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Projective invariant metrics and open convex regular cones. II

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Geometria differenziale. — *Projective invariant metrics and open convex regular cones.* II. Nota di FABIO PODESTÀ, presentata (*) dal Corrisp. E. VESENTINI.

ABSTRACT. — The aim of this work, which continues Part I with the same title, is to study a class of projective transformations of open, convex, regular cones in $\mathbf{R}^n$ and to prove a structure theorem for affine transformations of a restricted class of cones; we conclude with a version of the Schwarz Lemma holding for affine transformations.

KEY WORDS: Projective connections; Regular cones; Projective transformations.

RIASSUNTO. — *Metriche invarianti proiettive e coni aperti convessi regolari.* In questa Nota, proseguimento della Nota I dallo stesso titolo, si studia, nell'ambito dei coni aperti, convessi, regolari di $\mathbf{R}^n$, una classe di trasformazioni proiettive, nonché il gruppo delle trasformazioni affini, per il quale si fornisce un teorema di struttura ed un analogo del Lemma di Schwarz.

INTRODUCTION

This work, which continues Part I with the same title, is devoted to the study of projective transformations of open, convex, regular cones in $\mathbf{R}^n$: more precisely it will be shown (Theorem 4.1) that a particular class of projective automorphisms of the cones, introduced by Gentili ([4]), is a subgroup of the full projective transformation group; in view of the reduction theorem for projective transformations, proved in Part I, we furnish a structure theorem (Theorem 5.4) for affine transformations in the case of self-adjoint, affine-homogeneous and irreducible cones. We conclude with a version of the Schwarz Lemma holding for affine transformations (Theorem 5.4).

§ 4. A GROUP OF PROJECTIVE TRANSFORMATIONS

Gentili ([4]) has introduced a group of transformations acting on open convex regular cones Ω by considering the image $i(\Omega)$ under the embedding

$$(4.1)\qquad \begin{aligned} &i : \mathbf{R}^n \to \mathbf{P}^n \\ &i(x_1, \ldots, x_n) = (1, x_1, \ldots, x_n) \end{aligned}$$

(*) Nella seduta del 13 dicembre 1986.

(with $(x_0, x_1, \ldots, x_n)$ homogeneous coordinates on $\mathbf{P}^n$) and setting

$$\mathrm{GL}(\Omega, \mathbf{P}^n) = \{\psi \in \mathrm{PGL}(n, \mathbf{R}) \mid \psi(i(\Omega)) = i(\Omega)\}\,. \tag{4.2}$$

We want to prove that $\mathrm{GL}(\Omega, \mathbf{P}^n)$ is a subgroup of $\mathrm{Proj}(\Omega)$, when we read its action on Ω through the map i. The following lemma holds:

LEMMA 4.1. *Let* M *be a* C^∞*-manifold with symmetric connection: then a diffeomorphism is a projective transformation if and only if it maps geodesics into geodesics up to parametrization.*

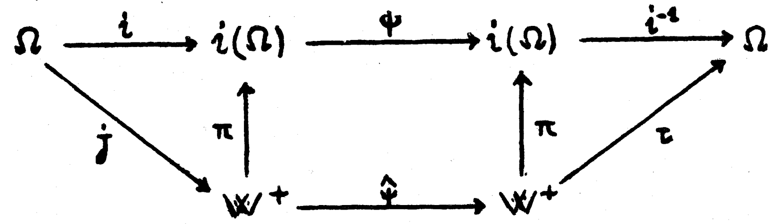
THEOREM 4.1. *Let* Ω *be an open convex regular cone in* $\mathbf{R}^n$. *Then* $\mathrm{GL}(\Omega, \mathbf{P}^n)$ *is a closed subgroup of* $\mathrm{Proj}(\Omega)$.

