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Integral representation and relaxation for functionals defined on measures


<http://www.bdim.eu/item?id=RLINA_1987_8_81_1_7_0>

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Abstract. — Given a separable metric locally compact space $\Omega$, and a positive finite non-atomic measure $\lambda$ on $\Omega$, we study the integral representation on the space of measures with bounded variation $\Omega$ of the lower semicontinuous envelope of the functional

$$F(u) = \int_{\Omega} f(x, u) \, d\lambda$$

with respect to the weak convergence of measures.

Key words: Relaxation; Integral representation; Measures.

Riassunto. — Rappresentazione integrale e rilassamento per funzionali definiti sulle misure. Dato uno spazio metrico localmente compatto a base numerabile $\Omega$ ed una misura $\lambda$ su tale spazio, positiva, finita e non atomica, si studia la rappresentazione integrale del funzionale ottenuto rilassando

$$F(u) = \int_{\Omega} f(x, u) \, d\lambda$$

nello spazio $M_n(\Omega)$ delle misure a variazione limitata su $\Omega$, rispetto alla topologia della convergenza debole di misure.

1. INTRODUCTION

In many problems of Calculus of Variations, given a functional $F$ defined on a topological space $(X, \tau)$, it is useful to introduce the so-called (sequentially) $\tau$-relaxed functional $\overline{F}$ defined by

$$\overline{F}(x) = \sup \{G(x) : G \text{ is sequentially } \tau \text{ - l.s.c.}, G \leq F\}.$$

where $G : X \rightarrow \mathbb{R}$ is said sequentially $\tau$-l.s.c. if and only if

$$G(x_\infty) \leq \liminf_{h \rightarrow +\infty} G(x_h)$$

for every sequence $(x_h) \subset X$ converging to $x_\infty \in X$ in the topology $\tau$.

(*) Pervenuta all'Accademia il 6 agosto 1986.
When F is an integral functional, it is interesting to find an integral representation for the relaxed functional $\overline{F}$. General results of this type have been obtained in the literature either when $\Omega$ is a bounded open subset of $\mathbb{R}^k$, $X$ is a Sobolev space $W^{1,p}(\Omega;\mathbb{R}^n)$, $\tau$ is the weak $W^{1,p}(\Omega;\mathbb{R}^n)$ topology (or the strong $L^p(\Omega;\mathbb{R}^n)$ topology) and

$$F(u) = \int_{\Omega} f(x, u, \nabla u) \, dx$$

(see for instance [1], [4], [9]), or when $X$ is a space $L^p(\Omega, \lambda;\mathbb{R}^n)$, $\tau$ is the weak $L^p(\Omega, \lambda;\mathbb{R}^n)$ topology (or the strong $L^p(\Omega, \lambda;\mathbb{R}^n)$ topology) and

$$F(u) = \int_{\Omega} f(x, u) \, d\lambda(x)$$

where $\lambda$ is a given measure on a separable locally compact metric space $\Omega$ (see for instance [3], [5], [7]).

In this paper we study the $\tau$-relaxation of functionals of the type (see Theorem 2.4)

$$F(\mu) = \begin{cases} \int_{\Omega} f(x, u) \, d\lambda(x) & \text{if } \mu = u \cdot \lambda \text{ with } u \in L^1(\Omega, \lambda;\mathbb{R}^n) \\ +\infty & \text{otherwise} \end{cases}$$

where $\mu$ belongs to the space $\mathcal{M}_n$ of the vector valued measures on $\Omega$ with bounded variation, $\tau$ is the weak topology of measures, $f : \Omega \times \mathbb{R}^n \to [0, +\infty]$ is a function (not necessarily measurable), and $\int^*$ denotes the upper integral. Similar results, under measurability hypotheses on $f$, have been obtained with different proofs in [2], [11], [14]. The proof of Theorem 2.4, which guarantees an integral representation for the relaxed functional, is based on an approach rather different from the one followed in [2], [11], [14], and relies on an abstract integral representation theorem for functionals $F(\mu, B)$, depending on measures $\mu \in \mathcal{M}_n$ and Borel sets $B \in \mathcal{B}(\Omega)$ (see Theorem 2.3).

