$$E_{11} = (a+\mu)(f_7+af_8)/a(f_3+af_4)$$

$$E_{12} = [(a+\mu)(f_7\mathrm{d}+f_8 b) + (\mu\mathrm{d}+b)(f_7+f_8 a)]/a(f_3+af_4)$$

$$E_{13} = [f_7 u(\mu+a) + \mu u(f_7+af_8) + (\mu\mathrm{d}+b)(f_7\mathrm{d}+f_8 b)]/a(f_3+af_4)$$

$$E_{14} = [f_7 u(\mu\mathrm{d}+b) + \mu u(f_7\mathrm{d}+f_8 b)]/a(f_3+af_4)$$

$$E_{15} = u\mu f_7/a(f_3+af_4)$$

$$f_7 = (n+3)\mu\left(\lambda+\frac{2}{3}\mu\right)n \quad , \quad f_8 = ((n+3)\lambda+(n+5)\mu)n$$

$$f_9 = n(n+2)\mu\left(\lambda+\frac{2}{3}\mu\right) \quad , \quad f_{10} = n(n+2)\lambda+(n^2+n-1)\mu$$

$$R_{13} = E_{12} - \frac{(a+\mu)(f_7+af_8)}{a^2(f_3+af_4)^2}[a(f_3\mathrm{d}+f_4 b) + b(f_3+af_4)] - \frac{\mu uf_7}{f_3 b}$$

$$R_{14} = E_{13} - \frac{(a+\mu)(f_7+af_8)}{a^2(f_3+af_4)^2}[auf_3 + b(\mathrm{d}f_3+bf_4)] - \frac{u\mu f_7}{f_3 b}\left[b+\frac{f_3\mathrm{d}+f_4 b}{f_3+af_4}\right]$$

$$R_{15} = E_{14} - \frac{(a+\mu)(f_7+af_8)}{a^2(f_3+af_4)^2}buf_3 - \frac{\mu uf_7}{bf_3}\frac{auf_3+b(\mathrm{d}f_3+bf_4)}{a(f_3+af_4)}$$

$$E_{31} = (a+\mu)(f_9+af_{10})/a(f_3+af_4)$$

$$E_{32} = [(a+\mu)(f_9\mathrm{d}+f_{10}b) + (\mu\mathrm{d}+b)(f_9+af_{10})]/a(f_3+af_4)$$

$$E_{33} = [f_9 u(a+\mu) + \mu u(f_9+af_{10}) + (\mu\mathrm{d}+b)(f_9+bf_{10})]/a(f_3+af_4)$$

$$E_{34} = [f_9 u(\mu\mathrm{d}+b) + \mu u(f_9+bf_{10})]/a(f_3+af_4)$$

$$E_{35} = f_9 u\mu/a(f_3+af_4)$$

$$R_{33} = E_{32} - \frac{(a+\mu)(f_9+af_{10})}{a^2(f_3+af_4)^2}[a(f_3\mathrm{d}+f_4 b) + b(f_3+af_4)] - \frac{\mu uf_9}{bf_3}$$

$$R_{34} = E_{33} - \frac{(a+\mu)(f_9+af_{10})}{a^2(f_3+af_4)^2}[auf_3 + b(\mathrm{d}f_3+bf_4)] - \frac{f_9\mu u}{bf_3}\left[\cdot b+\frac{f_3+f_4 b}{f_3+af_4}\right]$$

$$R_{35} = E_{34} - \frac{(a+\mu)(f_9+af_{10})}{a^2(f_3+af_4)^2}buf_3 - \frac{\mu uf_9}{bf_3}\frac{auf_3+b(\mathrm{d}f_3+bf_4)}{a(f_3+af_4)}$$

