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Meccanica dei continui. — *A completion of A. Bressan's work on axiomatic foundations of the Mach Painlevé type for various classical theories of continuous media.* Part 2. *Alternative completion of Bressan's work, fit for extension to special relativity.* Nota di ADRIANO MONTANARO, presentata (*) dal Corrisp. A. BRESSAN.

ABSTRACT. — The work [3] of axiomatization of various classical theories on continuous bodies from the Mach-Painlevè point of view, is completed here in a way which -unlike [4]- is suitable for extension to special relativity. The main reason of this is the fact that gravitation can be excluded in all the theories on continuous bodies considered here. Following [1], the notion of (physical) equivalence among affine inertial frames, and that of (physical isotropy of these frames are introduced; it is shown that the isotropic inertial frames equivalent to a fixed frame of this kind are those linked to this frame by a (proper) Galilean transformation. As in Part 1, the Euclidean physical metric on inertial spaces is consequently determined, without introducing it as a primitive notion. The treatment of Part 2 is referred to thermodynamic theories for continuous bodies and, as a particular case, to purely mechanic theories. In this last case, the primitive concepts are only the purely kinematical ones, presented in [3].

KEY WORDS: Axiomatization; Continuum; Thermodynamics.

RIASSUNTO. — *Un completamento del lavoro di A. Bressan sui fondamenti assiomatici alla Mach-Painlevè per varie teorie classiche dei mezzi continui. Parte 2. Completamento alternativo del lavoro di Bressan, adatto per la estensione alla relatività speciale.* In maniera alternativa a quanto fatto nella Parte 1 del presente lavoro (vedere [4]), si completa il lavoro [3] di assiomatizzazione alla Mach-Painlevè di varie teorie classiche di sistemi continui; ivi, tra l'altro, riguardo alla cinematica classica si arriva a definire i riferimenti inerziali affini.

Diversamente dalla Parte 1, qui non viene fatto uso delle forze gravitazionali e, seguendo [1], si introducono la nozione di equivalenza (fisica) tra riferimenti inerziali affini e quella di isotropia (fisica) di tali riferimenti; si dimostra che i riferimenti inerziali isotropi equivalenti ad un fissato tale riferimento, sono tutti e soli quelli legati a questo da una trasformazione Galileiana (propria); la metrica Euclidea fisica sugli spazi inerziali risulta quindi determinata, senza bisogno di introdurla come nozione primitiva.

La trattazione si riferisce ad una generica teoria termodinamica per sistemi continui o, come caso particolare, anche puramente meccanica; in questo ultimo caso i concetti primitivi assunti sono solo quelli puramente cinematici presentati in [3].

Tutte le teorie considerate nella presente parte possono escludere la gravitazione in quanto essa non viene mai usata; per questo, diversamente dalla Parte 1, la Parte 2 è adatta alla estensione alla relatività ristretta.

(*) Nella seduta del 20 giugno 1986.

N. 4. Physical equivalence between affine inertial frames in $\mathscr{T}_{r,s,0}$. Galilean principle of relativity expressed by them. Physical homogeneity and isotropy of space-time. Isotropic inertial frames and Galilean theorem of relativity expressed by them.

The whole of Part 2 refers to any theory $\mathscr{T}_{r,1,0}$ $(r = 0,1)$. By crossing out all references in them to temperature and related notions, one obtains the analogous considerations for $\mathscr{T}_{r,0,0}$ $(r = 0\,,1)$.

By Th. 5.3 (α) in [3], if $\varphi \equiv (x_\alpha)$ and $\psi \equiv (z_\alpha) \in$ CAIF [(1)] – see Def.s 5.1-2 and below Remark in N. 5 of [3] –, then the transformation $\psi \circ \varphi^{-1}$ is of the type

$$(4.1) \qquad \boldsymbol{z} = \mathbf{A}\,\boldsymbol{x} - \boldsymbol{b}\,x_0 - \boldsymbol{c}\,, \quad z_0 = \tau\,x_0 - c_0$$

for some $(\mathbf{A}\,, \boldsymbol{b}\,, \boldsymbol{c}\,, \tau\,, c_0) \in \mathrm{Lin}_* \times \mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{R}_* \times \mathbf{R}$,[(2)]

where

$$(4.2) \qquad \mathrm{Lin}_* = \{3 \times 3 \text{ real matrices } \mathbf{A} \mid \det \mathbf{A} \neq 0\}\,.$$

As in [4], if (4.1) represents $\psi \circ \varphi^{-1}$, then it is set

$$(4.3) \qquad \psi = \varphi^{\mathbf{A}\;\boldsymbol{b}\;\boldsymbol{c}}_{\;\;\tau\;c_0}$$

Following [1] the notion of physical equivalence is introduced.