Proof. Let $\pi : \mathbf{R}^{n+1} \to \mathbf{P}^n$ be the canonical projection and define

$$\mathrm{W}^{\pm} = \{(x_0, x_1, \ldots, x_n) \in \mathbf{R}^{n+1} \mid (\pm x_1, \ldots, \pm x_n) \in \Omega \text{ and } x_0 \in \mathbf{R}^*_{\pm}\} \tag{4.3}$$

Since Ω is regular we have that $\pi^{-1}(\Omega) = \mathrm{W}^+ \cup \mathrm{W}^-$. Let $\psi \in \mathrm{GL}(\Omega, \mathbf{P}^n)$ and $\hat\psi \in \mathrm{GL}(n+1, \mathbf{R})$ inducing ψ: then $\hat\psi(\mathrm{W}^+ \cup \mathrm{W}^-) = \mathrm{W}^+ \cup \mathrm{W}^-$: since we study only the action of ψ on Ω, we can suppose that $\hat\psi(\mathrm{W}^+) = \mathrm{W}^+$, changing $\hat\psi$ into $-\hat\psi$ if necessary. Consider now the map $j : \Omega \to \mathrm{W}^+$ given by $j(x_1, \ldots, x_n) = (1, x_1, \ldots, x_n)$ and the map $\tau : \mathrm{W}^+ \to \{1\} \times \Omega \cong \Omega$ given by $\tau(x_0, x_1, \ldots, x_n) = (1, x_1/x_0, \ldots, x_n/x_0)$, so that the following diagram is commutative

(4.4)



From (4.4) it follows that the action of ψ on Ω is given by $\tau \circ \hat\psi \circ j$. Now W^+ is an open convex regular cone and its characteristic metric G is given by the direct sum of the metric on Ω and the Poincaré metric on $\mathbf{R}^*_+$. Both the maps j and $\hat\psi$ preserve geodesics with their affine parameter, hence by Lemma 4.1 we need only prove that τ preserves geodesics (in fact with their affine parameter too). Let $\gamma : \mathbf{R} \to \mathrm{W}^+$ be a geodesic with affine parameter t and write $\gamma(t) = (\sigma(t), u(t))$ with $\sigma(t) \in \mathbf{R}^*_+$ and $u(t) \in \Omega\ \forall t \in \mathbf{R}$. Then it easy to see that

$$\begin{aligned} &\frac{\mathrm{d}^2 u^i}{\mathrm{d}t^2} + \sum_{j,k} \Gamma^i_{jk}(u(t))\, \frac{\mathrm{d}u^j}{\mathrm{d}t}(t)\, \frac{\mathrm{d}u^k}{\mathrm{d}t}(t) = 0 \\ &\frac{\mathrm{d}^2 \sigma}{\mathrm{d}t^2}(t) - \frac{1}{\sigma(t)} \left(\frac{\mathrm{d}\sigma}{\mathrm{d}t}(t)\right)^2 = 0 \end{aligned} \tag{4.5}$$

where (Γ^i_{jk}) are as usual the Christoffel symbols for the metric g. We are left to check that the curve $v(t) = u(t)/\sigma(t)$ $\forall t \in \mathbf{R}$ is a geodesic in Ω. In fact $\forall i = 1, \dots, n$ and $\forall t \in \mathbf{R}$

$$
\begin{aligned}
\frac{d^2 v_i}{dt^2}(t) &= \frac{1}{\sigma(t)} \frac{d^2 u^i}{dt^2}(t) - 2 \frac{1}{(\sigma(t))^2} \frac{d\sigma}{dt}(t) \frac{du^i}{dt}(t) + \\
&+ 2 \frac{1}{(\sigma(t))^3} \frac{d\sigma}{dt}(t) u^i(t) - \frac{1}{(\sigma(t))^2} \frac{d^2\sigma}{dt^2}(t) u^i(t).
\end{aligned} \tag{4.6}
$$