APPENDIX B

The LT^{-1} of (13), with $Hp = \eta p^z$, reduces to the LT^{-1} of

(B 1) $$(1/\mu + 1/\eta p^z)(U + \eta\, Bp^z)/(U + \eta\, Cp^z)$$

By first decomposing it as follows

$$(U/\eta\mu C + Bp^z/C\mu + Up^{-z}/\eta^2\, C + B/\eta C)/(p^z + U/\eta C)$$

then noting that (Caputo, 1984)

$$\chi(t) = LT^{-1}((U/\eta C + p^z)^{-1}) = \sin \pi z/\pi \int_0^\infty ((r^z \exp(-rt))/(r^{2z} + (2\,U/\eta C)\, r^z \cos \pi z + (U/\eta C)^2)\, dr$$

we find

(B 2) $$LT^{-1}((U/\eta\mu C)/(U/\eta C + p^z)) = (U/\eta\mu C)\, \chi(t)$$

$$LT^{-1}(Bp^z/(C\mu\,(U/\eta C + p^z))) = (B/\mu C)\,[\delta(t) - U\chi(t)/\eta C]$$

$$LT^{-1}(B/C\eta\,(U/\eta C + p^z)) = B\chi(t)/\eta C\,.$$

Using the same method of inversion (Caputo, 1984) we find also that

(B 3) $$(U/\eta^2 C)\,\rho(t) = LT^{-1}(U/\eta^2\, Cp^z\,(U/\eta C + p^z)) =$$

$$= \sin \pi z/\pi \int_0^\infty (2\, r^z \cos \pi z + U/\eta C) \exp(-rt)\, dr/r^z\,(r^{2z} + 2\,(U/\eta C)\, r^z \cos \pi z + (U/\eta C)^2) = [t^{z-1}/\Gamma(z) - \chi(t)]/\eta$$

which completes the LT^{-1} of (B 1).

APPENDIX C

We shall show that the reciprocity theorem of Betti (1872) is valid also when Hooke's law is substituted by (3) with $f_1 = 0$. To prove it, we shall follow the method of Graffi (1939) who gave the first extention of Betti's theorem to the dynamic case.

Let us consider two states of a body whose anelastic properties are described by (3) with $f_1 = 0$. Let the body forces, surface tractions and displacement components of the first state be ρf_{1i}, ψ_{1i}, and s_{1i}, also let ρf_{2i}, ψ_{2i} and s_{2i} be those of the second state; both states be such that $s_{1i} = \partial s_{1i}/\partial t = 0$, $s_{2i} = \partial s_2/\partial t = 0$ for $t = 0$, ρ is the density which may be assumed variable with the point. Let F_{1i}, F_{2i}, ϕ_{1i}, ϕ_{2i}, S_{1i}, S_{2i} be the LT of f_{1i}, f_{2i}, ψ_{1i}, ψ_{2i}, s_{1i}, s_{2i}.

By taking the LT of the equilibrium conditions we obtain

$$\rho p^2 S_{1i} = \rho F_{1i} + T_{1ij,j} \tag{C 1}$$

$$\rho p^2 S_{2i} = \rho F_{2i} + T_{2ij,j}$$

and also for the surface traction,

$$T_{1ij}\eta_j = \phi_{1i} \tag{C 2}$$

$$T_{2ij}\eta_j = \phi_{2i}$$

where η_j are the components of the unit vector normal to the boundary of the body.

By multiplying the first of (C 1) by S_{2i} and the second by S_{1i} and subtracting, then integrating over the volume V occupied by the body we obtain

$$\int_V \rho\,(F_{1i}S_{2i} - F_{2i}S_{1i})\,dV + \int_V (T_{1ij,j}S_{2i} - T_{2ij,j}S_{1i})\,dV = 0 \tag{C 3}$$

We may verify with lengthy but easy computations that from

$$(T_{1ij}S_{2i})_{,j} = T_{1ij,j}S_{2i} + T_{1ij,j}S_{2i,j}$$

$$(T_{2ij}S_{1i})_{,j} = T_{2ij,j}S_{1i} + T_{2ij,j}S_{1i,j}$$

subtracting and substituting the stress-strain relations we obtain

$$(T_{1ij}S_{2i})_{,j} - (T_{2ij}S_{1i})_{j} = T_{1ij,j}S_{1i} - T_{2ij,j}S_{2i}$$

and substituting in (C 3) we find

$$\int_V \rho\,(F_{1i}S_{2i} - F_{2i}S_{1i})\,dV = \int_\Sigma T_{1ij}S_{2i}n_j - T_{2ij}S_{1i}n_j)\,d\Sigma \tag{C 4}$$

where Σ is the boundary of V.

