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## On hyperbolic partial differential equations in Banach spaces

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**Equazioni a derivate parziali.** — *On hyperbolic partial differential equations in Banach spaces.* Nota (*) di BOGDAN RZEPECKI, presentata dal Corrisp. R. CONTI.

RIASSUNTO. — Viene dimostrata l'esistenza di soluzioni del problema di Darboux per l'equazione iperbolica $z''_{xy} = f(x, y, z, Z'_x, z_y)$ sul planiquarto $x \geq 0$, $y \geq 0$. Qui, $f$ è una funzione continua, con valori in uno spazio Banach che soddisfano alcune condizioni di regolarità espresse in termini della misura di non-compattezza $\alpha$.

We prove the existence of solutions of the Darboux problem for the hyperbolic equation $z''_{xy} = f(x, y, z, z'_x, z'_y)$ on the quarter-plane $x \geq 0$, $y \geq 0$. Here $f$ is a continuous function with values in a Banach space satisfying some regularity condition expressed in terms of the measure of noncompactness $\alpha$.

Throughout this paper $\mathbf{R}^+ = [0, \infty)$, $Q = \mathbf{R}^+ \times \mathbf{R}^+$, E is a Banach space with norm $\|\cdot\|$, and $f$ is an E-valued continuous function defined on the set $\Omega = Q \times E \times E \times E$. Further, let $\alpha$ denote the Kuratowski's measure of noncompactness in E (see e.g. [2]), and $C^1(Q, E)$ the standard Fréchet space of all E-valued functions $z$ continuous on Q together with their partial derivatives $z'_x$ and $z'_y$.

By (+) we shall denote the problem of finding a solution $z \in C^1(Q, E)$ of the partial differential equation of the hyperbolic type

$$z''_{xy} = f(x, y, z, z'_x, z'_y)$$

satisfying the initial conditions

$$z(x, 0) = 0 \quad , \quad z(0, y) = 0$$

for $x \geq 0$ and $y \geq 0$.

Hartman and Wintner [3] have shown that, for the existence of a solution of (+) with real function $f$, it suffices to suppose that $f$ is continuous bounded and satisfies a Lipschitz condition with respect to the two last variables. Using the fixed point theorem of Sadovskii ([4], Theorem 3.4.3) we prove the existence of solutions of (+) provided for $f$ some regularity condition expresed in terms of $\alpha$.

Denote by $S_\infty$ the set of all non-negative real sequences with standard partial order $\leq$; for $\xi, \eta \in S_\infty$ we write $\xi < \eta$ if $\xi \leq \eta$ and $\xi \neq \eta$. Let us state the Sadovskii theorem in the following form:

(*) Pervenuta all'Accademia il 28 ottobre 1986.

Let $\mathscr{V}$ be a closed convex subset of $C^1(Q, E)$, and $\Phi$ a function which maps each non-empty subset $Z \subset \mathscr{V}$ to a sequence $\Phi(Z) \in S_\infty$ sucht hat (1) $\Phi(\{z\} \cup Z) = \Phi(Z)$ for $z \in \mathscr{V}$, (2) $\Phi(\overline{\text{conv}}\, Z) = \Phi(Z)$ (here $\overline{\text{conv}}\, Z$ is the closed convex hull of Z), and (3) if $\Phi(Z) = \theta$ (the zero sequence) then $\bar{Z}$ is compact in $C^1(Q, E)$. Assume that F is a continuous mapping of $\mathscr{V}$ into itself and $\Phi(F[Z]) < \Phi(Z)$ whenever $\Phi(Z) > \theta$. Then F has a fixed point in $\mathscr{V}$.

For $Z \subset C^1(Q, E)$ we denote by $Z'_x$, $Z'_y$ and $Z(s, t)$, respectively, sets of all $z'_x$, $z'_y$, $z(s, t)$ such that $z \in Z$. The lemma below is an adaptation of the corresponding result of Ambrosetti [1].

LEMMA. *If* W *is a bounded equicontinuous subset of usual Banach space of continuous* E*-valued functions defined on a compact subset* P *of* Q, *then* $\alpha(\bigcup \{Z(x, y) : (x, y) \in P\}) = \sup_{(x,y) \in P} \alpha(Z(x, y))$.

Our result reads as follows.