Def. 4.1. *Assume that* φ *and* ψ *are in* CAIF; *they are said to be (physically) equivalent if the following condition holds : for every body* $\mathscr{B}$, $\boldsymbol{p}$ *is the* φ*-representation of a* C^2 *process that* $\mathscr{B}$ *can undergo in isolation – briefly a* $\mathrm{PIP}_{\mathscr{B}}$ *– if and only if* $\boldsymbol{p}$ *is the* ψ*-representation of a* $\mathrm{PIP}_{\mathscr{B}}$ – see Def. 2.4 in [4] –

Ax. 4.1. (Galilean principle of relativity for CAIF). *Each* $\varphi \in$ CAIF *is physically equivalent to every other* $\psi \in$ CAIF *for which* (4.1,3) *hold with* $\mathbf{A} = \mathbf{1}$ *and* $\tau = 1$.

Def. 4.1 and Ax. 4.1 imply the following:

Th. 4.1. *If* $\boldsymbol{p}$ *is the* φ*-representation of a* $\mathrm{PIP}_{\mathscr{B}}$, *then for every* $(\boldsymbol{v}\,, \boldsymbol{s}\,, u) \in \mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{R}$, $\{\boldsymbol{x} = \boldsymbol{f}\,(\boldsymbol{y}\,, t + u) + \boldsymbol{v}t + \boldsymbol{s}\,, \theta = \zeta\,(\boldsymbol{y}\,, t + u)\}_{(\boldsymbol{x},t) \in \Upsilon^*(\mathscr{P}) \times \mathbf{R}}$ *– see* (2.3) *in* [4] *– is the* φ*-representation of a* $\mathrm{PIP}_{\mathscr{B}}$.

Again from [1]:

(1) Greek [Latin] indices are meant to run from 0 [1] to 3.
(2) $\mathbf{R}_*^+ = \{x \in \mathbf{R} \mid x > 0\}$.

DEF. 4.2. *Space-time is said to be*

(a) *spatially (physically) homogeneous if each* $\varphi \in$ CAIF *is physically equivalent to every translate of its* (i.e. $\varphi^{1}{}_{1}{}^{0}{}_{0}{}^{c}$ *is physically equivalent to* $\varphi\ \forall\ \mathbf{c} \in \mathbf{R}^3$);

(b) *(physically) homogeneous with respect to time if any two arbitrary affine inertial frames which differ only by the time origin are physically equivalent;*

(c) *physically homogeneous if it is homogeneous both spatially and with respect to time.*

DEF. 4.3. *For* $\varphi \in$ CAIF, $\mathbf{Q} \in$ Orth+ (3) *and* $\varphi_{\mathbf{Q}} = \varphi^{\mathbf{Q}}{}_{1}{}^{0}{}_{0}{}^{0}$ ($\in$ CAIF, *see* (4.1–3)), $\varphi_{\mathbf{Q}}$ *is called the* **Q**-*rotate of frame* φ.

DEF. 4.4. $\varphi \in$ CAIF *is said to be physically isotropic (with respect to its origin) if* $\varphi_{\mathbf{Q}}$ *is physically equivalent to* φ, $\forall \mathbf{Q} \in$ Orth+.

Ax. 4.2. (Existence). *There exists some* $\varphi \in$ CIIF.

