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Boundedness results of solutions to the equation $x''' + ax'' + g(x)x' + h(x) = p(t)$ without the hypothesis $h(x) \operatorname{sgn} x \geq 0$ for $|x| > R$.

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Equazioni differenziali. — *Boundedness results of solutions to the equation* $x''' + ax'' + g(x)x' + h(x) = p(t)$ *without the hypothesis* $h(x) \operatorname{sgn} x \geq 0$ *for* $|x| > R$. Nota di JAN ANDRES, presentata (*) dal Corrisp. R. CONTI.

RIASSUNTO. — Per l'equazione differenziale ordinaria non lineare del 3° ordine indicata nel titolo, studiata da numerosi autori sotto l'ipotesi $h(x) \operatorname{sgn} x \geq 0$ per $|x| > R$, si dimostra l'esistenza di almeno una soluzione limitata sopprimendo l'ipotesi suddetta.

1. HISTORICAL REMARKS

About some fifteen years ago there was still under consideration a very actual question at that time of boundedness of solutions to the following Liénard-type third order equations

$$x''' + ax'' + g(x)x' + h(x) = p(t), \tag{1}$$

where a is a positive real, $g(x)$, $h(x) \in C^1(R^1)$ and $p(t) \in C^1(R^+)$.

Assuming either (see e.g. [1-5])

$$b \leq g(x) \leq G \qquad (b\text{, G–suitable positive reals}) \tag{2}$$

or [1, 6, 7] $G(x)/x \geq b$ for all $x \in R^1$, where $G(x) = \int^x g(s)\,ds$,

$$\limsup_{|x|\to\infty} |h(x)| < \infty \tag{3}$$

together wi'h

$$\liminf_{|x|\to\infty} h(x) \operatorname{sgn} x > 0 \tag{4}$$

or

$$\limsup_{|x|\to\infty} h'(x) < ab \tag{5}$$

(*) Nella seduta del 29 novembre 1986.

together with

(6) $$\lim_{|x|\to\infty} h(x) \operatorname{sgn} x = \infty$$

respectively, and

(7) $$\limsup_{t\to\infty} |p(t)| < \infty ,$$

(8) $$\left| \int_0^\infty p(t)\, dt \right| < \infty ,$$

many dissipativity results then had been carried out.

Furthermore, J.O.C. Ezeilo (jointly with H.O. Tejumola) pointed out [2] that (6) is superfluous with respect to (4) and (5) and the same author also noticed [3] that the conditions (7) and (8) are under (6) interchangeable.

K.E. Swick has succeeded [4] moreover in replacing (5) by a more liberal restriction, namely

$$\liminf_{|x|\to\infty} a\,(a\mathrm{G}(x)/x - 2h(x)/x) - (a^2/2 - \mathrm{G}(x)/x - \alpha h(x)/x)^2$$

with a suitable constant $\alpha > 2(a^2 + b)/a(a^2 + 2b)$. The same author still has studied [5] the more general case (cf. (4), (6))

(9) $$h(x) \operatorname{sgn} x \geq 0 \qquad \text{for } |x| > \mathrm{R} \quad (\text{R-suitable positive real})$$

in spite of replacing (7), (8) by

(10) $$\int_0^\infty |p(t)|\, \mathrm{d}t < \infty$$

in order to obtain the following result:

(11) $$\lim_{t\to\infty} x(t) = \overline{x} , \qquad \lim_{t\to\infty} x'(t) = \lim_{t\to\infty} x''(t) = 0 ,$$

satisfied for all solutions of (1) with $\overline{x}$ being the appropriate zero points of $h(x)$; but note that so far only (9) has been used everywhere (see also [6, 7] to get the Lagrange-like stability.

REMARK 1. *Recently the present author has shown* [8] *that also the oscillatory restoring term without* (9) *may imply the Lagrange-like stability of* (1), *provided the distances between the zero points of* $h(x)$ *are large enough.*

On the other side, J. Voráček has proved [7] that (1) admits a solution tending to the infinity for $t \to \infty$, provided besides (2), (3), (7), (8) the reversal condition to (9), namely

$$(12)\qquad \limsup_{|x|\to\infty} h(x)\,\mathrm{sgn}\,x < -\varepsilon \qquad (\varepsilon\text{-suitable positive real}).$$

A natural problem arises: whether there exists besides unbounded solutions of (1) a bounded one as well, taking into account the same restrictions. In the following text we will give an affirmative answer to this question in the special case $g(x) \equiv b$. However, we would like to mention something before.

