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The Normal Gravity Field in Space and a Simplified Model

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Geodesia. — *The Normal Gravity Field in Space and a Simplified Model.* Nota di ALBERTO BENAVIDEZ (*) e MICHELE CAPUTO (*), presentata (**) dal Socio M. CAPUTO.

RIASSUNTO. — Si estende allo spazio il campo di gravità normale secondo la teoria di Pizzetti–Somigliana usando il Sistema Geodetico di Riferimento (GRS 80), adottato dall'Assemblea dell'UGGI di Camberra nel 1979 per il calcolo de parametri che forniscono il modulo di gravità normale e la direzione della verticale. Per il calcolo della gravità al polo ed all'equatore si sono usate le forme chiuse dei teoremi di Pizzetti e Clairaut; tuttavia la differenza coi valori del GRS 80, sull'ellissoide di riferimento, è trascurabile.

Si presenta inoltre un modello semplificato del campo gravitazionale e si verifica che la differenza col campo gravitazionale normale è meno di 0.8 mgal per il modulo del vettore gravitazionale, meno di 1.67 m per la distanza fra le corrispondenti superficie equipotenziali misurata sulla normale ad una delle due superficie.

Si suggerisce inoltre un metodo rigoroso per il calcolo delle anomalie all'aria libera.

Tutte le formule usate sono espresse in forma chiusa.

The well known Pizzetti-Somigliana theory leads to the potential W of the gravity field of the Earth assuming that W is constant on an ellipsoid of revolution E_0 of semiaxes a_1 and a_3 $(a_1 > a_3)$. With reference to a cartesian x_1, x_2, x_3, and polar r, θ, μ and elliptic λ, ψ, μ reference frames we have

$$
\begin{aligned}
x_1 &= [(a_1^2 + \lambda)/d] \cos\psi \cos\mu = r \sin\theta \cos\mu \\
x_2 &= [(a_2^2 + \lambda)/d] \cos\psi \sin\mu = r \sin\theta \sin\mu \\
x_3 &= [(a_3^2 + \lambda)/d] \sin\psi = r \cos\theta \qquad (1) \\
d^2 &= a_1^2 \cos^2\psi \cos^2\mu + a_2^2 \cos^2\psi \sin^2\mu + a_3^2 \sin^2\psi + \lambda \\
r^2 &= x_1^2 + x_2^2 + x_3^2 \quad , \qquad s^2 = x_1^2 + x_2^2
\end{aligned}
$$

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(**) Nella seduta del 29 novembre 1986.

W is

$$(2)\qquad W = (G/\varepsilon\, a_3)\,(M - 2/3\, K_2)\,\tan^{-1} E + (GK_2/a_3^3\,\varepsilon^3)\,\{[\tan^{-1} E + - E/(1 + E^2)]\, s^2 + 2\,(E - \tan^{-1} E)\, x_3^2\} + \omega^2 s^2/2$$

$$g = g_{a1}\,[1 + (\alpha - f - f\,\alpha)\sin^2\psi]/[1 - f\,(2 - f)\sin^2\psi]^{1/2}$$

where

$$a_2 = a_1 \quad , \quad f = (a_1 - a_3)/a_1 \quad , \quad \alpha = (g_{a3} - g_{a1})/g_{a1}$$

$$\lambda = \{r^2 - a_3^2\,(2 + \varepsilon^2) + [r^4 + 2\, a_3^2\, \varepsilon^2\, r^2\,(2\cos^2\theta - 1) + a_3^4\,\varepsilon^4]^{1/2}\}/2$$

$$(3)\qquad E^2 = (a_1^2 - a_3^2)/(a_3^2 + \lambda) \quad , \quad \varepsilon^2 = (a_1^2 - a_3^2)/a_3^2$$

$$GK_2 = - \varepsilon^3\, a_3^3\, \omega^2\,(1 + \varepsilon^2)/2\,[(3 + \varepsilon^2)\tan^{-1}\varepsilon - 3\,\varepsilon]$$

$$C_{2n} = (-1)^n\,\{1 + 8\, n\, K_2/[3\,(2\,n + 3)\, M]\}\, f^n\,(2 - f)^n/(2\,n + 1)\,.$$