The class of these frames will be denoted by CIIF (*classical isotropic inertial rames*). Again from [1]:

DEF. 4.5. *Space-time is said to be*

(a) *physically isotropic at the event-point* $\mathscr{E}$ *if there exists a* $\varphi \in$ CIIF *for which* $\varphi(\mathscr{E}) = (0, 0, 0, 0)$;

(b) *physically isotropic if it is physically isotropic at each event-point.*

Def.s 4.4-5 and Ax.s 4.1-2 imply the following theorem.

Th. 4.2 (α) (Galilean principle of relativity expressed for CIIF). *If* $\varphi \in$ CIIF *and* $\psi \circ \varphi^{-1}$ *is a (proper) Galilean transformation* (4) *then* ψ ($\in$ CIIF) *is physically equivalent to* φ.

(β) *Space-time is physically homogeneous and isotropic.*

Now, differently from [1] where an axiom more powerful than Ax. 4.2 is stated – see [1, Ax. 28.1, p. 177] –, it remains to prove the converse of Th. 4.2 (α) — i.e. Th. 5.1 below –, which cannot be deduced without enunciating next Ax.s 5.1-2.

Next theorem is the version of Th. 4.1 for isotropic inertial frames.

Th. 4.3. *If* $\varphi \in$ CAIF, φ *is physically isotropic* $\Longleftrightarrow$ *any of conditions* (α) *and* (β) *below holds.*

(3) As in [4] assume that Orth $= \{A \in \mathrm{Lin}_* \mid AA^* = \mathbf{1}\}$–see (4.2)–, $\mathrm{Lin}_*^+ = \{A \in \mathrm{Lin}_* \mid \det A > 0\}$. Orth+ $= \{A \in \mathrm{Lin}_*^+ \mid AA^* = \mathbf{1}\}$, where A^* denotes the transpose of A, AA^* is the usual product between matrices, and $\mathbf{1}$ is the identity matrix.

(4) The transformation $\psi \circ \varphi^{-1}$ expressed by (4.1) is said to be (proper) *Galilean* if $A \in$ Orth+ and $\tau = 1$.

(α) [(β)] *For each body* $\mathscr{B}$, *for each* $\mathrm{PIP}_{\mathscr{B}}$ – see in Def. 4.1 –, *whose* φ-*representation is* $\{x = f(y, t), \theta = \zeta(y, t)\}_{(y,t)\in\Upsilon^*(\mathscr{P})\times\mathbf{R}}$, *and for each* $\mathbf{R} \in \mathrm{Orth}^+$ $[(\mathbf{R}, v, s, u) \in \mathrm{Orth}^+ \times \mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{R}]$, *the* $\{x = \mathbf{R} f(y, t), \theta = \zeta(y, t)\} \ldots$ $[\{x = \mathbf{R} f(y, t+u) + v t + s, \theta = \zeta(y, t+u)\} \ldots]$, *is the* φ-*representation of a* $\mathrm{PIP}_{\mathscr{B}}$ (5).

N. 5. Properties of co-ordinate transformations implied by the physical equivalence of two isotropic inertial frames

Assume that (i) $p = \{x = f(y, t), \theta = \zeta(y, t)\}_{(y,t)\in\Upsilon^*(\mathscr{P})\times\mathbf{R}}$ and (ii) $(\mathbf{A}, \alpha) \in \mathrm{Lin}_* \times \mathbf{R}^+_*$. I set

$$p_{\mathbf{A},\alpha} = \{x = \mathbf{A} f(y, \alpha t), \theta = \zeta(y, \alpha t\}_{(y,t)\in\Upsilon^*(\mathscr{P})\times\mathbf{R}} \tag{5.1}$$

Lemma 5.1. *Assume that* (i) φ *and* ψ *are physically equivalent* CAIFs, (*classical affine inertial frames*), *and* (ii) $\psi = \varphi^{\mathbf{A}}{}_{\tau}{}^{b}{}_{c_0}{}^{c}$ *for some* $(\mathbf{A}, b, c, \tau, c_0) \in \mathrm{Lin}_* \times \mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{R}^+_* \times \mathbf{R}$ – *see* (4.1-3) –. *Then for each body* $\mathscr{B}$, *if* p *is the* φ-*representation of a process in* $\mathrm{PIP}_{\mathscr{B}}$ *then both* $p_{\mathbf{A},\tau^{-1}}$ *and* $p_{\mathbf{A}^{-1},\tau}$ *are the* φ-*representations of a process in* $\mathrm{PIP}_{\mathscr{B}}$.