By (4.5) we obtain that

$$
\begin{aligned}
\frac{d^2 v^i}{dt^2}(t) &= -\frac{1}{\sigma(t)} \sum \Gamma^i_{jk}(u(t)) \frac{du^j}{dt}(t) \frac{du^k}{dt}(t) + \\
&- 2 \frac{1}{(\sigma(t))^2} \frac{d\sigma}{dt}(t) \frac{du^i}{dt}(t) + \frac{1}{(\sigma(t))^3} \left(\frac{d\sigma}{dt}(t)\right)^2 u^i(t).
\end{aligned} \tag{4.7}
$$

We recall that $\forall x \in \Omega$ and $\forall t \in \mathbf{R}^*_+$ $\Gamma^i_{jk}(tx) = \frac{1}{t} \Gamma^i_{jk}(x)$ $\forall i, j, k = 1 \dots, n$ and so from (4.7) we obtain that

$$
\begin{aligned}
&\frac{d^2 v^i}{dt^2}(t) + \sum_{j,k} \Gamma^i_{jk}(v(t)) \frac{dv^j}{dt}(t) \frac{dv^k}{dt}(t) = \frac{d^2 v^i}{dt^2}(t) + \\
&+ \sigma(t) \sum_{j,l} \Gamma^i_{jl}(u(t)) \left[\frac{1}{(\sigma(t))^2} \frac{du^j}{dt}(t) \frac{du^l}{dt}(t) + \right. \\
&\left. + \frac{1}{(\sigma(t))^4} \left(\frac{d\sigma}{dt}(t)\right)^2 u^j(t) u^l(t)\right] + \\
&- 2\sigma(t) \sum_{j,l} \Gamma^i_{jl}(u(t)) \frac{1}{(\sigma(t))} \frac{d\sigma}{dt}(t) u^j(t) \frac{du^l}{dt}(t) = \\
&= \frac{d^2 v^i}{dt^2}(t) + \frac{1}{\sigma(t)} \sum_{j,l} \Gamma^i_{jl}(u(t)) \frac{du^j}{dt}(t) \frac{du^l}{dt}(t) + \\
&+ \frac{1}{(\sigma(t))^3} \left(\frac{d\sigma}{dt}(t)\right)^2 u^i(t) + 2 \frac{1}{(\sigma(t))^2} \frac{d\sigma}{dt}(t) \cdot \frac{du^i}{dt}(t).
\end{aligned} \tag{4.8}
$$

From (4.8) and (4.6) it follows that v is a geodesic with affine parameter t.

Q.E.D.

§ 5. THE GROUP OF AFFINE TRANSFORMATIONS OF AN IRREDUCIBLE CONE

In view of Theorem 2.2 if the open convex regular cone Ω in $\mathbf{R}^n$ is self-adjoint, affinely-homogeneous and irreducible, then $\operatorname{Proj}(\Omega) = \operatorname{Aff}(\Omega)$. We establish now a structure theorem about the group $\mathrm{Aff}(\Omega)$, under the same hypotheses on Ω.

THEOREM 5.1. (*a*) If $\{F_\lambda\}_{\lambda\in\mathbf{R}_+^*}$ *denotes the foliation* (3.6), *then if* $\psi\in\mathrm{Aff}(\Omega)$ $\forall\lambda\in\mathbf{R}_+^*\ \exists\mu\in\mathbf{R}_+^*$ *such that* $\psi(F_\lambda)=F_\mu$

(*b*) *Let* $\psi\in\mathrm{Aff}(\Omega)$. *Then* $\psi\in\mathrm{Iso}(\Omega)$ *if and only if there exists at least one point* $q\in\Omega$ *at which*

$$(5.1)\qquad d\psi_q(q)=\psi(q)\quad \textit{or}\quad d\psi_q(q)=-\psi(q)$$

(*under the usual identification* $T\Omega_x=\mathbf{R}^n$).

Proof. (*a*) For every $x\in\Omega$ let $T\Omega_x=\bigoplus_{i=0}^{k}T_x^{(i)}$ be the De Rham decomposition, where $T_x^{(0)}$ is the "euclidean" subspace.