2. D′-CLASS AND THE AUTONOMOUS EQUATION (1)

DEFINITION. *We say that* (1) *has* D′-*property* (*in the sense of Levinson*) *if such a constant* D′ *exists that*

$$\limsup_{t\to\infty} (|x'(t)| + |x''(t)|) < D'$$

holds for all solutions $x(t)$ of (1).

It can be easily verified either by the Liapunov-Yoshizawa function

$$(13)\qquad U(x', x'') = (a^2 + 2b)\,x'^2 + 2\,ax'\,x'' + 2\,x''^2$$

or by virtue of the Cauchy formula (see [6]) and hence also by the planar geometrical methods used in the (x, y)-phase-space for

$$\frac{dx''}{dx'} = -a - b\,\frac{x'}{x''} - \frac{1}{x''}\,(h(x(t)) - p(t))$$

that (1) has under (3), (7) the D′-property for $g(x) \equiv b > 0$.

However, we remember here an analytical approach by J. Voráček [7] leading to the same result even for more general than (1) equations, when (2) is satisfied with $G < a^2$, namely

$$(14)\qquad 0 < b \le g(x) \le G < a^2 \qquad \text{for all} \quad x \in R^1 .$$

LEMMA 1. *If the conditions* (3), (7) *and* (14) *are satisfied, then* (1) *has the* D′-*property.*

CONSEQUENCE 1. *If the assumptions of Lemma* 1 *are satisfied for* $p(t) \equiv 0$ *together with* (9), *where* $R = 0$ *and*

$$(15)\qquad h'(0) > 0\,,\ ag(0) - h'(0) > 0\,,\ g'(0) = 0\,,$$

then (11) *holds for all the solutions of* (1).

Proof. At first we will verify the Lagrange-like stability of (1).

For this aim let us assume (on the contrary) that $x(t)$ is an unbounded solution of (1).

Since we have according to Lemma 1 that

$|x'(t)| < D'$, $|x''(t)| < D'$ for large enough t, say $t \geq T$, substituting $x(t)$ into (1), integrating the obtained identity from T to t and multiplying it by sgn x, we get the following inequality:

$$b(|x(t)| - |x(T)|) \leq \left| \int_{x(T)}^{x(t)} g(s)\, ds \right| < - \int_{T}^{t} |h(x(s))|\, ds + 2(a+1) D' \leq$$
$$\leq 2(a+1) D'$$

a contradiction to $\limsup_{t\to\infty} |x(t)| = \infty$.

Thus all the solutions of (1) must be bounded (without any loss of generality we arrive at the same statement under (8) in the non-autonomous case).

Using the same argument as above, we still obtain the relation

$$\left| \int_0^\infty h(x(t))\, dt \right| \leq G(|x(T)| + \lim_{t\to\infty} |x(t)|) + 2(a+1) D' < \infty,$$

leading (for more details see [8]) either to (11) or to

$$\liminf_{t\to\infty} |x(t)| = 0 < \limsup_{t\to\infty} |x(t)|.$$

The latter possibility can be however neglected under (15) with respect to the asymptotical stability of the origin (see [9]). This completes the proof.

CONSEQUENCE 2. *Let the assumptions of Lemma* 1 *be satisfied for* $p(t) \equiv \mathbf{0}$ *and let the function* $h(x)$ *be oscillatory everywhere with isolated zero points* $\bar{x}$.

If there exist such positive constants ε, R *that the condition*

$$ag(x) - h'(x) \geq \varepsilon \tag{16}$$

holds for $|x| > R$ *with* $g'(\bar{x}) = \mathbf{0}$, *then all the solutions of* (1) *are bounded.*

Proof. Remember again that Lemma 1 implies the existence of such a constant D′ that every solution $x(t)$ of (1) satisfies the relation

$$|x'(t)| \leq D' \text{ for } t \geq T_x \quad (T_x\text{-large enough}). \tag{17}$$

Furthermore, it is clear that either the situation of Consequence 1 appears (not to be analysed here) or such sequences of the asymptotically stable (for more details see [9]) zero points $\bar{x}_{\pm i}$ with $\lim \bar{x}_{\pm i} = \pm \infty$, namely $\{\bar{x}_i\}$, $\{\bar{x}_{-i}\}$, can be found that their basins of attraction are determined (see [9]) by means of a suitable positive constant δ_x from

$$|g'(x(t))\, x'(t)| \leq \delta_x \tag{18}$$

and by the validity of $h(x) \operatorname{sgn}(x - \bar{x}) > 0$.