Proof. By Ax. 4.1 it is not restrictive to assume that φ and ψ are mutually joint and with the same origin, i.e. that $\psi \cdot \varphi^{-1}$ is

$$z = \mathbf{A} x, z_0 = \tau x_0 \text{ with } (\mathbf{A}, \tau) \in \mathrm{Lin}_* \times \mathbf{R}^+_*, \text{ – see (4.1-2) –.} \tag{5.2}$$

In addition let $p = < f, \zeta >$ be the φ-representation of a $\mathrm{PIP}_{\mathscr{B}}$ – see (2.3) and Def. 2.4 in [4] –, say P; then

$$\{z = \mathbf{A} f(y, \tau^{-1} z_0), \theta = \zeta(y, \tau^{-1} z_0)]\}_{(y,z0)\in\Upsilon^*(\mathscr{P})\times\mathbf{R}} \tag{5.3}$$

is the ψ-representtion of P (by (5.2)). The analogue for $\varphi \equiv (x_\alpha)$ of P's representation (5.3) in $\psi \equiv (z_\alpha)$ is

$$p_{\mathbf{A},\tau^{-1}} = \{x = \mathbf{A} f(y, \tau^{-1} x_0), \theta = \zeta(y, \tau^{-1} x_0)\} \ldots \qquad \text{(see (5.1).} \tag{5.4}$$

As φ and ψ are physically equivalent, $p_{\mathbf{A},\tau^{-1}}$ is the φ-representation of a $\mathrm{PIP}_{\mathscr{B}}$. q.e.d.

(5) The $\Rightarrow$ part follows from Def.s 4.1,4 and Ax. 4.1. To prove the $\Leftarrow$ part, choose $\mathbf{Q} \in \mathrm{Orth}^+$ and consider (i) $\varphi_{\mathrm{Q}} \equiv (\tilde{x}_\alpha)$–see Def. 4.3–, (ii) a body $\mathscr{B}$ and (iii) a $\mathrm{PIP}_{\mathscr{B}}$ whose φ-representation is $< \mathbf{f}, \zeta >$–see (2.3) in [4]–. I want to show that φ is isotropic. By (α), $[x = \mathbf{Q}^* f(y, t), \theta = \zeta(y, t)] \ldots$ is the φ-representation of a $\mathrm{PIP}_{\mathscr{B}}$. Hence $[\tilde{x} = f(y, \tilde{x}_0), \theta = \zeta(y, \tilde{x}_0)\} \ldots (\tilde{x}_0 = t)$, is the φ_{Q}-representation of a $\mathrm{PIP}_{\mathscr{B}}$. On the other hand, if $\{\tilde{x}_0 = g(y, \tilde{x}_0), \theta = \zeta(y, \tilde{x}_0)\} \ldots$ is the φ_{Q} representation of a $\mathrm{PIP}_{\mathscr{B}}$, then $\{x = \mathbf{Q}^* g(y, t), \theta = \zeta(y, t)\} \ldots$ is its φ-represetaion, and by (α) $\{x = g(y, t)$ $\theta = \zeta(y, t)\} \ldots$ is the φ-representation of a $\mathrm{PIP}_{\mathscr{B}}$. q.e.d.

Consider a process P for a body $\mathscr{B}$ which (i) is a rest process up to instant 0, and in which (ii) the acceleration field $\boldsymbol{a}$ is $\neq \mathbf{0}$ after 0. Consider the frame $\varphi^{1\,0\,0}_{\tau\,0}$, for $0 < \tau \neq 1$ – see (4.3) – and assume that it is physically equivalent to φ. Let $\mathscr{B}' \supseteq \mathscr{B}$ be the body which is isolated when $\mathscr{B}$ undergoes P, and let $P' \in PIP_{\mathscr{B}'}$ be the process which coincides with P on $\mathscr{B}$ – see Ax. 6.2 in [3] –; let $\boldsymbol{p}$ be the φ-representation of P'. Then, for $t \leq 0$ P' coincides with the process $\tilde{P}'$ whose φ-representation is $\boldsymbol{p}_{1,\tau}$ (which by Lemma 5.1 is physically possible); and after 0 we have $\mathbf{0} \neq \boldsymbol{a} \neq \tilde{\boldsymbol{a}}$, where $\tilde{\boldsymbol{a}}$ is the acceleration field in $\tilde{P}'$. This justifies the following