THEOREM. *Let* $f$ *be uniformly continuous on bounded subsets of* $\Omega$, *and let* $\|f(x, y, z, p, q)\| \leq G(x, y, \|z\|, \|p\|, \|q\|)$ *for* $(x, y, z, p, q) \in \Omega$. *Suppose that for each bounded subset* D *of* $\Omega$ *and a compact subset* P *of* Q *there exist constants, respectively,* L(D) *and* M(P) *such that*

$$\|f(x, y, z, p_1, q_1) - f(x, y, z, p_2, q_2)\| \leq$$
$$\leq L(D) \cdot \max[\|p_1 - p_2\|, \|q_1 - q_2\|] \text{ on } D$$

*and*

$$\alpha(f[P \times W_1 \times W_2 \times W_3]) \leq M(P) \cdot \max[\alpha(W_1), \alpha(W_2), \alpha(W_3)]$$

*for all bounded subsets* $W$, $W_2$, $W_3$ *of* E. *Assume in addition that* G *is a continuous function which is nondecreasing with respect to the three last variables and the scalar integral equation*

$$g(x, y) = \int_0^x \int_0^y G(s, t, g(s, t), g'_x(s, t), g'_y(s, t))\, ds\, dt$$

*has a solution* $g \in C^1(Q, \mathbf{R}^+)$.

*Under the above hypotheses,* (+) *has at least one solution on* Q.

*Proof.* Fix a positive integer $n$. Put $Q_n = [0, n] \times [0, n]$. Let

$$a_n = \sup_{(s,t) \in Q_n} g(s, t),\ b_n = \sup_{(s,t) \in Q_n} g'_x(s, t),\ c_n = \sup_{(s,t) \in Q_n} g'_y(s, t),$$
$$d_n = \sup_{(s,t) \in Q_n} G(s, t, g(s, t), g'_x(s, t), g'_y(s, t))\, ds\, dt$$

and

$$D_n = \left\{(x, y, z, p, q) \in Q : (x, y) \in Q_n, \|z\| \leq a_n, \|p\| \leq b_n, \|q\| \leq c_n\right\}.$$

Let us denote by $\delta_n(\varepsilon)$ $(\varepsilon \geq 0)$ the supremum of the numbers $\|f(x_1, y_1, z_1, p_1, q_1) - f(x_2, y_2, z_2, p_2, q_2)\|$ such that $\max[|x_1 - x_2|, |y_1 - y_2|, \|z_1 - z_2\|, \|p_1 - p_2\|, \|q_1 - q_2\|] \leq \varepsilon$ whenever $(x_i, y_i, z_i, p_i, q_i) \in D_n$ $(i = 1,2)$. We define

$$\sigma_n(x, y, \varepsilon) = n \cdot \delta_n(h_n \varepsilon) \cdot \exp(L(D_n) y),$$

$$\tau_n(x, y, \varepsilon) = n \cdot \delta_n(h_n \varepsilon) \cdot \exp(L(D_n) x)$$

for $x, y, \varepsilon$ in $\mathbf{R}^+$, where $h_n = \max(1, b_n, c_n, d_n)$.

Let $\mathscr{V}_0$ be the set of all $z \in C^1(Q, E)$ with $\|z(s, t)\| \leq g(s, t)$, $\|z'_x(s, t)\| \leq g'_x(s, t)$ and $\|z'_y(s, t)\| \leq g'_y(s, t)$ on Q. By $\mathscr{V}$ we represent the set of all $z \in \mathscr{V}_0$ such that

$$\|z'_x(s + \varepsilon, t) - z'_x(s, t)\| \leq \sigma_n(s, t, \varepsilon), \|z'_y(s + \varepsilon, t) - z'_y(s, t)\| \leq d_n \varepsilon$$

and

$$\|z'_y(s, t + \varepsilon) - z'_y(s, t)\| \leq \tau_n(s, t, \varepsilon), \|z'_x(s, t + \varepsilon) - z'_x(s, t)\| \leq d_n \varepsilon$$

$(n = 1, 2, \ldots)$ for $(s, t)$ in $Q_n$ and with sufficiently small $\varepsilon > 0$. It is not hard to see that $\mathscr{V}$ is a convex closed bounded subset of $C^1(Q, E)$, and $\mathscr{V}$ $\mathscr{V}'_x$, $\mathscr{V}'_y$ are almost equicontinuous on Q.

Putting

$$(F z)(x, y) = \int_0^x \int_0^y f(s, t, z(s, t), z'_x(s, t), z'_y(s, t)) \, ds \, dt$$

we shall write (+) in the form $z = F z$ (in $\mathscr{V}$), because F is a continuous mapping of $\mathscr{V}$ into itself.

Let

$$\varphi_n(Z) = \max\Big[\sup_{(s,t)\in Q_n} \exp(-r_n(s + t))\, \alpha(Z(s, t)),$$
$$\sup_{(s,t)\in Q_n} \exp(-r_n(s + t))\, \alpha(z'_x(s, t)),$$
$$\sup_{(s,t)\in Q_n} \exp(-r_n(s + t))\, \alpha(Z'_y(s, t))\Big].$$

$(n = 1, 2, \ldots)$ for a nonempty subset Z of $\mathscr{V}$, where $r_n > 0$. Now we shall show the basic inequality: $\min(r_n, r_n^2) \cdot \varphi_n(F[Z]) \leq 3 M_n \cdot \varphi_n(Z)$ with $M_n = M(Q_n)$.

