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Analisi matematica. — *Estimates near the boundary for second order derivatives of solutions of the Dirichlet problem for the biharmonic equation.* Nota (*) di Vladimir A. Kondratiev e Olga A. Oleinik, presentata dal Socio G. Fichera.

Riassunto. — Per ogni soluzione della (1) nel dominio limitato Ω, appartenente a $H_0^2(\Omega)$ e soddisfacente le condizioni (2), si dimostra la maggiorazione (5), valida nell'intorno di ogni punto x^0 del contorno; si consente a $\partial\Omega$ di essere singolare in x^0.

This paper gives an answer to a question posed by Prof. G. Fichera in May 1985 at the Conference dedicated to Prof. M. Picone and Prof. L. Tonelli, organized by the Accademia Nazionale dei Lincei.

We consider a weak solution of the Dirichlet problem for the equation

$$\Delta\Delta u = \sum_{j=1}^{2} \frac{\partial f_j}{\partial x_j} \tag{1}$$

in an arbitrary bounded domain Ω in R^2, where

$$\Delta \equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}, \quad f_j \in L^p(\Omega), \; p > 2,$$

with the boundary conditions

$$u|_{\partial\Omega} = 0, \quad \operatorname{grad} u|_{\partial\Omega} = 0, \tag{2}$$

$\partial\Omega$ is the boundary of Ω. We study weak solutions of problem (1), (2) which belong to the space $H_0^2(\Omega)$. The space $H_0^2(\Omega)$ is defined as a completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\| u \|_2 \equiv \left(\int_\Omega \sum_{|\alpha| \le 2} | \mathscr{D}^\alpha u |^2 \, dx \right)^{\frac{1}{2}},$$

where

$$\alpha = (\alpha_1, \alpha_2), \; |\alpha| = \alpha_1 + \alpha_2, \; \mathscr{D}^\alpha \equiv \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}},$$

$C_0^\infty(\Omega)$ is the class of infinitely differentiable functions with compact support in Ω.

(*) Pervenuta all'Accademia l'11 agosto 1986.

In papers [1], [2] best possible estimates for the modulus of a weak solution of (1), (2) and its first derivatives near the boundary are given, the precise Hölder space $C^{1+\delta}(\Omega)$ is found which contains weak solutions of (1), (2) under some conditions on the geometry of $\partial\Omega$, (see also [3], [4]). Estimates for the derivatives of any order near a singular point of the boundary $\partial\Omega$ for solutions of the elasticity system are given in [5]. Estimates of the same kind are valid for solutions of the biharmonic equation. In particular, if the origin $0 \in \partial\Omega$ and the intersection of Ω with the circle $|x| = t$ is not empty for $t \leq T$, $T = \text{const} > 0$, $f_j \in L^p$, $p > 2$, then for a weak solution of problem (1), (2) the estimates

$$(3) \qquad |u(x)| \leq C_1 |x|^{3/2}, \left|\frac{\partial u(x)}{\partial x_j}\right| \leq C_2 |x|^{1/2}, \quad j = 1, 2, \qquad |x| \leq \frac{T}{2},$$

are valid, $C_1, C_2 = \text{const}$. In (3) one cannot take $\frac{3}{2} + \varepsilon$ ($\varepsilon = \text{const} > 0$) instead of $\frac{3}{2}$ in the first inequality, and $\frac{1}{2} + \varepsilon$ instead of $\frac{1}{2}$ in the second inequality. In this sense estimates (3) are best possible (see [1], [2]) (1).

THEOREM. *Let 0 be the origin,* $0 \in \partial\Omega$. *Suppose that the following conditions are satisfied:*

1) *the intersection of* $\partial\Omega$ *with the circle* $|x| = t$ *for* $t \leq T$, $T = \text{const} > 0$, *is not empty;*