Ax. 5.1. *For $\tau \neq 1$ the frame $\varphi^{1\,0\,0}_{\tau\,0}$ is not physically equivalent to φ.*

Remark. Ax. 5.1 can be derived from the deterministic Ax. 5.1* below, existence Ax. 5.1** below, and Lemma 5.1 (β) (hence from Ax. 6.2 in [3]).

Ax. 5.1* (Principle of determinism in $\mathscr{T}_{r,s,0}$). *Assume that* (i) $\varphi = (x_\alpha) \in$ $\in$ CAIF, $\xi \in \mathbf{R}$, (ii) $\mathscr{B} = \mathbf{B}_{\mathscr{P}}$ *is a body, and* (iii) P *is a process for* $\mathscr{B}$; *then, there exists at most one process* P_ξ *that* $\mathscr{B}$ *can undergo in isolation and which coincides with* P *up to instant* ξ.

Ax. 5.1** (Existence). *Given a body* $\mathscr{B}$, $\varphi \in$ CAIF, *and* $\xi \in \mathbf{R}$, *there exists a process* $P_\xi \in PP_{\mathscr{B}}$ *which* (i) *fails to be a φ-rest process, but* (ii) *coincides with such a process up to* ξ.

Now let $\mathscr{B}$ be a body, $\varphi \in$ CAIF and assume that for $\tau > 0$ the frame $\varphi^{1\,0\,0}_{\tau\,0}$ is physically equivalent to φ. By Ax. 5.1** there exists a $PP_{\mathscr{B}}$, say P_0, satisfying conditions (i) and (ii). Let $\mathscr{B}' \supseteq \mathscr{B}$ be the body which is isolated when $\mathscr{B}$ undergoes P_0, let $P_0' \in PIP_{\mathscr{B}'}$ be the process which coincides with P_0 on $\mathscr{B}$ — see Ax. 6.2 in [3] —, and let $\boldsymbol{p}$ be its φ-representation. By Lemma 5.1, $\boldsymbol{p}_{1,\tau}$ is the φ-representation of a $PIP_{\mathscr{B}'}$, say $P'_{0,\tau}$; $P'_{0,\tau}$ and P_0' are processes which coincide up to 0, and, if $\tau \neq 1$, they do not coincide after 0, contradicting Ax. 5.1*. Hence $\tau = 1$, i.e. Ax. 5.1 follows from Ax.s 5.1* and 5.1**.

* * *

Below, the existence of a suitable body which is capable of exactly one rest process, up to constant translations and rotations, is postulated.

Ax. 5.2 (Existence). *Let φ be in* CIIF. *For some body* $\mathscr{B}$ (i) *some φ-rest process* P_0 *is a* $PIP_{\mathscr{B}}$; *furthermore* (ii) *if* $\{\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{y}), \theta = \zeta(\boldsymbol{y})\}_{\boldsymbol{y} \in \Upsilon^*(\mathscr{P})}$ *is the φ-representation of* P_0, *then each other φ-rest process that includes the same temperature distribution and can be undergone by* $\mathscr{B}$ *in isolation, has a φ-repretentation of the type* $\{\boldsymbol{x} = \mathbf{Q}\boldsymbol{f}(\boldsymbol{y}) + \boldsymbol{s}, \theta = \zeta(\boldsymbol{y})\}_{\boldsymbol{y} \in \Upsilon^*(\mathscr{P})}$ *for some* $(\mathbf{Q}, \boldsymbol{s}) \in \mathrm{Orth}^+ \times \mathbf{R}^3$.