LEMMA. $T_x^{(0)}=L_x$.

Proof. First of all we note that $\dim T_x^{(0)}\leq 1$, because otherwise there would be two distinct directions on which the Ricci tensor would vanish, contradicting Theorem 3.1. Since (Ω, g) is a symmetric space, at every point $x\in\Omega$ the algebra of holonomy is generated by $\{R(X,Y)\mid X,Y\in T\Omega_x\}$, where R is the curvature tensor, we have only to prove that $\forall x\in\Omega$

$$(5.2)\qquad \sum_j R^i_{jkl}(x)\,x^j=0\qquad \forall i,k,l=1,\dots,n$$

to obtain that $T_x^{(0)}\supseteq L_x$. Formula (5.2) can be derived directly from (3.9).
Q.E.D.

By a classical result of differential geometry (see e.g. Kobayashi-Nomizu ([7])) every affinity preserves, in the De Rham decomposition, the subspaces, $T_x^{(0)}$ and $\bigoplus_{i=1}^{k}T_x^{(i)}$: let now ψ be any affinity and $q\in\Omega$ any point of the cone, with W, W_1 the leaves through q and $\psi(q)$ respectively. If $q'\in W$, let τ be an arc of geodesic joining q and q': then τ lies in W, because W is totally geodesic. The image of τ under ψ is a geodesic with initial vector perpendicular to $T^{(0)}_{\psi(q)}=L_{\psi(q)}$ (by the Lemma), so that it lies entirely in W_1 and $\psi(q')\in W_1$.

(*b*) Let ψ be any affinity; ψ establishes an isometry between any leaf W and its image $\psi(W)$: indeed let $\Lambda\in\mathrm{Aut}(\Omega)$ be such that $\Lambda(W)=\psi(W)$ and consider $\psi^{-1}\circ\Lambda$ that maps W onto itself; since every leaf is Einstein with the induced metric, it follows that $\psi^{-1}\circ\Lambda$ is an isometry of W. Hence condition (5.1) is equivalent to say that ψ is an isometry at the point q. Our assertion follows then from a well known result about affinities (see e.g. Kobayashi-Nomizu ([7])). Q.E.D.

Since every affinity preserves the euclidean supspace $T_x^{(0)} = L_x$ (by the Lemma), we have that $\forall \psi \in \mathrm{Aff}(\Omega)$, $\forall x \in \Omega$, $\exists\, d_x \in \mathbf{R}^*$ such that

$$d\psi_x(x) = d_x \cdot \psi(x) . \tag{5.3}$$

It is a straightforward matter to verify that the map $d^\psi : \Omega \to \mathbf{R}^*$ defined by

$$d^\psi(x) = d_x \tag{5.4}$$

is of class C^∞. But much more is true:

THEOREM 5.2. *The function* d^ψ *is constant* ($\psi \in \mathrm{Aff}(\Omega)$).

Proof. From (5.3) we obtain

$$\| d\psi_x(x) \|^2_{\psi(x)} = d_x\, n \tag{5.5}$$

since

$$\sum_{i,j} g_{ij}(x)\, x^i x^j = n \qquad \forall x \in \Omega . \tag{5.6}$$

In order to prove the theorem, we compute

$$\begin{aligned} \frac{\partial}{\partial x^i} \| d\psi_x(x) \|^2_{\psi(x)} &= \frac{\partial}{\partial x^i} \Big(\sum_{jhkl} g_{jl}(\psi(x)) \big(\frac{\partial \psi^j}{\partial x^h}(x) \frac{\partial \psi^l}{\partial x^k}(x)\, x^h x^k \big) \Big) = \\ &= \sum_{jhklm} \frac{\partial g_{jl}}{\partial x^m}(\psi(x)) \frac{\partial \psi^m}{\partial x^i}(x) \frac{\partial \psi^j}{\partial x^h}(x) \frac{\partial \psi^l}{\partial x^k}(x)\, x^h x^k + 2 \sum_{jhl} g_{jl}(\psi(x)) \cdot \\ &\cdot \frac{\partial \psi^j}{\partial x^h}(x) \frac{\partial \psi^l}{\partial x^i}(x)\, x^h + 2 \sum_{jlhk} g_{jl}(\psi(x)) \frac{\partial^2 \psi^j}{\partial x^i \partial x^h}(x) \frac{\partial \psi^l}{\partial x^k}(x)\, x^h x^k . \end{aligned} \tag{5.7}$$