From W one obtains (e.g. Caputo 1967) the s anu x_3 components of g:

$$g_s = s\, G\,(M + 4\, K_2/3)\, d^2/(a_1^2 + \lambda)^2\,(a_3^2 + \lambda)^{1/2} - 2\, s\, GK_2\,[\tan^{-1} E - - E/(1 + E^2)]/a_3^3\,\varepsilon^3 - \omega^2 s$$

$$g_{x3} = x_3\, G\,(M + 4\, K_2/3)\, d^2/[(a_1^2 + \lambda)\,(a_3^2 + \lambda)^{3/2}] - 4\, x_3\, K_2\, G\,(E - \tan^{-1} E)/a_3^3\,\varepsilon^3$$

One obtains also the closed form expression of the Pizzetti and of the Clairaut theorems on the reference surface $E_0\,(\lambda = 0)$,

$$2\, g_{a1}/a_1 + g_{a3}/a_3 = 3\, GM/a_1^2\, a_3 - 2\,\omega^2$$

$$g_{a3}/g_{a1} = a_3/a_1 + 2\,\omega^2\, a_3\,\varepsilon^2\,(\varepsilon - \tan^{-1}\varepsilon)/\{g_{a1}\,[(3 + \varepsilon^2)\tan^{-1}\varepsilon - 3\,\varepsilon]\}\,.$$

The Geodetic Reference System 1980 adopted at the General Assembly of IUGG in Canberra (1979) is (SI units):

$$C_{20} = 0.00108263$$

$$a_1 = 6378137\, m$$

$$(5)\qquad MG = .39860050 \cdot 10^{15}\, m^3\, s^{-2}$$

$$\omega = .7292115 \cdot 10^{-4}\,\text{rad}\, s^{-1}$$

$$W_0 = 62{,}636{,}860.850\, m^2\, s^{-2}$$

With these values we obtain, using (3) and the Pizzetti's and Clairaut's theorems,

$$
\begin{aligned}
f^{-1} &= 298.257222101 \\
GK_2 &= -\,38477404975\ m^3\ s^{-2} \\
a_3 &= 6356752.3141\ m \\
g_{a1} &= 9.7803267715\ m\ s^{-2} \\
g_{a3} &= 9.8321863685 m\ s^{-2}\ .
\end{aligned}
\tag{6}
$$

With the values (6), we obtain also (2) and (4) for practical uses

$$
g = 9.7803267715\ \frac{1 + 0.0019318513533 \sin^2 \psi}{(1 - 0.0066943800229 \sin^2 \psi)^{1/2}}
\tag{7}
$$

$$
\begin{aligned}
g_s &= \{-\,8.0519073375 \cdot 10^{-4} [(1 + E^2) \operatorname{ctg}^2 \theta + 1]\, E^3/[(1 + E^2) \\
&((1 + E^2)^2 \operatorname{ctg}^2 \theta + 1)] + 5.4148782435 \cdot 10^{-3} [\tan^{-1} E - E/(1 + E^2)] - \\
&-\,.53174941173 \cdot 10^{-8}\}\, r \sin \theta \\
g_{r3} &= \{-\,8.0519073375 \cdot 10^{-4} [(1 + E^2) \operatorname{ctg}^2 \theta + 1]\, E^3/[(1 + E^2)^2 \\
&\operatorname{ctg}^2 \theta + 1] + 1.0829756487 \cdot 10^{-2} (E - \tan^{-1} E)\}\, r \cos \theta
\end{aligned}
\tag{8}
$$

which extend rigorously the normal gravity field in the space outside the reference ellipsoid E_0. λ is obtained from (3) as function of r and θ.