Th. 5.1. *Assume that φ and ψ are physically equivalent* CIIFs. *Then the transformation $\psi \circ \varphi^{-1}$ is (proper) Galilean.*

Indeed by Ax. 4.1 it is not restrictive to suppose that $\psi \circ \varphi^{-1}$ is

(5.5) $\quad \boldsymbol{z} = \mathbf{A}\,\boldsymbol{x}\,, z_0 = \tau\, x_0$ for some $(\mathbf{A}, \tau) \in \mathrm{Lin}_* \times \mathbf{R}^+_*$ – see (4.1,2) –.

Choose $\mathscr{B}$ and P_0 as in Ax. 5.2 and let $\{\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{y}), \theta = \zeta(\boldsymbol{y})\}_{\boldsymbol{y}\in\gamma^*(\mathscr{P})}$ be the φ-representation of P_0; as ψ and φ are physically equivalent, $\{\boldsymbol{z} = \boldsymbol{f}(\boldsymbol{y}), \theta = \zeta(\boldsymbol{y})\} \ldots$ is the ψ-representation of another $\mathrm{PIP}_{\mathscr{B}}$, say $\tilde{P}_0$, as in Ax. 5.2 (i); since $\tilde{P}_0$ has the φ-representation $\{\boldsymbol{x} = \mathbf{A}^{-1}\boldsymbol{f}(\boldsymbol{y}), \theta = \zeta(\boldsymbol{y})\} \ldots$, by Ax. 5.2 (ii) $\mathbf{A}^{-1}$ and $\mathbf{A}$ are in Orth$^+$. Consider the $\mathbf{A}$-rotate $\varphi_{\mathbf{A}}$ of φ –see Def.s 4.3,4 –; as ψ and $\varphi' = \varphi_{\mathbf{A}}$ are physically equivalent and $\psi = (\varphi')^{1\,0\,0}_{\tau\,0}$, by Ax. 5.1 $\tau = 1$, i.e. $\psi \circ \varphi^{-1}$ is (proper) Galilean. q.e.d.

Th. 4.2 (α) and Th. 5.1 imply the following corollary:

COR. 5.1. *φ and ψ are physically equivalent* CIIFs *if and only if the transformation $\psi \circ \varphi^{-1}$ is (proper) Galilean.*

At this point the physical Euclidean metric can be introduced on each inertial space Σ_φ and on each φ-Inst_t (6) exactly as this is made at the end of N. 3 in [4] below the proof of Th. 3.3, except that one only has to replace CGIIF with CIIF and "gravitationally" with "physically".

Th. 5.2. *Assume that $\varphi \equiv (x_\alpha) \in$* CIIF *and $\psi \equiv (x_\alpha) \in$* CAIF. *Then $\psi \in$* CIIF *if and only if there exist $\tau, \delta > 0$ and $(\mathbf{Q}, \boldsymbol{b}, \boldsymbol{c}, c_0) \in$* Orth$^+ \times \mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{R}$ *for which $\psi \circ \varphi^{-1}$ is*

$$\boldsymbol{z} = e\,\delta\,\mathbf{Q}\,\boldsymbol{x} - \boldsymbol{b}\,x_0 - \boldsymbol{c}, \qquad z_0 = \tau\,x_0 - c_0 \qquad (e = \pm 1). \tag{5.6}$$

Proof. Assume $\varphi, \psi \in$ CIIF; by Ax. 4.1, the isotropy of φ and ψ, Th. 4.2 (α), and the polar decomposition theorem, one can assume that $\psi \circ \varphi^{-1}$ is

$$\boldsymbol{z} = e\,\mathbf{A}\,\boldsymbol{x}, \qquad z_0 = \tau\,x_0, \tag{5.7}$$