Because ψ is an affinity we can write that
$\forall x \in \Omega$ and $\forall h, j, l = 1, \dots, n$

$$\frac{\partial \psi^h}{\partial x^j \partial x^l}(x) = \sum_m \Gamma^m_{jl}(x) \frac{\partial \psi^h}{\partial x^m}(x) - \sum_{m,k} \Gamma^h_{km}(\psi(x)) \frac{\partial \psi^k}{\partial x^j}(x) \frac{\partial \psi^m}{\partial x^l}(x) . \tag{5.8}$$

Replacing (5.8) into (5.7) and using (5.6) we obtain that

$$\frac{\partial}{\partial x^i} d_x^2 = 0 \qquad \forall i = 1, \dots, n \qquad \text{Q.E.D.}$$

So we get a homomorphism $\mathrm{d} : \mathrm{Aff}(\Omega) \to \mathbf{R}_+^*$

$$\mathrm{d}(\psi) = |\mathrm{d}^\psi| \tag{5.9}$$

where we have used the same notation to indicate the function d^ψ and its constant value. By Theorem 5.1 the kernel of this homomorphism is exactly $\mathrm{Iso}(\Omega)$.

THEOREM 5.3. *The homomorphism* d *is surjective.*

Proof. Fix $c \in \mathbf{R}_+^*$ and choose any isometry I of the leaf F_1. We now put

$$x \in \Omega \qquad \psi(x) = \exp\{c \log[(\phi(x))^{-1/n}]\}\, \mathrm{I}(\phi(x)^{1/n} x). \tag{5.10}$$

It is easy to see that ψ is a diffeomorphism of Ω and that

$$\forall x \in \Omega \qquad \forall t \in \mathbf{R}_+^* \qquad \psi(tx) = \exp(c \log t)\, \psi(x). \tag{5.11}$$

The fact that ψ is an affinity will complete the proof, since (5.11) implies that $\mathrm{d}(\psi) = c$. To establish this last fact, we read the action of ψ on $\mathrm{F}_1 \times \mathbf{R}_+^*$ through the isometry ψ_1 (see (3.7)) considering

$$\hat{\psi} = \psi_1 \circ \psi \circ \psi_1^{-1} : \mathrm{F}_1 \times \mathbf{R}_+^* \to \mathrm{F}_1 \times \mathbf{R}_+^*.$$

Then $\forall r \in \mathbf{R}_+^*$, $\forall w \in \mathrm{F}_1$

$$\hat{\psi}(w, r) = (\mathrm{I}(w), r^c). \tag{5.12}$$

Since the map $r \to r^c$ of $\mathbf{R}_+^*$ onto itself is an affinity, when we endow $\mathbf{R}_+^*$ with the metric ds^2 (see Theorem 3.2), our assertion follows from (5.12). Q.E.D.

THEOREM 5.4. *Let the cone Ω be self-adjoint, homogeneous and irreducible. Then*

(*a*) $\mathrm{Iso}(\Omega)$ *is a closed, normal subgroup of* $\mathrm{Aff}(\Omega)$ *and the quotient group is isomorphic to* $(\mathbf{R}_+^* \cdot)$.

(*b*) *For any affinity* ψ, *if* $\psi^n \in \mathrm{Iso}(\Omega)$ *for some* $n \in \mathbf{Z}$, *then* $\psi \in \mathrm{Iso}(\Omega)$.