Formulae (6), (7) and (8) substitute formulae (34.9) of Caputo (1967) which were computed using the 1964 Geodetic Reference System (e.g. see Caputo 1967).

The use of (8) may be seen considering that (7) and (8) include the variation of the axifugal acceleration with height above the reference ellipsoid which is usually neglected in the dynamical levelling computations.

One may also use these formulas to eliminate the free air correction in the estimate of the gravity anomalies to be used in applied research. Let us consider the following expression

$$
g\,(P_2) - g_0\,(P_2) - BC
\tag{9}
$$

where $g\,(P)$ is the observed gravity, $g_0\,(P)$ is the value of the reference field in P and BC is the Bouguer correction; by applying the free air correction (FA) to both values we obtain the equivalent expression where P_1 is the projection of P_2 on the reference surface along the isozenithal

$$\begin{aligned} & g(P_2) - g_0(P_2) - BC = \\ (10) \qquad & = g(P_2) + FA - (g_0(P_2) + FA) - BC = \\ & = g(P_2) + FA - BC - g_0(P_1) = \Delta g \end{aligned}$$

which is the classic expression. Although the computation of $g_0(P_2)$ in (9) requires the knowledge of the coordinates of P_2, (9) is simpler than the classic expression. Moreover, the value of $g_0(P_1)$ is generally computed assuming for P_1 the same latitude of P_2, which is not rigorous. The use of (9) without the consideration of FA is more correct because it avoids the problem of downward projection, and also includes the correction of the axifugal acceleration caused by the variation of distance from the Earth axis of rotation.

The values of C_{2n} of the Pizzetti Somigliana field coincide with those used in IAG (1971) namely

$$(11) \qquad J_{2n} = (-1)^{n+1}\, 3\, e^{2n} (1 - n + 5\, n\, J_2/e^2)/((2\, n + 1)(2\, n + 3))$$

where $e^2 = \varepsilon^2 a_3^2/a_1^2$ which may be derived from (3) by expressing K_2/M in terms of C_{20} (obtained from (3) with $n = 1$) and $e^2 = f(2 - f)$ and then substituting in (3) for $n > 1$ (Caputo, 1967).

We may also note that the formula used in the IAG (1971) to obtain f from C_{20} coincides with that of the Pizzetti-Somigliana theory (that is (11) with $n = 1$) and obviously the values of the Pizzetti-Somigliana potential obtained from (2) coincide with those of the IAG (1971) because they represent the solution of the same boundary value problem in a different coordinate systems.

A difference between the numerical values of the Pizzetti-Somigliana field presented here and the GRS 80 is in the computation of g at the equator and the poles of the reference surface; in fact the values presented here are obtained from the rigorous closed form expression of the Pizzetti and Clairaut theorems. Here we give also rigorous closed formulas for the value of gravity in space.

The simplified space field

The formulas given here for the reference space gravity field according to the Pizzetti-Somigliana theory are simple. However, the values of W (Caputo 1967) are constant on surfaces which depart significantly from ellipsoids of revolution with the exception of E_0. The same applies to the case of the gravitational field where the axifugal term is disregarded.

These complications may be avoided by using the field introduced by Caputo (1966) in which it is assumed that the reference surface of the gravitational field is an ellipsoid of revolution E_0^* of semiaxes a_1^*, a_3^*, $(f^* = (a_1^* - a_3^*)/a_1^*)$.

The field resulting from E_0^* (Caputo 1967) has equipotential surfaces which are ellipsoids of revolutions confocal to E_0^*, the lines of force are hyperbolas confocal to E_0^*, and the isozenithal lines are hyperbolas (obviously not confocal to E_0^*). Other interesting properties of this field are found in its tidal field (Caputo 1985).