2) *there exists* $\beta = \text{const} > 0$, $\beta < 1$, *such that for any* $x^0 \in \partial\Omega$ *and* $|x^0| < \frac{1}{2} T$, $x^0 \neq 0$, *the intersection of* $\partial\Omega$ *with the disk* $|x - x^0| < \beta |x^0|$ *contains a curve* S_{x^0} *whose end-points belong to the boundary of the disk,* $x^0 \in S_{x^0}$, *the curve* S_{x^0} *has the form*

$$x_1 = \varphi_2(x_2) \qquad \text{or} \quad x_2 = \varphi_1(x_1),$$

where

$$(4) \quad |\varphi_j'(x_j)| \leq C_3, \ |\varphi_j''(x_j)| \leq C_4 |x^0|^{-1}, \ |\varphi_j'''(x_j)| \leq C_5 |x^0|^{-2}, j = 1, 2,$$

and constants C_3, C_4, C_5 *do not depend on* x^0; *either two domains, bounded by* S_{x^0} *and a part of the circle* $|x - x^0| = \beta |x^0|$, *belong to* Ω, *or one of them belongs to* Ω *and* S_{x^0} *belongs to the boundary of* $R^2 \setminus \bar{\Omega}$. *Then there exists a constant* $C_6 > 0$ *which does not depend on* u, f_1, f_2 *and such that*

$$(5) \qquad \left|\frac{\partial^2 u(x)}{\partial x_i \partial x_j}\right| \leq C_6 |x|^{-1/2} \left(\int_\Omega \sum_{j=1}^{2} |f_j|^p \, dx\right)^{1/p}, \quad i, j = 1, 2,$$

(1) For the second inequality (3) we need an additional assumption: the intersection of $\partial\Omega$ and $|x - x^0| = \rho$ is not empty for $\rho < |x^0|/2$ and for any x^0 with $|x^0| < T/2$.

for $|x| < \frac{1}{4} T$, $x \in \Omega$, $x \neq 0$. *Estimate* (5) *is best possible.*

Proof. In [1], [2] *it is proved that the estimate*

$$(6) \qquad |u(x)| \leq C_7 |x|^{3/2} \left(\int_\Omega \sum_{j=1} |f_j|^p \, dx \right)^{1/p}, \ p > 1, \ |x| \leq \frac{1}{2} T,$$

is valid under the first condition of this theorem, where the constant C_7 does not depend on f_1, f_2.

Suppose that $y \in \Omega$ and the disk $K_y = \left\{ x : |x - y| < \frac{1}{4} \beta |y| \right\}$ does not intersect $\partial\Omega$. Let us introduce new independent variables

$$x' = \frac{x}{|y|} .$$

In these variables equation (1) has the form

$$\Delta\Delta u = |y|^3 \sum_{j=1}^{2} \frac{\partial f_j}{\partial x'_j}$$

in the disk

$$K'_y = \left\{ x' : |x' - y'| < \frac{1}{4} \beta \right\}, \ y' = \frac{y}{|y|} .$$

It follows from the interior estimates for elliptic equations [6] and the imbedding theorems [7] that

$$\left| \frac{\partial^2 u(y')}{\partial x' \, \partial x'_j} \right| \leq C_8 \left(\int_{|x' - y'| < 1/8\beta} \sum_{|\alpha| \leq 3} |\mathcal{D}^\alpha u|^p \, dx' \right)^{1/p} \leq$$

$$\leq C_9 \left[|y|^3 \left(\int_{K'_y} \sum_{j=1}^{2} |f_j|^p \, dx \right)^{1/p} + \left(\int_{K'_y} |u|^p \, dx' \right)^{1/p} \right]$$

and therefore in variables x we have

$$(7) \quad \left| \frac{\partial^2 u(y)}{\partial x_i \, \partial x_j} \right| \leq C_9 \left[|y|^{1-(2/p)} \left(\int_{K_y} \sum_{j=1}^{2} |f_j|^p \, dx \right)^{1/p} + |y|^{-2-(2/p)} \left(\int_{K_y} |u|^p \, dx \right)^{1/p} \right].$$

Using estimate (6) and the condition $1 - \frac{2}{p} > 0$ we get from (7) that

$$\left|\frac{\partial^2 u(y)}{\partial x_i\, \partial x_j}\right| \leq C_{10} \mid y \mid^{-1/2} \left(\int_\Omega \sum_{j=1}^{2} \mid f_j \mid^p dx\right)^{1/p},$$

Suppose now that $y \in \Omega$, but the disk $K_y = \left\{x : \mid x - y \mid \leq \frac{\beta}{4} \mid y \mid\right\}$ has a non-empty intersection with $\partial \Omega$. Let y_* be a point of $\partial \Omega$ and $\mid y - y_* \mid = \rho(y, \partial \Omega)$, where $\rho(y, A)$ is the distance between y and A. Then the disk $B_{y_*} = \{x : \mid x - y_* \mid < \beta \mid y_* \mid\}$ contains y and, according to condition 2) of the Theorem, S_{y_*} satisfies conditions (4).