(*c*) *Let* γ_t *be any* 1-*parameter subgroup of affinities. If* $\gamma_{t_0} \in \mathrm{Iso}(\Omega)$ *for some* $t_0 \in \mathbf{R}^*$, *then* $\gamma_t \in \mathrm{Iso}(\Omega)$ $\forall t \in \mathbf{R}$.

(*d*) *If* ψ *is an affinity and if for some distinct* $\lambda_1, \lambda_2 \in \mathbf{R}_+^*$ $\psi(\mathrm{F}_{\lambda_1}) = \mathrm{F}_{\lambda_2}$ *then* $\psi \in \mathrm{Iso}(\Omega)$.

(*e*) *Let* ψ *be an affinity and put* $\mathrm{Fix}\,\psi = \{x \in \Omega \mid \psi(x) = x\}$. *If* $\mathrm{card}\,\{\phi(x) \mid x \in \mathrm{Fix}\,\psi\} \geq 2$, *then* $\psi \in \mathrm{Iso}(\Omega)$.

Proof. (*a*) follows immediately from Theorem 5.3

(*b*) If $\psi^n \in \mathrm{Iso}(\Omega)$ then $d(\psi^n) = (d(\psi))^n = 1$, hence $d(\psi) = 1$ and $\psi \in \mathrm{Iso}(\Omega)$.

(*c*) If $\hat{\gamma}_t$ is defined as $\hat{\gamma}_t = d \circ \gamma_t : \mathbf{R} \to \mathbf{R}^*_+$, then there is $a \in \mathbf{R}$ such that $\hat{\gamma}_t = \exp(at)\ \forall t \in \mathbf{R}$. Since $\exp(at_0) = 1$ and $t_0 \neq 0$, we have $a = 0$, tence $\gamma_t \in \mathrm{Iso}(\Omega)\ \forall t \in \mathbf{R}$.

(*d*) By integrating the relation (5.3), we get

$$\forall x \in \Omega,\ \forall t \in \mathbf{R}^*_+ \psi(tx) = \exp(d^\psi \log t)\, \psi(x). \tag{5.13}$$

Pick any $x \in F_{\lambda_1}$ and choose $t \in \mathbf{R}$ such that $t x \in F_{\lambda_2}$. Being $\lambda_1 \neq \lambda_2$, then $t \neq 1$. So $t^{-n} \phi(x) = \phi(t x) = \phi(\psi(tx)) = \phi(\exp(d^\psi \operatorname{logt})\, \psi(x)) = \exp(d \operatorname{logt})^{-n} \phi(\psi(x)) = \exp(d^\psi \operatorname{logt})^{-n} \phi(x)$

Hence

$$t = \exp(d^\psi \operatorname{logt}) \qquad t \neq 1$$

so that $d^\psi = 1$ and is an isometry.

(*e*) follows from (*d*). Q.E.D

We can now establish an analogue of the Schwarz Lemma holding for affine transformations.

THEOREM 5.5. *Let Ω be as in Theorem 5.4 and ψ any affine transformation If d_Ω denotes the Riemannian distance on Ω, one of the following relations is true.*

1) $\forall x, y \in \Omega \quad d_\Omega(\psi(x), \psi(y)) \leq d_\Omega(x, y)$;

2) $\forall x, y \in \Omega \quad d_\Omega(\psi(\sigma), \psi(y)) \geq d_\Omega(x, y)$.

Moreover if $d_\Omega(\psi(x), \psi(y)) = d_\Omega(x, y)$ for some distinct points $x, y \in \Omega$, then either $\phi(x) = \phi(y)$ or $\psi \in \mathrm{Iso}(\Omega)$.