The potential of this simplified space field, based on a coordinate system similar to (1) where $a_1, a_3, \lambda, E, \psi$ are replaced by $a_1^*, a_3^*, \lambda^*, E^*, \psi^*$ respectively, is

$$V^* = (MG/(a_1^{*2} - a_3^{*2})^{1/2}) \tan^{-1} E^*$$

$$E^{*2} = (a_1^{*2} - a_3^{*2})/(a_3^{*2} + \lambda^*) \, , \quad \varepsilon^* = (a_1^{*2} - a_3^{*2})^{1/2}/a_3^*$$

the gravity g^* in space is

$$g^* = MG\,(1 + E^{*2} \cos^2 \psi^*)^{1/2}\, E^{*2}/[\varepsilon^{*2}\, a_3^{*2}\,(1 + E^{*2})] \, .$$

The extension of the Pizzetti and Clairaut theorems leads to

$$g_{a_1}^* (a_1^{*2} + \lambda^*)^{1/2} (a_3^{*2} + \lambda^*)^{1/2} + g_{a_3}^* (a_1^{*2} + \lambda^*) = 2\,MG$$

$$g_{a_1}^* /(a_1^{*2} + \lambda^*)^{1/2} = g_{a_3}^* /(a_3^{*2} + \lambda^*)^{1/2} \, .$$

The Fourier coordinates of V^* are

$$C_{2n}^* = (-1)^n\, [(2 - f^*)\, f^*]^n/(2\,n + 1)$$

$$C_{2n}^* = (-1)^n\, (3\, C_{20}^*)^n/(2\,n + 1) \, .$$

The relation between θ and ψ^* is

$$\tan \psi^* = \frac{a_1^{*2} + \lambda^*}{a_3^{*2} + \lambda^*} \operatorname{ctg} \theta \, .$$

The relation between f^* and f is

$$f^* = 1 - \{1 - f\,(2 - f)\,[1 - 4\, \varepsilon^3\, a_3^3\, \omega^2\, (1 + \varepsilon^2)/15\, MG\, ((3 + \\ + \varepsilon^2) \tan^{-1} \varepsilon - 3\, \varepsilon)]\}^{1/2}$$

Assuming $C_{20}^* = C_{20}$, $a_1^* = a_1$ and MG of GRS 80 we find

$$a_3^* = 6367770.8324\, m$$

$$f^* = 0.0016252657444$$

$$GM/(a_3^{*2}\, \varepsilon^{*2}) = 3016.8161206; \; GM/(a_3^{*3}\, \varepsilon^{*3}) = 0.0082995460806$$

$$V_0^* = 62528695.197$$

$$g^* = 3016.8161206\, (1 + E^{*2} \cos^2 \psi^*)^{1/2}\, E^{*2}/(1 + E^{*2})$$

$$g_s^* = \left\{ \left[0.0082995460806\, ((1 + E^{*2}) \operatorname{ctg}^2 \theta + 1)\, E^{*3} \right] / \left[(1 + E^{*2})\, ((1 + \\ + E^{*2})^2 \operatorname{ctg}^2 \theta + 1) \right] \right\} r \sin \theta$$

$$g^*_{x3} = \left\{ 0.0082995460806 \, [((1 + E^{*2}) \operatorname{ctg}^2 \theta + 1) \, E^{*3}/(1 + (1 + E^{*2})^2 \operatorname{ctg}^2 \theta)] \right\} r \cos\theta$$

$$E^{*2} = .13212621654 \cdot 10^{12}/(.40548505374 \cdot 10^{14} + \lambda^*) \; ; \; V^* = 1.0965876226 \cdot$$

$$\cdot \, 10^9 \tan^{-1} (0.13212621654 \cdot 10^{12}/(0.40548505374 \cdot 10^{14} + \lambda^*))^{1/2}$$

$$a_3^{*2} + \lambda^* = r^2 (1 + a_3^{*2} \, \varepsilon^{*2}/r^2 + [(1 - a_3^{*2} \, \varepsilon^{*2}/r^2)^2 + 4 (\varepsilon^{*2} \, a_3^{*2} \cos^2 \theta)/ r^2]^{1/2})/2$$

$$a_1^{*2} + \lambda^* = r^2 (1 - a_3^{*2} \, \varepsilon^{*2}/r^2 + [(1 - a_3^{*2} \, \varepsilon^{*2}/r^2)^2 + 4 (\varepsilon^{*2} \, a_3^{*2} \cos^2 \theta)/ r^2]^{1/2})/2 \, .$$

