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Estimates near the boundary for second order derivatives of solutions of the Dirichlet problem for the biharmonic equation


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Riassunto. — Per ogni soluzione della (1) nel dominio limitato Ω, appartenente a $H^2_0(Ω)$ e soddisfacente le condizioni (2), si dimostra la maggiorazione (5), valida nell'intorno di ogni punto $x^0$ del contorno; si consente a $∂Ω$ di essere singolare in $x^0$.

This paper gives an answer to a question posed by Prof. G. Fichera in May 1985 at the Conference dedicated to Prof. M. Picone and Prof. L. Tonelli, organized by the Accademia Nazionale dei Lincei.

We consider a weak solution of the Dirichlet problem for the equation

\begin{equation}
\Delta \Delta u = \sum_{j=1}^{n} \frac{\partial f_j}{\partial x_j}
\end{equation}

in an arbitrary bounded domain $Ω$ in $\mathbb{R}^2$, where

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}, \quad f_j \in L^p(Ω), \quad p > 2,$$

with the boundary conditions

\begin{equation}
u \mid \partial Ω = 0, \quad \text{grad } u \mid \partial Ω = 0,
\end{equation}

$∂Ω$ is the boundary of $Ω$. We study weak solutions of problem (1), (2) which belong to the space $H^2_0(Ω)$. The space $H^2_0(Ω)$ is defined as a completion of $C_0^∞(Ω)$ with respect to the norm

$$\| u \|_2 = \left( \int_Ω \sum_{|α| \leq 2} |D^α u|^2 \, dx \right)^{\frac{1}{2}},$$

where

$$α = (α_1, α_2), \quad |α| = α_1 + α_2, \quad D^α = \frac{∂^{|α|}}{∂x_1^{α_1} ∂x_2^{α_2}},$$

$C_0^∞(Ω)$ is the class of infinitely differentiable functions with compact support in $Ω$.

(*) Pervenuta all'Accademia l'11 agosto 1986.
In papers [1], [2] best possible estimates for the modulus of a weak solution of (1), (2) and its first derivatives near the boundary are given, the precise Hölder space $C^{1+s}(\Omega)$ is found which contains weak solutions of (1), (2) under some conditions on the geometry of $\partial \Omega$, (see also [3], [4]). Estimates for the derivatives of any order near a singular point of the boundary $\partial \Omega$ for solutions of the elasticity system are given in [5]. Estimates of the same kind are valid for solutions of the biharmonic equation. In particular, if the origin $0 \in \partial \Omega$ and the intersection of $\Omega$ with the circle $|x| = t$ is not empty for $t \leq T$, $T = \text{const} > 0$, $f_j \in L^p$, $p > 2$, then for a weak solution of problem (1), (2) the estimates

$$(3) \quad |u(x)| \leq C_1 \frac{|x|^{3/2}}{3!}, \quad \left| \frac{\partial u(x)}{\partial x_j} \right| \leq C_2 \frac{|x|^{1/2}}{3!}, \quad j = 1, 2, \quad |x| \leq \frac{T}{2},$$

are valid, $C_1, C_2 = \text{const}$. In (3) one cannot take $\frac{3}{2} + \epsilon (\epsilon = \text{const} > 0)$ instead of $\frac{3}{2}$ in the first inequality, and $\frac{1}{2} + \epsilon$ instead of $\frac{1}{2}$ in the second inequality. In this sense estimates (3) are best possible (see [1], [2]).

**Theorem.** Let $0$ be the origin, $0 \in \partial \Omega$. Suppose that the following conditions are satisfied:

1) the intersection of $\partial \Omega$ with the circle $|x| = t$ for $t \leq T$, $T = \text{const} > 0$, is not empty;

2) there exists $\beta = \text{const} > 0$, $\beta < 1$, such that for any $x^0 \in \partial \Omega$ and $|x^0| < \frac{T}{2}$, $x^0 \neq 0$, the intersection of $\partial \Omega$ with the disk $|x - x^0| < \beta |x^0|$ contains a curve $S_{x^0}$ whose end-points belong to the boundary of the disk, $x^0 \in S_{x^0}$, the curve $S_{x^0}$ has the form

$$x_1 = \varphi_2(x_2) \quad \text{or} \quad x_2 = \varphi_1(x_1),$$

where

$$(4) \quad |\varphi_j(x_j)| \leq C_3, \quad |\varphi_j''(x_j)| \leq C_4 |x^0|^{-1}, \quad |\varphi_j'''(x_j)| \leq C_5 |x^0|^{-2}, \quad j = 1, 2,$$

and constants $C_3, C_4, C_5$ do not depend on $x^0$; either two domains, bounded by $S_{x^0}$ and a part of the circle $|x - x^0| = \beta |x^0|$, belong to $\Omega$, or one of them belongs to $\Omega$ and $S_{x^0}$ belongs to the boundary of $R^2 \setminus \Omega$. Then there exists a constant $C_6 > 0$ which does not depend on $u, f_1, f_2$ and such that

$$(5) \quad \left| \frac{\partial^2 u(x)}{\partial x_i \partial x_j} \right| \leq C_6 |x|^{-1/2} \left( \int_{\Omega} \left( \sum_{j=1}^{2} |f_j|^{p} \right)^{1/p} \right)^{1/p}, \quad i, j = 1, 2,$$