From here the proof follows exactly as in Graffi's paper (1939). We note that, since ρ is a function of position, concerning the velocity of seismic waves v_p, v_s, the theorem is valid also for those inhomogeneous bodies with the velocities v_p and v_s subject to one constrain such as $v_p = \sqrt{3}\,v_s$.

Let us now assume that

$$\begin{aligned} f_{1i} &= G(t)\,a_{1i}\,, \quad \psi_{1i} = G(t)\,b_{1i} \\ f_{2i} &= G(t)\,a_{2i}\,, \quad \psi_{2i} = G(t)\,b_{2i} \end{aligned} \tag{C 5}$$

where a_{1i}, a_{2i}, b_{1i}, b_{2i} are functions only of position and independent of time, and $G(t)$ is a function of time only.

Substituting (C 5) in (C 4) we have

$$\int_V \rho \left\{ a_{1i} \left(\int_0^\infty G(t) \exp(-pt)\,dt \right) \left(\int_0^\infty s_{2i} \exp(-pt)\,dt \right) \right\} dV$$

$$- \int_V \rho \left\{ a_{2i} \left(\int_0^\infty G(t) \exp(-pt)\,dt \right) \cdot \left(\int_0^\infty s_{1i} \exp(-pt)\,dt \right) \right\} dV =$$

$$= \int_\Sigma \left\{ b_{1i} \left(\int_0^\infty G(t) \exp(-pt)\,dt \right) \left(\int_0^\infty s_{2i} \exp(-pt)\,dt \right) \right\} d\Sigma$$

$$- \int_\Sigma \left\{ b_{2i} \int_0^\infty G(t) \exp(-pt)\,dt \right) \left(\int_0^\infty s_{1i} \exp(-pt)\,dt \right) \right\} d\Sigma$$

or changing the integration order between dt and dV and eliminating

$\int_0^\infty \mathrm{G}(t) \exp(-pt)\, \mathrm{d}t$ as factor

$$\int_0^\infty \exp(-pt) \left\{ \int_{\mathrm{V}} \rho\, (a_{1i} s_{2i} - a_{2i} s_{1i})\, \mathrm{dV} + \int_{\Sigma} (b_{1i}\, s_{2i} - b_{2i} s_{1i})\, \mathrm{d}\Sigma \right\} = 0$$

and finally

$$\int_{\mathrm{V}} \rho\, (a_{1i}\, \mathrm{G}(t)\, s_{2i} - a_{2i}\, \mathrm{G}(t)\, s_{1i})\, \mathrm{dV} + \int_{\Sigma} (b_{1i}\, \mathrm{G}(t)\, s_{2i} - b_{2i}\, \mathrm{G}(t)\, s_{1i})\, \mathrm{d}\Sigma = 0$$

which proves that Betti's theorem is valid also when the stress strain relations are expressed by (3) with $f_1 = 0$.

APPENDIX D

The solution for Maxwell rheology is obtained by setting $\mathrm{H} = \nu$, $\mathrm{F} = 0$ in (4), where ν is the viscosity, and in (13). We obtain, by taking in LT^{-1} of (13) and considering D_n in (13) independent of p, which implies that the force applied to the boundary, in the time domain, is represented by $\delta(t)$,

$$s_1 = \sum_0^\infty {}_n \left\{ -\left(\frac{r}{r_0}\right)^2 \left[\frac{[n(n+1)\lambda + (n+1)(n+2)\mu]\,\delta(t)}{\mu\,[(2n^2+4n+3)\lambda + 2(n^2+n+1)\mu]} + \right.\right.$$