Therefore since we have (for $t \geq T_x$)

$$\lim_{x(t) \to \bar{x}} g'(x(t))\, x'(t) = 0$$

for an unbounded solution $x(t)$ of (1) and a certain zero point $\bar{x}$ of $h(x)$ with respect to (17) and $g'(\bar{x}) = 0$, the relation (18) will be realized in a small enough neighbourhood of $\bar{x}$ and consequently $x(t)$ will be attracted by it. Thus (1) is stable in the sense of Lagrange, which was to be proved.

REMARK 2. *More precisely, only one of two possibilities from the proof of Consequence* 1 *can be satisfied for every solution of* (1) (see [8]). *Hence* (11) *holds for all the solutions of* (1), *when* $g(x) \equiv b$, *with respect to* (18).

3. EXISTENCE OF A BOUNDED SOLUTION UNDER (12)

Now we come to the most controversial case via the asymptotic Poincaré boundary value problem.

LEMMA 2. *If all the solutions of* (1) *satisfying a one-parameter family of boundary conditions*

$$x(T) - x(0) = x'(T) - x'(0) = x''(T) - x''(0) = 0 \tag{19}$$

are " a priori " uniformly bounded together with their two first derivatives independently of $T \in (0, \infty)$, *then* (1) *admits a bounded solution, provided* (12) *holds with* $\varepsilon = |p(0)|$.

Proof. The proof concerning the solvability of (1), (19) for a finite $T \in R^1$ ($T = \mu\omega$, where $\mu \in (0, 1>$) can be found e.g. in [10] and the one of the limit case for $T \to \infty$ is then guaranteed by the lemma of Krasnosel'skii [11, pp. 178-180].

THEOREM. *Under the assumptions* (3), (7), (8), (12) *with* $\varepsilon = |p(0)|$ *and* $g(x) \equiv b > 0$ *the equation* (1) *admits a bounded solution.*

Proof. According to Lemma 2 it is sufficient to prove the uniform " a priori " boundedness of the solutions of (1), (19) together with their two first derivatives. For this purpose we will proceed by the well-known Yoshizawa's technique [12] of Liapunov functions.

Since the time-derivative of (13) with respect to (1) reads for $g(x) \equiv b$:

$$U_{(1)}(x', x'') = -2ax''^2 - 2abx'^2 + (4x'' + 2ax')(p(t) - h(x)),$$

such a positive constant S must exist that $U'_{(1)}(x', x'')$ is negative definite for $|x'| + |x''| > S$, $t \in R^+$, $x \in R^1$ and simultaneously that

$$\inf_{|x'|+|x''| \geq S_0} U(x', x'') > \sup_{|x'|+|x''|=S} U(x', x'') \tag{20}$$

is holding for some $S_0 > S$ with respect to $U(x', x'') \to \infty$ for $|x'| + |x''| \to \infty$.

Since the bound S_0 is a uniform one with respect to (20), only two troublesome possibilities may occur; either the relation $|x'(t)| + |x''(t)| > S$ holds for $x(t)$ of (1), (19) on all the interval $\langle 0, T\rangle$ or it is satisfied for $0 \leq t < T_0 \leq T$ with $|x'(T_0)| + |x''(T_0)| = S$. Both possibilities however contradict to (19), because in the first case we come to $U(x'(T), x''(T)) < U(x'(0), x''(0))$, while in the second one to $|x'(T)| + |x''(T)| \leq S_0 < |x(0)| + |x''(0)|$ (for $S < |x'(0)| + |x''(0)| \leq S_0$ we might get some bound $S_1 > S_0$, when replacing S_0 by S_1 and S by S_0 in (19), analogically).