where $\mathbf{A}$ is a diagonal matrix with $a_{rr} = \delta_r > 0$ $(r = 1, 2, 3)$, $\tau > 0$, and $e = \pm 1$ - see (4.1) -. Let $\mathscr{B}$ and P_0 be as in Ax. 5.2; if $\{\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{y}), \theta = \zeta(\boldsymbol{y})\}_{\boldsymbol{y}\in\gamma^*(\mathscr{P})}$ is the φ-representation of P_0, then its ψ-representation is $\{\boldsymbol{z} = e\,\mathbf{A}\,\boldsymbol{f}(\boldsymbol{y}), \theta = \zeta(\boldsymbol{y})\} \ldots$ Choose an $\mathbf{R}$ in Orth$^+$ that carries the z_1-axis onto the z_2-axis. As $\psi \in$ CIIF, by Th. 4.3 $\{\boldsymbol{z} = e\,\mathbf{R}\,\mathbf{A}\,\boldsymbol{f}(\boldsymbol{y}), \theta = \zeta(\boldsymbol{y})\} \ldots$, is the ψ-representation of a $\mathrm{PIP}_{\mathscr{B}}$ $\tilde{P}_0$. Its φ-representation, by (5.7), is $\{\boldsymbol{x} = \mathbf{A}^{-1}\,\mathbf{R}\,\mathbf{A}\,\boldsymbol{f}(\boldsymbol{y}), \theta = \zeta(\boldsymbol{y})\} \ldots$ As $\psi \in$ CIIF, again by Th. 4.3, the $\{\boldsymbol{x} = \boldsymbol{g}(\boldsymbol{y}), \theta = \zeta(\boldsymbol{y})\} \ldots$ with

$$\boldsymbol{g}(\boldsymbol{y}) = \mathbf{R}^{-1}\,\mathbf{A}^{-1}\,\mathbf{R}\,\mathbf{A}\,\boldsymbol{f}(\boldsymbol{y}), \tag{5.8}$$

is the φ-representation of a $\hat{P}_0 \in \mathrm{PIP}_{\mathscr{B}}$. Now choose $\boldsymbol{y}_0, \boldsymbol{y}_1, \boldsymbol{y}_2 \in \gamma^*(\mathscr{P})$ such that, for $\boldsymbol{x}_i = \boldsymbol{f}(\boldsymbol{y}_i)$, $i = 1, 2$, it results that (i) $\boldsymbol{x}_i - \boldsymbol{x}_0$ is parallel to the x_i-

(6) Σ_φ denotes the set of those PWIMPs that have zero φ-velocity–see [3, Def. 3.5] and [2, p. 173]–; furthermore, for $t \in \mathbf{R}$, φ-$\mathrm{Inst}_t =_{\mathrm{D}} \{\mathscr{E} \in \mathrm{EP} \mid \varphi_0(\mathscr{E}) = t\}$ is said to be the *φ-instant of absciss t*–see [2, (5.7)]–, where EP is the set of event points, i.e. space-time.

axis ($i = 1, 2$), and (ii) $\mathbf{R}(\boldsymbol{x}_1 - \boldsymbol{x}_0) = \boldsymbol{x}_2 - \boldsymbol{x}_0$. Then [7]

$$|\boldsymbol{g}(\boldsymbol{y}_1) - \boldsymbol{g}(\boldsymbol{y}_0)| = |\mathbf{R}^{-1}\mathbf{A}^{-1}\mathbf{R}\mathbf{A}(\boldsymbol{x}_1 - \boldsymbol{x}_0)| = |\mathbf{R}^{-1}\mathbf{A}^{-1}\mathbf{R}\,\delta_1(\boldsymbol{x}_1 - \boldsymbol{x}_0)| =$$
$$= \delta_1 |\mathbf{R}^{-1}\mathbf{A}^{-1}(\boldsymbol{x}_2 - \boldsymbol{x}_0)| = \delta_1 \delta_2^{-1} |\mathbf{R}^{-1}(\boldsymbol{x}_2 - \boldsymbol{x}_0)| = \delta_1 \delta_2^{-1} |\boldsymbol{f}(\boldsymbol{y}_1) - \boldsymbol{f}(\boldsymbol{y}_0)|$$

– see (5.8) and (i), (ii) above –. Therefore $\delta_1 \neq \delta_2$ implies that the motions in the processes P_0 and $\hat{\mathrm{P}}_0$ are not congruent, contradicting Ax. 5.2. Hence $\delta_1 = \delta_2$; and by analogous reasons, $\delta_2 = \delta_3$.