The integral representation theorem given in this paper is the natural generalization of the theorem given in [3] concerning functionals $F(u, B)$, depending on $u \in L^1(\Omega, \lambda;\mathbb{R}^n)$ and $B \in \mathcal{B}(\Omega)$. Finally, in the last section of the paper we show some examples for which it is possible to compute explicitly the relaxed functional.

2. STATEMENT OF THE RESULTS.

In this section $(\Omega, \mathcal{B}, \lambda)$ will denote a measure space, where $\Omega$ is a separable metric locally compact space, $\mathcal{B}$ is the $\sigma$-algebra of the Borel subsets of $\Omega$, and $\lambda : \mathcal{B} \to [0, +\infty[$ is a positive, non-atomic, finite measure.
For every vector measure \( \mu : B \to \mathbb{R}^n \) and every \( B \in B \) the variation of \( \mu \) on \( B \) is defined by
\[
| \mu | (B) = \sup \left\{ \sum_{h=1}^{\infty} | \mu (B_h) | : B_h \in B, \bigcup_{h=1}^{\infty} B_h \subseteq B, B_h \text{ pairwise disjoint} \right\},
\]

We consider the following spaces:

- \( M_n \) the space of all vector measures \( \mu : B \to \mathbb{R}^n \) with finite variation on \( \Omega \);
- \( L_n^p \) the space of all \( \lambda \)-measurable functions \( u : \Omega \to \mathbb{R}^n \) with \( \int_{\Omega} | u |^p \, d\lambda < + \infty \);
- \( C_n^0 \) the space of all continuous functions \( u : \Omega \to \mathbb{R}^n \) "vanishing on the boundary", that is for every \( \varepsilon > 0 \) there exists a compact subset \( K_\varepsilon \) of \( \Omega \) such that \( | u(x) | < \varepsilon \) for all \( x \in \Omega - K_\varepsilon \).

The space \( M_n \) can be identified with the dual space of \( C_n^0 \) by the duality (see [13], page 40)
\[
\langle \mu, u \rangle_{\Omega} = \sum_{i=1}^{n} \int_{\Omega} u_i \, d\mu_i \quad (u \in C_n^0, \mu \in M_n),
\]
so that a sequence \( (\mu_h) \) in \( M_n \) is weak*-convergent to \( \mu \in M_n \) if and only if
\[
\langle \mu_h, u \rangle_{\Omega} \to \langle \mu, u \rangle_{\Omega} \quad \text{for every } u \in C_n^0.
\]

In the following, given \( u \in L_n^1 \), we denote by \( u \cdot \lambda \) the measure of \( M_n \) defined by
\[
(u \cdot \lambda) (B) = \int_{B} u \, d\lambda \quad \text{for every } B \in B.
\]

**Definition 2.1.** We say that \( \mu \in M_n \) is absolutely continuous with respect to \( \lambda \) (and we write \( \mu \ll \lambda \)) if
\[
| \mu | (B) = 0 \quad \text{whenever } B \in B \text{ and } \lambda (B) = 0.
\]

We say that \( \mu \in M_n \) is singular with respect to \( \lambda \) (and we write \( \mu \perp \lambda \)) if
\[
| \mu | (\Omega - B) = 0 \quad \text{for a suitable } B \in B \text{ with } \lambda (B) = 0.
\]

It is well-known that every absolutely continuous measure \( \mu \in M_n \) is representable in the form \( \mu = a \cdot \lambda \) for a suitable \( a \in L_n^1 \); moreover, the following Le-
besgue-Nyikodim decomposition result for measures of $\mathbf{M}_n$ holds (see [13] page 122).

**Proposition 2.2.** For every $\mu \in \mathbf{M}_n$ there exist a unique function $a \in L^1_n$ and a unique measure $\mu^s \in \mathbf{M}_n$ such that

i) $\mu = a \cdot \lambda + \mu^s$;

ii) $\mu^s$ is singular with respect to $\lambda$.