We introduce new variables $x' = \dfrac{x}{\mid y_* \mid}$. In these variables equation (1) has the form

$$\Delta\Delta u = \mid y_* \mid^3 \sum_{j=1}^{2} \frac{\partial f_j}{\partial x'_j}$$

in the disk $\mid x' - y'_* \mid < \beta$, $x \in \Omega$. The curve S_{y_*} in the new variables is given by the equations

$$x'_1 \mid y_* \mid = \varphi_2 (x'_2 \mid y_* \mid) \quad \text{or} \quad x'_2 \mid y_* \mid = \varphi_1 (x'_1 \mid y_* \mid).$$

It is easy to see that according to (4) S_{y_*}, which we denote by S'_{y_*} in the new variables, belongs to class C^3 and S'_{y_*} is defined by a function who se norm in C^3 is bounded uniformly with respect to y_*. We denote by G_β the domain, bounded by the circle $\mid x' - y'_* \mid = \beta$ and S'_{y_*}, and containing y'. It is known (see [8], [9]) that for $p > 2$

$$\left(\int_{G_{\beta/2}} \sum_{\mid \alpha \mid \leq 3} \mid \mathcal{D}^\alpha u \mid^p dx'\right)^{1/p} \leq C_{11} \left[\mid y_* \mid^3 \left(\int_{G_\beta} \sum_{j=1}^{2} \mid f_j \mid^p dx'\right)^{1/p} + \left(\int_{G_\beta} \mid u \mid^p dx'\right)^{1/p}\right]$$

where the constant C_{11} does not depend on y'_*. It follows from the imbedding theorems that for $x' \in G_{\beta/2}$

$$\left|\frac{\partial^2 u(x')}{\partial x'_i\, \partial x'_j}\right| \leq C_{12} \left[\mid y_* \mid^3 \left(\int_{G_\beta} \sum_{j=1}^{2} \mid f_j \mid^p dx'\right)^{1/p} + \left(\int_{G_\beta} \mid u \mid^p d\ \right)^{1/p}\right]. \tag{8}$$

We write the inequality (8) in the variables x and get

$$\left|\frac{\partial^2 u(x)}{\partial x_i\, \partial x_j}\right| \leq C_{12} \left[\mid y_* \mid^{1-(2/p)} \left(\int_\Omega \sum_{j=1}^{2} \mid f_j \mid^p dx\right)^{1/p} + \left(\int_{\Omega \cap By_*} \mid u \mid^p dx\right)^{1/p}\right],$$

$$x \in \left\{x : \mid x - y_* \mid < \frac{\beta \mid y \mid}{2}\right\}.$$

Using (6) to estimate the last integral, we obtain

$$\left|\frac{\partial^2 u(x)}{\partial x_{i*} \partial x_j}\right| \leq C_{13} \, | \, y_* \, |^{-1/2} \left(\int_\Omega \sum_{j=1}^{2} | \, f_j \, |^p \, dx\right)^{1/p}, \; p > 2, \qquad i, j = 1,2. \tag{9}$$

Since $| \, y - y_* \, | < \dfrac{\beta \, | \, y \, |}{4}$, $| \, y \, | < \dfrac{4}{3} \, | \, y_* \, |$, we have from (9)

$$\left|\frac{\partial^2 u(y)}{\partial x_i \, \partial x_j}\right| \leq C_{14} \, | \, y \, |^{-1/2} \left(\int_\Omega \sum_{j=1}^{2} | \, f_j \, |^p \, dx\right)^{1/p}.$$

This means that the estimate (5) is valid. The theorem is proved.

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