Conversely assume now (i) $\varphi \in$ CIIF, $\psi \in$ CAIF, and (ii) (5.6) holds for some τ, $\delta > 0$ and $\mathrm{Q} \in \mathrm{Orth}^+$ (having assumed, as usual, $\boldsymbol{b} = \boldsymbol{c} = \mathbf{0}$, $c_0 = 0$). To prove that ψ is isotropic, choose a body $\mathscr{B}$ and a $\mathrm{P} \in \mathrm{PIP}_{\mathscr{B}}$ ψ-represented by $\{\boldsymbol{z} = \boldsymbol{f}(\boldsymbol{y}, z_0), \theta = \zeta(\boldsymbol{y}, z_0)\}_{(\boldsymbol{y}, z_0) \in \Upsilon^*(\mathscr{P}) \times \mathbf{R}}$. By (5.6) its φ-representation is $\{\boldsymbol{x} = e\,\delta^{-1}\mathbf{Q}^*\boldsymbol{f}(\boldsymbol{y}, \tau x_0), \theta = \zeta(\boldsymbol{y}, \tau x_0)\} \ldots$ As φ is isotropic, by Th. 4.3, for each $\hat{\mathbf{R}} \in \mathrm{Orth}^+$ $\{\boldsymbol{x} = e\,\hat{\mathbf{R}}\,\delta^{-1}\mathbf{Q}^*\boldsymbol{f}(\boldsymbol{y}, \tau x_0), \theta = \zeta(\boldsymbol{y}, \tau x_0\} \ldots$ is the φ-representation of a $\mathrm{PIP}_{\mathscr{B}}$, whose ψ-representation is $\{\boldsymbol{z} = \mathbf{Q}\hat{\mathbf{R}}\mathbf{Q}^*\boldsymbol{f}(\boldsymbol{y}, z_0), \theta = \zeta(\boldsymbol{y}, z_0)\} \ldots$ Chosing $\hat{\mathbf{R}} = \mathbf{Q}^*\mathbf{R}\mathbf{Q}$ with $\mathbf{R} \in \mathrm{Orth}^+$, it follows that $\{\boldsymbol{z} = \mathbf{R}\boldsymbol{f}(\boldsymbol{y}, z_0), \theta = \zeta(\boldsymbol{y}, z_0)\} \ldots$ is the ψ-representation of a $\mathrm{PIP}_{\mathscr{B}}$. By arbitrariness of $\mathscr{B}$, P, $\mathbf{R}$, and Def. 4.4, Th. 4.3, one conclude that ψ is isotropic. q.e.d.

N. 6. SOME CONNECTIONS BETWEEN PARTS 1 AND 2

Within the theory developed in [4] one can define the notions of gravitational homogeneity and gravitational isotropy of space-time in a way similar to that used here, in N. 4, where physical homogeneity and physical isotropy are defined. One only has to substitute " CAIF " and " physically " with " CGIIF " and " gravitationally " respectively in the Def.s 4.2 (*a*)-(*c*) and the Def. 4.5 within [4]. Then Th. 3.3 (δ) and Ax. 3.3 in [4], which are the analogues of Th. 4.2 (α) and Ax. 4.2 here respectively, imply that *space-time is gravitationally homogeneous and isotropic*, which assertion is the analogue of Th. 4.2 (β) here. Referring only to theories of the type $\mathscr{T}_{1,s,0}$ ($s = 0, 1$), by Th. 5.2 and its analogue Th. 3.3 (α) in [4], it follows that

(α) *assume that there exists a* φ *in* CIIF $\cap$ CGIIF *and that* $\psi \in$ CAIF; *then* $\psi \in$ CIIF *if and only if* $\psi \in$ CGIIF *and* ψ *and* φ *have the same orientation.*

It is trivial to prove that CIIF $\subseteq$ CGIIF. By Ax. 4.2 there is some φ in CIIF $\cap$ CGIIF, so that by (α):

(β) *if* φ *and* ψ *are gravitationally equivalent, then they are also physically equivalent if and only if they have the same orientation.*

Lastly observe that the deterministic proposition connecting of Ax. 4.1 turns out to be a theorem within the theory developed in [4] – see [4, Th. 3.3 (γ)] –.

(7) $|\boldsymbol{x} - \boldsymbol{x}'|$ is the Euclidean metric.

References

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