$$\left. + \frac{n(n+1)}{(2n^2+4n+3)\nu} \left\{ 1 - \frac{(2n^2+7n+6)(n-1)\mu^2 e^{-t\left[\frac{\left(\lambda+\frac{2}{3}\mu\right)\frac{\mu}{\nu}}{\lambda+\frac{2n^{2}+2n+2}{2n\;+4+3}\mu}\right]}}{3\,[(2n^2+4n+3)\lambda + 2(n^2+n+1)\mu]^2} \right\} \right] +$$

$$-(1-\delta_{1n}) \left[\frac{[n^2(n+2)\lambda + n(n+1)^2\mu]\,\delta(t)}{[(2n^2+4n+3)\lambda + 2(n^2+n+1)\mu](n-1)\mu} + \right.$$

$$+ \frac{n^2(n+2)}{(n-1)(2n^2+4n+3)\nu} \left\{ 1 - \frac{2(n-1)}{3(n+2)} \right.$$

$$\left.\left.\left. \frac{(n^2+5n+3)\mu^2 e^{-t\left[\frac{\left(\lambda+\frac{2}{3}\mu\right)\frac{\mu}{\nu}}{\lambda+\frac{2n^2+2n+2}{2n^2+4n+3}\mu}\right]}}{[(2n^2+4n+3)\lambda + 2(n^2+n+1)\mu]^2} \right\} \right] \right\}$$

$$\cdot \left(\frac{\quad}{r_0}\right)^{n-2} \frac{r}{2}\, \mathrm{Y}_{n0}\, \mathrm{D}_n\,.$$

$$
\text{(D 1)} \quad s_2 = \sum_{1}^{\infty} {}_n \Bigg\{ - \left(\frac{r}{r_0}\right)^2 \Bigg[\frac{[(n+3)\lambda + (n+5)\mu]\,\delta(t)}{[(2n^2+4n+3)\lambda + 2(n^2+n+1)\mu]\,\mu} +
$$

$$
+ \frac{n+3}{(2n^2+4n+3)\nu} \left\{ 1 + \frac{2(2n^2+5n+3)\,n(n-1)\,\mu^2 e^{-t\left[\frac{\left(\lambda+\frac{2}{3}\mu\right)\frac{\mu}{\nu}}{\lambda+\frac{2n^2+2n+2}{2n^2+4n+3}\mu}\right]}}{(n+3)\,[(2n^2+4n+3)\lambda + 2(n^2+n+1)\mu]^2} \right\} \Bigg] +
$$

$$
- (1 - \delta_{1n}) \Bigg[\frac{[n(n+2)\lambda + (n^2+n-1)\mu]\,\delta(t)}{[(2n^2+4n+3)\lambda + 2(n^2+n+1)\mu](n-1)\mu} +
$$

$$
- \frac{n(n+2)}{(n-1)(2n^2+4n+3)\nu}
$$

$$
\left\{ 1 - \frac{2(n-1)(n^2+5n+3)\,\mu^2 e^{-t\left[\frac{\left(\lambda+\frac{2}{3}\mu\right)\frac{\mu}{\nu}}{\lambda+\frac{2(n^2+n+1)}{2n^2+4n+3}\mu}\right]}}{3(n+2)\,[(2n^2+4n+3)\lambda + 2(n^2+n+1)\mu]^2} \right\} \Bigg] \Bigg\}
$$

$$
\cdot \left(\frac{r}{r_0}\right)^{n-2} \frac{r}{2} \frac{\partial Y_{n0}}{\partial \theta} D_n
$$

The effect of a constant force applied for a time T is obtained by convolving (D 1) with a Box in the time domain with duration T.