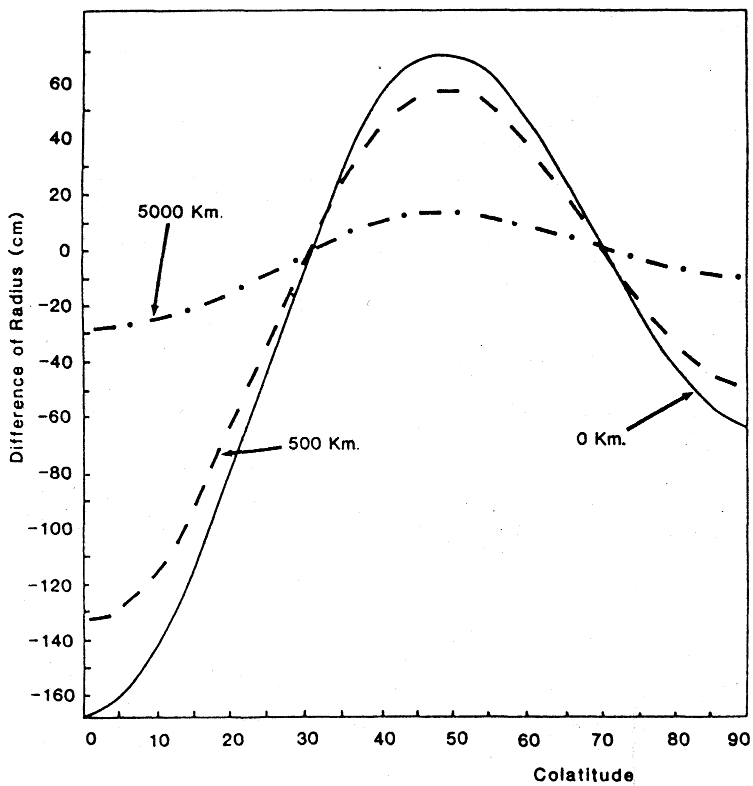

Fig. 1. – Distance between the equipotential surfaces of the normal gravitational field and the corresponding surface of the simplified model measured along the radius to the centre of mass—as function of the geocentric co-latitude. To obtain the value measured along the normal to any of the two surfaces one should multiply by the cosine of angle between the normal and the direction to the centre which is at least cos (0.094°). The values are given on equipotential surfaces with elevation of 500 km and 5,000 km above the reference equipotential surface at the equator, as function of the geocentric co-latitude.

The reference equipotential surface E_0^*, with these numerical values, ($V_0^* = 62528695.197$) departs from $\bar{E}_0$-equipotential surface of the normal gravitational field through $(a_1 , 0 , 0)$, i.e. with $W_0 - \frac{\omega^2 a_1^2}{2} = V_0 = 62528701.340$ –, by at most 1.3 meter. The values of moduli of the gravitational vector on $\bar{E}_0$ depart by at most 1.0 mgal with respect to the case of Pizzetti-Somigliana field where the axifugal term has been removed, and the latitudes depart by at most $7.0 \cdot 10^{-7}$ radians.