To this end, fix $(x, y)$ in $Q_n$. Let

$$\mu_0(s, t) = \alpha(Z(s, t)), \mu_1(s, t) = \alpha(Z'_x(s, t)), \mu_2(s, t) = \alpha(Z'_y(s, t))$$

for $(s, t) \in Q$. For any given $\varepsilon > 0$ there exists a $\delta > 0$ such that $(s', t')$, $(s'', t'') \in [0, x] \times [0, y]$ with $|s' - s''| < \delta$ and $|t' - t''| < \delta$ implies $3 M_n |\mu_k(s', t') - \mu_k(s'', t'')| < \varepsilon$ $(k = 0, 1, 2)$. We divide the intervals $[0, x]$ and $[0, y]$ into $m$ parts

$$x_0 = 0 < x_1 < \dots < x_m = x, \quad y_0 = 0 < y_1 < \dots < y_m = y$$

in such a way that $\max |x_i - x_{i-1}| < \delta$ and $\max |y_i - y_{i-1}| < \delta$. Put:

$$P_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j],$$
$$W_{ij}^{(0)} = \cup \{Z(s, t) : (s, t) \in P_{ij}\},$$
$$W_{ij}^{(1)} = \cup \{Z'_x(s, t) : (s, t) \in P_{ij}\} \text{ and}$$
$$W_{ij}^{(2)} = \cup \{Z'_y(s, t) : (s, t) \in P_{ij}\}$$

for $i, j = 1, 2, \dots, m$. By Lemma we obtain

$$\alpha(W_{ij}^{(k)}) = \sup_{(s,t) \in P_{ij}} \mu_k(s, t) \ (k = 0, 1, 2);$$

let $(s_i^{(k)}, t_j^{(k)})$ be a point in $P_{ij}$ with $\alpha(W_{ij}^{(k)}) = \mu_k(s_i^{(k)}, t_j^{(k)})$.

Let $\mu(s, t) = \mu_0(s, t) + \mu_1(s, t) + \mu_2(s, t)$ on $Q$.

Applying the integral mean value theorem, we get

$$\alpha(F[Z](x, y)) \le \alpha\left(\sum_{i,j=1}^{m} \operatorname{mes}(P_{ij}) \overline{\operatorname{conv}} (f[Q_n \times W_{ij}^{(0)} \times W_{ij}^{(1)} \times W_{ij}^{(2)}])\right) \le$$
$$\le M_n \sum_{i,j=1}^{m} \operatorname{mes}(P_{ij}) \max[\alpha(W_{ij}^{(0)}), \alpha(W_{ij}^{(1)}), \alpha(W_{ij}^{(2)})] \le$$
$$\le M_n \left\{ \int_0^x \int_0^y \mu(s, t)\, ds\, dt + \right.$$
$$\left. + \sum_{k=0}^{2} \sum_{i,j=1}^{m} \int_{P_{ij}} |\mu_k(s, t) - \mu_k(s_i^{(k)}, t_j^{(k)})|\, ds\, dt \right\} <$$
$$< M_n \cdot \int_0^x \int_0^y \mu(s, t)\, ds\, dt + \varepsilon xy;$$

therefore

$$\alpha (F [Z] (x , y)) \leq M_n \cdot \int_0^x \int_0^y \mu (s , t) ds dt \leq$$

$$\leq 3 M_n \cdot \varphi_n (Z) \cdot \int_0^x \int_0^y \exp (r_n (s + t)) ds dt$$

and consequently

$$r_n^{-2} \cdot \sup_{(x,y) \in Q_n} \exp (- r_n (x + y)) \alpha (F [Z] (x , y)) \leq 3 M_n \cdot \varphi_n (Z) .$$

Arguments similar to the above imply that

$$r_n^{-1} \cdot \sup_{(x,y) \in Q_n} \exp (- r_n (x + y)) \max [ \alpha (F [Z]'_x (x , y)) , \alpha (F [Z]'_y (x , y)) ] \leq$$

$$\leq 3 M_n \cdot \varphi_n (Z) ,$$

and our inequality is proved.

Define:

$\Phi (Z) = (\varphi_1 (Z) , \varphi_2 (Z) , \dots)$ for a non-empty subset Z of $\mathscr{V}$. Evidently, $\Phi (Z) \in S_\infty$. By properties of $\alpha$ and Ascoli theorem the function $\Phi$ satisfy conditions (1)-(3) listed above.

Assume $r_n > \max (1 , 3 M_n)$ for $n \geq 1$. Then from our basic inequality it follows that $\Phi (F [Z]) < \Phi (Z)$ whenever $\Phi (Z) > \theta$. Thus all assumptions of Sadovskii's fixed point theorem are satisfied, F has a fixed point in $\mathscr{V}$ and the proof is complete.

## References

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[4] B.N. Sadovskii (1972) – *Limit compact and condensing operators*, « Russian Math. Surveys », *27*, 86-144.