The function $a$ is often indicated by $\frac{d\mu}{d\lambda}$.

For proper convex functions $f : \mathbb{R}^n \to ]-\infty, + \infty]$ we define as usual the recession function of $f$ (see [12]) by

$$f^\infty(s) = \lim_{t \to +\infty} \frac{f(w + ts)}{t}$$

for every $s \in \mathbb{R}^n$,

where $w$ is any point in $\mathbb{R}^n$ such that $f(w) < + \infty$; in fact, the definition above is actually independent of the choice of $w$.

We are now in a position to state our integral representation result.

**Theorem 2.3.** Let $\Phi : \mathbf{M}_n \times \mathcal{B} \to ]-\infty, + \infty]$ be a functional satisfying the following properties:

i) $\Phi$ is $\mathcal{B}$-local (that is $\Phi(\mu, B) = \Phi(\nu, B)$ whenever $\mu, \nu \in \mathbf{M}_n$, $B \in \mathcal{B}$, and $|\mu - \nu|(B) = 0$);

ii) for every $\mu \in \mathbf{M}_n$ the set function $\Phi(\mu, \cdot)$ is finitely additive;

iii) the functional $\Phi(\cdot, \Omega)$ is convex and sequentially lower semicontinuous with respect to the weak*-convergence on $\mathbf{M}_n$;

iv) there exists $u_0 \in L^1_n$ such that $\Phi(u_0, B) < + \infty$ for every $B \in \mathcal{B}$;

v) for every $\mu \in \mathbf{M}_n$ singular with respect to $\lambda$ the function $t \to \Phi(u_0 + t\mu, \Omega)$ is positively 1-homogeneous.

Then, there exists a Borel function $\phi : \Omega \times \mathbb{R}^n \to ]-\infty, + \infty]$ such that

a) for every $x \in \Omega$ the function $\phi(x, \cdot)$ is convex and lower semicontinuous on $\mathbb{R}^n$;

b) there exist $a \in L^1_n$ and $b \geq 0$ such that $\phi(x, s) \geq -b|s| + a(x)$ for $\lambda$-a.e. $x \in \Omega$ and for all $s \in \mathbb{R}^n$;

c) the following integral representation formula holds for every $\mu \in \mathbf{M}_n$:

$$\Phi(\mu, B) - \Phi(u_0, B) = \int_B \phi(x, \frac{d\mu}{d\lambda}) \ d\lambda + \int_B \phi^\infty(x, \frac{d\mu^s}{d|\mu|}) \ d|\mu|$$
where \( \mu = \frac{d\lambda}{d\lambda} \cdot \lambda + \mu^s \) is the Lebesgue-Nykdim decomposition of \( \mu \), and for every \( x \in \Omega \), \( \phi^\infty(x, \cdot) \) is the recession function of \( \phi(x, \cdot) \).

d) the function \( \Phi^\infty(x, s) \) is lower semicontinuous in \( (x, s) \).

By using the integral representation theorem above, we can solve the relaxation problem which can be stated as follows. Let \( f : \Omega \times \mathbb{R}^n \to [0, +\infty] \) be a given function (note that no measurability hypotheses are required); for every \( \mu \in M_n \) define

\[
F(\mu) = \begin{cases} 
\int_{\Omega} f(x, u(x)) \, d\lambda(x) & \text{if } \mu = u \cdot \lambda \text{ with } u \in L^1_n \\
+\infty & \text{otherwise},
\end{cases}
\]

where \( \int^* \) denotes the upper integral. We are interested in the characterization of the greatest functional \( \Phi \) on \( M_n \) which is sequentially \( w^* \)-l.s.c. and less than or equal to \( F \); in particular, we want to write \( \Phi \) in the form

\[
\Phi(\mu) = \int_{\Omega} \phi^s(x, \frac{d\mu}{d\lambda}) \, d\lambda(x) + \int_{\Omega} \phi^\infty(x, \frac{d\mu^s}{d\mu}) \, d|\mu|
\]

for a suitable integrand \( \phi \). The following result holds.