The parameters of the simplified space field were computed also assuming $V^* = V_0$. It is shown that this equipotential surface departs from $\bar{E}_0$ at most 1.7 *m*, the moduli of the gravitation vectors differ less than 0.8 mgals and the latitudes depart by at most 7×10^{-7} rad as it may be seen in Fig. 1, 2 and 3. In the surrounding space these differences decrease easily as it is shown in the same figures for $V = V^* = 57978820.723$ (with $\lambda^* = 6628137 \times 10^6$) and $V = V^* = 35038102.610$ (with $\lambda^* = 8878137 \times 10^7$) where the equipotential of the simplified model are at an elevation above the equator of 500 km and 5,000 km, respectively.

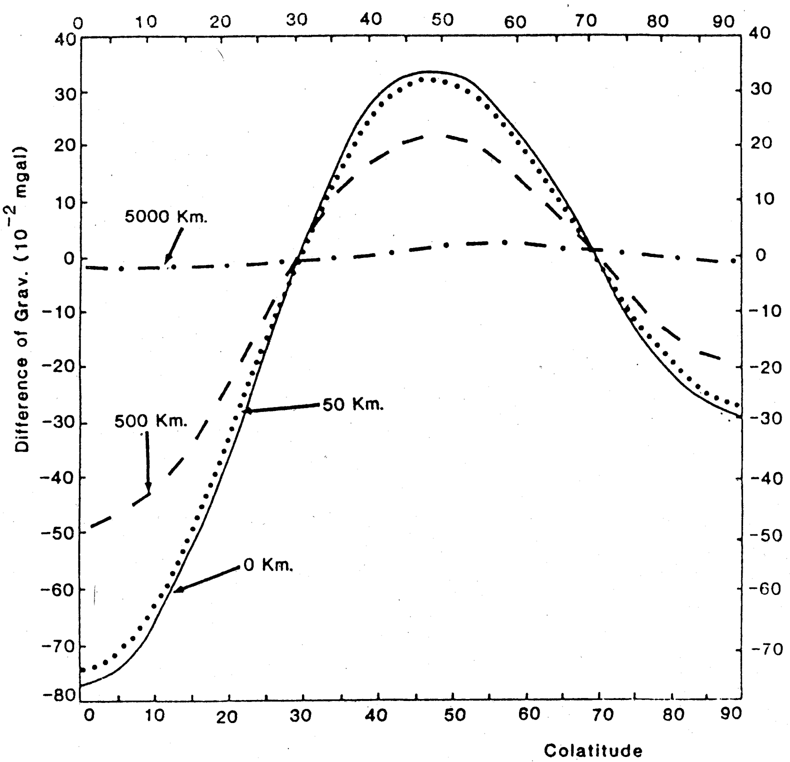

Fig. 2. – Difference between the moduli of the gravitationa vecto:s of the normal and the simplified fields on the equipotential surfaces with elevation of 50 km, 500 km, 5,000 km, above the reference equipotential surface, at the equator as function of the geocentric co-latitude.

An interesting property of the potential V^* is that the angle γ of the tidal reference frame with the canonical one, to first approximation in $\psi^* - \theta$, is (Caputo 1985).

$$\gamma \sim 1/3\,(\psi^* - \theta) = \bar{f}/3\, K_1$$

where $\bar{f}$ is the curvature of the line of force and K_1 the curvature of the meridian, $\psi^* - \theta$ reaches its largest value on E_0^* and is less than 1.8 millirad as it appears in Figure 4.