In the convolution, for $t > T$, the $\delta(t)$ term gives no contribution, each term independent of t gives a term proportional to T which is almost independent of n in s_1 and proportional to n^{-1} in s_2; finally there is the term depending on t formed by a factor exponentially increasing with T which, for n large, is proportional to n^{-1} in s_1 and almost proportional to n^{-2} in s_2. This factor is multiplied by an exponential in t; the exponent is almost the same in all terms; its dependence on n is represented in fig. 1 by the triangles.

The strong dependence on n of all the terms of s_2 is due to the structure of the mathematical solution (e.g. Caputo, 1961).

It is thus clear that the most important term is that depending on t and that, since the time components of all the terms of the series are almost wave number independent we may factor them out of the series which therefore depends only on the size of the cap of the sphere where the load is applied.

This implies that when the solution is found for a given history of the load and a particular geometry of the condition on the boundary, then the solution may be easily extended to other boundary conditions, with the same history; this is easily understood when the boundary conditions are expressed as in (23).

Obviously the solution (D 1) allows to find solutions for any time history of the same geometric boundary condition.

Another interesting case is that of a constant force applied to a region of the surface of the sphere which represents the case of mountain ranges. In this case the convolution of (D 1) with a constant has a constant term given by the $\delta(t)$ function and also by the exponential, a creep function, given by the constant term and a negative exponential of the type seen in (D 1).

The most important term is the creep with a rate of the order $r_0 D_n/2\nu$. For mountain ranges such as the Apennines, which were formed about 10 million years ago and now seem to decrease in height about 1 mm/year, or less, and have a relevant lack of isostasy, which would imply that the viscosity of the mantle plays an importat role, this creep rate would require a too-large Maxwellian viscosity unless active tectonic forces support the load of the mountain range.

Other interesting cases are the Apalachian and the Northern portion of the Rocky Mountains whose formation terminated 45 Myr and 225 Myr ago respectively and are isostatically compensated. With their present height these ranges seem to have extremely high viscosities, much in excess of the values generally accepted for the crust. The alternate possibility here is that the Maxwell rheology is not appropriate for all the crust of the Earth because the tectonic forces here apparently are not active.

APPENDIX E

We will find here the solution of the equations of elasticity for a layered sphere subject to surface tractions and to body forces.

Let us assume that in each shell i of a layered sphere, we may expand the components of the body force in series of spherical harmonics as follows

$$f_{1i} = D_{00i} + D_{10i}\, r^{-1}\, Y_{10} + D_{11i}\, r^{-1}\, Y_{11} + D_{12i}\, r^{-1}\, Y_{12} + \sum_{2}^{\infty}{}_n \sum_{0}^{2n}{}_k\, D_{nki}\, r^{-2}\, Y_{nk}$$

(E 1)

$$\begin{aligned} f_{2i} = {} & s_{10i}\, r^{-1} \frac{\partial Y_{10}}{\partial\theta} + S_{11i}\, r^{-1} \frac{\partial Y_{11}}{\partial\theta} + S_{12i}\, r^{-1} \frac{\partial Y_{12}}{\partial\theta} + \\ & + \frac{T_{10i}}{\sin\theta}\, r^{-1} \frac{\partial Y_{10}}{\partial\varphi} + \frac{T_{11i}}{\sin\theta}\, r^{-1} \frac{\partial Y_{11}}{\partial\varphi} + \frac{T_{12i}}{\sin\theta}\, r^{-1} \frac{\partial Y_{12}}{\partial\varphi} + \\ & + \sum_{2}^{\infty}{}_n \sum_{0}^{2n}{}_k\, S_{nki}\, r^{-2} \frac{\partial Y_{nk}}{\partial\theta} + \sum_{2}^{\infty}{}_n \sum_{0}^{2n}{}_k\, T_{nki} \frac{r^{-2}}{\sin\theta} \frac{\partial Y_{nk}}{\partial\varphi} \end{aligned}$$