Proof. Pick any two points $x, y \in \Omega$ and let γ be the minimizing geodesic joining x and y with arc parameter s, so that $\gamma(0) = x$ and $\gamma(s_0) = y$. Then

$$d_\Omega(\psi(x), \psi(y)) \leq \int_0^{s_0} \| d\psi_{\gamma(s)}\, \dot{\gamma}(s) \|_{\psi \circ \gamma}\, ds. \tag{5.14}$$

For every fixed $s \in [0, s_0]$ we decompose $\dot{\gamma}(s)$ as follows

$$\dot{\gamma}(s) = \dot{\gamma}_1(s) + \dot{\gamma}_2(s) \qquad \text{with} \tag{5.15}$$

$$\dot{\gamma}_1(s) \in L_{\gamma(s)} \quad \text{and} \quad \dot{\gamma}_2(s) \perp \dot{\gamma}_1(s).$$

Since ψ establishes an isometry between leaves, we ahve

$$
\begin{aligned}
(5.16)\qquad \| d\psi_{\gamma(s)} (\dot\gamma(s))\|^2_{\psi\circ\gamma(s)} &= \| d\psi_{\gamma(s)} (\dot\gamma_1(s)) \|^2_{\psi\circ\gamma(s)} + \| d\psi_{\gamma(s)} (\dot\gamma_2(s))\|^2_{\psi\circ\gamma(s)} = \\
&= \| \dot\gamma_1(s) \|^2_{\gamma(s)} + \| d\psi_{\gamma(s)} (\dot\gamma_2(s))\|^2_{\psi\circ\gamma(s)}
\end{aligned}
$$

Let now $t_0 \in \mathbf{R}$ be such that $\dot\gamma_2(s) = t_0\, \gamma(s)$ (s is fixed); such a real number t_0 exists because $\dot\gamma_2(s) \perp L_{\gamma(s)}$. Hence (5.3) and (5.6) imply that

$$
\begin{aligned}
(5.17)\qquad \| d\psi_{\gamma(s)} (\dot\gamma_2(s)) \|^2_{\psi\circ\gamma(s)} &= t_0^2 \| d\psi_{\gamma(s)} (\gamma(s)) \|^2_{\psi\circ\gamma(s)} = \\
&= t_0^2\, d^{\psi 2} \| \psi \circ \gamma(s) \|^2_{\psi\circ\gamma(s)} = t_0^2\, n\, (d(\psi))^2 .
\end{aligned}
$$

Noting that

$$
(5.18)\qquad t_0^2\, n = t_0^2 \| \gamma(s) \|_{\gamma(s)} = \| \dot\gamma(s) \|_{\gamma(s)}
$$

we obtain from (5.16), (5.17), (5.18) that

$$
(5.19)\qquad \| d\psi_{\gamma(s)} (\dot\gamma(s)) \|^2_{\psi\circ\gamma(s)} = \| \dot\gamma_1(s) \|^2_{\gamma(s)} + (d(\psi))^2 \| \dot\gamma_2(s) \|^2_{\gamma(s)} .
$$

If $d(\psi) = 1$ both (1) and (2) are true because ψ is an isometry. Suppose $d(\psi) < 1$: Then (5.19) and (5.14) imply that

$$
(5.20)\qquad d_\Omega (\psi(x), \psi(y)) \le \int_0^{s_0} \| \dot\gamma(s) \|_{\gamma(s)}\, ds = d_\Omega(x, y) .
$$

If $d(\psi) > 1$, we can apply the above argument to ψ^{-1} to obtain (2). Let now

$$
d_\Omega (\psi(x), \psi(y)) = d_\Omega(x, y) \qquad \text{for some } x \neq y \in \Omega .
$$

Then (5.14) and (5.19) imply that one of the following two possibilities occurs:

a) $d(\psi) = 1$ or

b) $\| \dot\gamma_2(s) \|_{\gamma(s)} = 0 \quad \forall s \in [0, s_0]$.

If (*a*) occurs, then $\psi \in \mathrm{Iso}(\Omega)$; if (*b*) occurs, then $\dot\gamma(s)$ is tangent to the leaf through $\gamma(s)$ and so x and y lie on the same leaf. Q.E.D.

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