Although only $|x'(t)| + |x''(t)| \leq S_1$ can be satisfied for all the solutions of (1), (19) as we could see, $x(t)$ may be yet arbitrary. Therefore let us consider still another Liapunov function, namely

$$2V(t, x, x', x'') = 2\int_0^x h(s)\, ds + (bx + ax' + x'' - \int_0^t p(s)\, ds)^2,$$

implying obviously the existence of such a positive constant R that the relation

$$\inf_{|x| \geq R_0} V(t, x, x', x'') > \sup_{|x|=R} V(t, x, x', x'') \tag{21}$$

holds for some $R_0 > R$, $|x'| + |x''| \leq S_1$ and $t \in R^+$.

Similarly, since its time-derivative with respect to (1), namely

$$V'_{(1)}(t, x, x', x'') = -bh(x)x - h(x)(x'' + (a+1)x' - \int_0^t p(s)\, ds),$$

is this time positive definite under our assumptions for $|x| > R$ and $|x'| + |x''| \leq S_1$, $t \in R^+$, a uniform "a priori" boundedness result is given. Indeed, because otherwise if there exists such a point $t_0 \in \langle 0, T)$ with $|x(t_0| \geq R$, and such a first point $T_0 \in (t_0, T\rangle$ with $|x(T_0)| = R$, then it should be satisfied $V(t_0) > V(T_0)$ according to (21) and $V(t_0) < V(T_0)$ with respect to the positive definiteness of $V_1(t, x, x', x'')$ altogether. Moreover, $|x(t)| > R_0$ for all $t \in \langle 0, T \rangle$ would imply that $V(T) > V(0)$, a contradiction to (19). This completes the proof.

REMARK 3. *Recently we have proved* [13] *that such a bounded solution belongs under* (10) *to the class* $L_2 < 0, \infty)$, *provided* $h(x) \operatorname{sgn} x < 0$ *for* $x \neq 0$.

4. EQUATIONS LIKE (1) WITHOUT D′-PROPERTY

Equations without D′-property have been considered only rarely (see e.g. [13, 14]) and mainly similar dichotomy results have been obtained for them.

REFERENCES

[1] R. REISSIG, G. SANSONE and R. CONTI (1969) – *Nichtlineare Differentialgleichungen höherer Ordnung*. Cremonese, Roma.

[2] J.O.C. EZEILO and H.O. TEJUMOLA (1973) – *Boundedness theorems for certain third order equations*. « Atti Accad. Naz. Lincei », (8), *55*, 194-201.

[3] J.O.C. EZEILO (1968) – *On the boundedness of solutions of the equation* $x''' + a x'' + f(x) x' + g(x) = p(t)$. « Ann. Mat. Pura Appl. », *4*, 80, 281-299.

[4] K.E. SWICK (1974) – *Boundedness and stability for nonlinear third order differential equations*. « Atti Accad. Naz. Lincei », (8), *56*, 859-865.

[5] K.E. SWICK (1970) – *Asymptotic behavior of the solutions of certain third order differential equations*. « SIAM J. Appl. Math. », *19*, 96-102.

[6] J. VORÁČEK (1966) – *Einige Bemerkungen über eine nichtlineare Differentialgleichungen dritten Ordnung*. « Arch. Math. », *2*, 19-26.

[7] J. VORÁČEK (1970) – *Über eine nichtlineare Differentialgleichung dritter Ordnung*. « Czech. Math. J. », *20*, 207-219.

[8] J. ANDRES (1986) – *Boundedness of solutions of the third order differential equation with the oscillatory restoring and forcing terms*. « Czech. Math. J. », *1*, 1-6.

[9] J. ANDRES (1986) – *On stability and instability of the roots of the oscillatory function in a certain nonlinear differential equation of the third order*. « Čas. pěst. mat. », *3*, 225-229.

[10] R. REISSIG (1973/74) – *Phasenraum-Methoden zum Studium nichtlinearerer Differentialgleichungen*. « Jber. Deutch. Math.-Verein », *75* (3), 1, 130-139.

[11] M.A. KRASNOSEL'SKI (1966) – *Translation operator along the trajectories of differential equations*. « Nauka, Moscow » (in Russian).

[12] T. YOSHIZAWA (1966) – *Stability theory by Liapunov's second method*. « Math. Soc. Japan », Tokyo.

[13] J. ANDRES – *Dichotomies for solutions of a certain third order nonlinear differential equation which is not from the class* D. To appear in « Fasc. Math. ».

[14] L.R. ANDERSON (1970) – *Integral manifolds of a class of third order autonomous differential equations*. « J. Diff. Eqs. », *7*, 274-286.