$$\begin{aligned} f_{3i} = {} & \frac{r^{-1}}{\sin\theta} \left[S_{10i} \frac{\partial Y_{10}}{\partial\varphi} + S_{11i} \frac{\partial Y_{11}}{\partial\varphi} + S_{12i} \frac{\partial Y_{12}}{\partial\varphi} \right] + \\ & - r^{-1} \left[T_{10i} \frac{\partial Y_{10}}{\partial\theta} + T_{11i} \frac{\partial Y_{11}}{\partial\theta} + T_{12i} \frac{\partial Y_{12}}{\partial\theta} \right] + \\ & + \sum_{2}^{\infty}{}_n \sum_{0}^{2n}{}_k \frac{S_{nki}}{\sin\theta}\, r^{-2} \frac{\partial Y_{nk}}{\partial\varphi} - \sum_{2}^{\infty}{}_n \sum_{0}^{2n}{}_k\, r^{-2}\, T_{nki} \frac{\partial Y_{nk}}{\partial\theta} \end{aligned}$$

The r^{-1} and constant terms for $n = 0$ and $n = 1$ in (E 1) are introduced to make the solution for $n = 1$ and $n = 0$ (with non zero body forces) independent from those of the case of zero body forces; D_{nki}, S_{nki} and T_{nki} are constant.

Using the same procedure of Aquaro (1949) we find that a particular solution of the equations governing the elastic deformation of the shell when the body forces are expressed by (E 1) may be given as in (6) and (7) where U_{nki} V_{nki} W_{nki} are solution of the following system of differential equations for $n > 1$

(E 2)
$$\frac{\lambda + 2\mu}{r}\left[r^{-2}\frac{\partial}{\partial r}(r^2 U_{nki}) - n(n+1)r^{-1}V_{nki}\right] + $$
$$+ \mu r^{-1}\left[\frac{\partial^2(rV_{nki})}{\partial r^2} - \frac{\partial U_{nki}}{\partial r}\right] + S_{nki} r^{-2} = 0$$

$$(\lambda + 2\mu)\frac{\partial}{\partial r}\left[r^{-2}\frac{\partial}{\partial r}(r^2 U_{nki}) - n(n+1)r^{-1}V_{nki}\right] - \mu n(n+1)r^{-2}\left[U_{nki} - \right.$$
$$\left. - \frac{\partial(rV_{nki})}{\partial r}\right] + D_{nki} r^{-2} = 0$$

$$\frac{\mu}{r}\frac{\partial^2(rW_{nki})}{\partial r^2} - \frac{\mu n(n+1}{r^2}W_{nki} + T_{nki} r^{-2} = 0$$

which is a simplified version of system (6) of Caputo (1963). A solution (for $n > 1$) is

$$W_{nki} = T_{nki}/n(n+1)\mu$$

(E 3)
$$V_{nki} = -\frac{2(\lambda + 2\mu)D_{nki} + (\lambda + 2\mu + n(n+1)\mu)S_{nki}}{(\lambda + 2\mu)[\lambda + 4\mu - n(n+1)\mu]n(n+1)}$$

$$U_{nki} = -\frac{(\lambda + 2\mu)D_{nki} + (\lambda + 3\mu)S_{nki}}{(\lambda + 2\mu)(6 - 5n - 5n^2)\mu}$$

For $n = 0$, $U_{00i} = - D_{00i} r^2/4(\lambda + 2\mu)$

U_{1ki}, V_{1ki}, W_{1ki} constants are not solutions of system (E 2) with $n = 1$, we therefore try solutions proportional to r and find that T_{1ki} must be nil and

$$U_{1ki} = -\frac{(\lambda + \mu)D_{1ki} - 2\mu S_{1ki}}{4(\lambda + 2\mu)\mu}r$$

$$V_{1ki} = +\frac{2\mu S_{1ki} - (3\lambda + 5\mu)D_{1ki}}{8(\lambda + 2\mu)\mu}r$$

$$W_{1ki} = 0 .$$

The particular solution for the case of the layered spherical shell, when the body forces in all shells are expressed as in (E 1) with different values of D_{nji}, S_{nji}, T_{nji} in each shell, may be found with the method used by Caputo (1961).

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