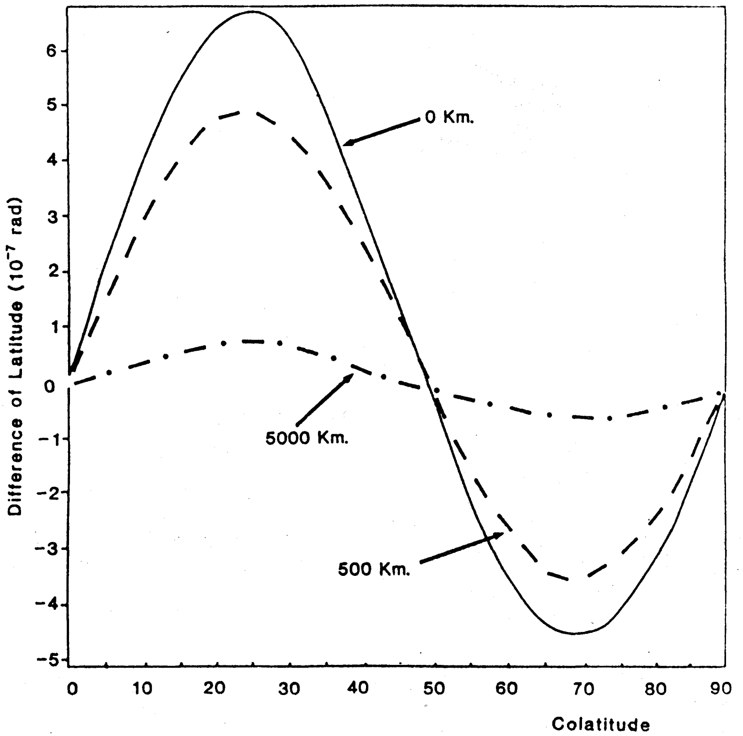

Fig. 3. - Difference between the directions of the gravitational vectors of the normal and the simplified fields on the equipotential surfaces with elevation of 500 km, 5,000 km, above the reference equipotential surface, at the equator as function of the geocentric co-latitude.

Conclusions

The values of the modulus of gravity of the normal gravity field of the Earth and of the direction of the vertical derived from the GRS 80 are extended in space according to the Pizzetti-Somigliana theory.

The difference between the set of formulas used here to obtain, from GM, C_{20}, a_1, and ω, the other parameters of the gravity field and the set used in IAG 1971 concerns only the values of the modulus gravity on the reference surface and in the surrounding space. In fact, here the values of g at the pole (g_{a3}) and at the equator (g_{a1}) are derived using the two rigorous, closed form expressions of the Pizzetti and Clairaut theorems.

A simpler model of the gravitational field of the Earth in space is also presented. It is shown that the difference between the modulus of the gravita-

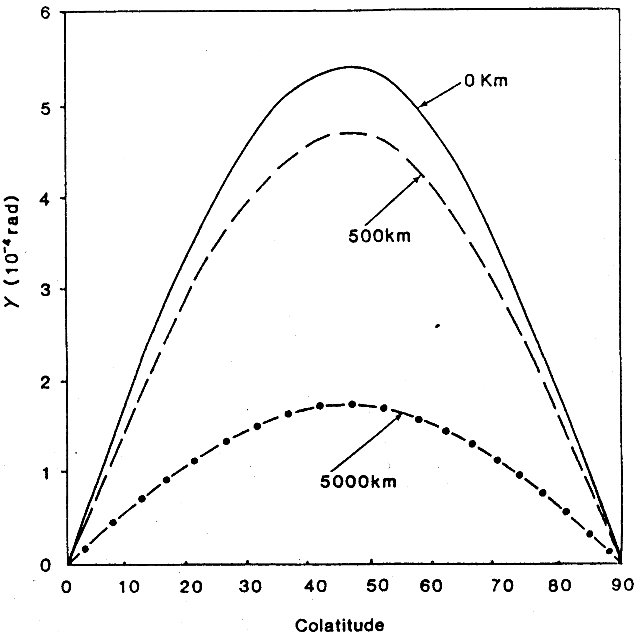

Fig. 4. – Angle γ for various heights of the equipotentials surface above the equator for the simplified model.

tion vector of this field and that of the normal gravitational field is less than 1 mgal and that the distance between the equipotential surfaces, taken on the normal to any of the two of them, is less than 1.7 m.

We also suggest a simpler and more accurate way of expressing the gravity anomalies.

All the formulas used are expressed in close form.

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