**Theorem 2.4.** Assume that the functional \( F \) defined in (2.1) is finite in at least one \( u_0 \in L^1_n \). Then there exists a Borel function \( \phi(x, s) \), convex and lower semicontinuous in \( s \), such that (2.2) holds for every \( \mu \in M_n \). Moreover, \( \phi^\infty(x, s) \) is lower semicontinuous in \( (x, s) \).

### 3. Some Examples

In [10] Olech found a characterization of all integrands \( \phi \) such that the functional

\[
\Phi(\mu) = \int_{\Omega} \phi(x, u) \, d\lambda
\]

is sequentially \( w^* \)-\( M_n \) lower semicontinuous on the space \( L^1_n \). His result is that \( \Phi \) is sequentially \( w^* \)-\( M_n \) lower semicontinuous if and only if there exist a sequence of functions \( a_h \in L^1_n \) and a sequence of functions \( b_h \in C^0_n \) such that

\[
\phi(x, u) = \sup \{a_h(x) + \langle b_h(x), u \rangle : h \in N\} \quad \forall u \in \mathbb{R}^n
\]
for $\lambda$-a.e. $x \in \Omega$. By using this result, it is possible to find an explicit characterization of the integrand given by Theorem 2.4 in some interesting cases (see also for instance [6], [8]).

**Example 1.** Let $f : \Omega \times \mathbb{R}^n \to \mathbb{R}$ be the function

$$f(x, s) = a(x) |s|$$

where $a : \Omega \to [0, +\infty]$ is a measurable function. Then the relaxed functional $\Phi$ of Theorem 2.4 can be represented in the form (2.2) with $\phi$ given by

$$\phi(x, s) = \tilde{a}(x) |s|$$

where $\tilde{a}$ is the greatest lower semicontinuous function on $\Omega$ less than or equal to $a$ almost everywhere on $\Omega$. If $a \in L^1_{\text{loc}}(\Omega)$, it is easy to see that the following formula holds

$$\tilde{a}(x) = \liminf_{s \to x} \limsup_{s' \to 0} \frac{1}{B_p(y)} \int_{B_p(y)} a(t) \, dt$$

for every $x \in \Omega$.

**Example 2.** Let $a \in L^1$ and let $f : \Omega \times \mathbb{R}^n \to \mathbb{R}$ be the function

$$f(x, s) = a(x) \sqrt{1 + |s|^2}.$$  

If we denote by $\tilde{a}$ the function defined in (3.1), then the relaxed function $\Phi$ given by Theorem 2.4 can be represented in the form (2.2) with $\phi$ given by

$$\phi(x, s) = \begin{cases} 
\tilde{a}(x) \sqrt{1 + |s|^2} & \text{if } a(x) \leq \tilde{a}(x) \\
\frac{a(x) \sqrt{1 + |s|^2}}{\tilde{a}(x) - a(x)} & \text{if } a(x) > \tilde{a}(x) \text{ and } |s| \leq \frac{\tilde{a}(x) - a(x)}{\tilde{a}(x) - a(x)} \\
\tilde{a}(x) |s| + \sqrt{a^2(x) - \tilde{a}^2(x)} & \text{otherwise.}
\end{cases}$$

**Example 3.** Let $a : \Omega \to [0, +\infty]$ be a measurable function and let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be the function

$$f(x, s) = a(x) |s|^p$$

(with $p > 1$).

Then, the relaxed functional $\Phi$ given by Theorem 2.4 can be represented in the form (2.2) with $\phi$ given by

$$\phi(x, s) = a^*(x) |s|^p$$
where $a^*$ is the function

$$
\begin{align*}
\text{if } x \in \Omega - U & \\
0 & \\
+ \infty & \text{if } x \in U \text{ and } a(x) = 0 \\
a(x) & \text{otherwise}
\end{align*}
$$

and $U$ is the greatest open subset of $\Omega$ such that $a^{1/(1-p)} \in L^1_{\text{loc}}(U)$.

REFERENCES