(1) For the second inequality (3) we need an additional assumption: the intersection of $\partial \Omega$ and $|x - x^0| = \rho$ is not empty for $\rho < |x^0|/2$ and for any $x^0$ with $|x^0| < T/2$. 


for $|x| < \frac{1}{4} T$, $x \in \Omega$, $x \neq 0$. Estimate (5) is best possible.

**Proof.** In [1], [2] it is proved that the estimate

$$|u(x)| \leq C_7 |x|^{\beta/2} \left( \int_\Omega \sum_{j=1}^2 |f_j|^p \, dx \right)^{1/p}, \quad p > 1, \quad |x| \leq \frac{1}{2} T,$$

is valid under the first condition of this theorem, where the constant $C_7$ does not depend on $f_1, f_2$.

Suppose that $y \in \Omega$ and the disk $K_y = \{x : |x - y| < \frac{1}{4} \beta |y|\}$ does not intersect $\partial \Omega$. Let us introduce new independent variables

$$x' = \frac{x}{|y|}.$$

In these variables equation (1) has the form

$$\Delta \Delta u = |y|^3 \sum_{j=1}^3 \frac{\partial f_j}{\partial x_j},$$

in the disk

$$K'_y = \left\{x' : |x' - y'| < \frac{1}{4} \beta \right\}, \quad y' = \frac{y}{|y|}.$$

It follows from the interior estimates for elliptic equations [6] and the imbedding theorems [7] that

$$\left| \frac{\partial^3 u(y')}{\partial x_1 \partial x_2} \right| \leq C_8 \left( \int_{|x' - y'| < 1/8 \beta} \sum_{|\alpha| \leq 3} |\Delta^\alpha u|^p \, dx' \right)^{1/p} \leq$$

$$\leq C_9 \left[ |y|^3 \left( \int_{K'_y} \sum_{j=1}^2 |f_j|^p \, dx \right)^{1/p} + \left( \int_{K'_y} |u|^p \, dx \right)^{1/p} \right]$$

and therefore in variables $x$ we have

$$\left| \frac{\partial^2 u(x)}{\partial x_1 \partial x_j} \right| \leq C_9 \left[ |x|^{-1 - (2/p)} \left( \int_{K_y} \sum_{j=1}^2 |f_j|^p \, dx \right)^{1/p} + |y|^{-2 - (2/p)} \left( \int_{K_y} |u|^p \, dx \right)^{1/p} \right].$$

Using estimate (6) and the condition $1 - \frac{2}{p} > 0$ we get from (7) that
Suppose now that \( y \in \Omega \), but the disk \( K_y = \left\{ x : |x - y| \leq \frac{\beta}{4} |y| \right\} \) has a non-empty intersection with \( \partial \Omega \). Let \( y^* \) be a point of \( \partial \Omega \) and \( |y - y^*| = \rho(y, \partial \Omega) \), where \( \rho(y, A) \) is the distance between \( y \) and \( A \). Then the disk \( B_{y^*} = \{ x : |x - y^*| < \beta |y^*| \} \) contains \( y \) and, according to condition 2) of the Theorem, \( S_{y^*} \) satisfies conditions (4).

We introduce new variables \( x' = \frac{x}{|y^*|} \). In these variables equation (1) has the form

\[
\Delta \Delta u = \left| y^* \right|^3 \sum_{j=1}^{2} \frac{\partial f_j}{\partial x_j}
\]

in the disk \( |x' - y^*_1| < \beta, x \in \Omega \). The curve \( S_{y^*} \) in the new variables is given by the equations

\[
x'_1 |y^*_1| = \varphi_2 (x'_2 |y^*_1|) \quad \text{or} \quad x'_2 |y^*_1| = \varphi_1 (x'_1 |y^*_1|).
\]

It is easy to see that according to (4) \( S_{y^*} \), which we denote by \( S'_{y^*} \) in the new variables, belongs to class \( C^3 \) and \( S'_{y^*} \) is defined by a function whose norm in \( C^3 \) is bounded uniformly with respect to \( y^* \). We denote by \( G^*_\beta \) the domain, bounded by the circle \( |x' - y^*_1| = \beta \) and \( S'_{y^*} \), and containing \( y' \). It is known (see [8], [9]) that for \( p > 2 \)

\[
\left( \int_{G^*_\beta} \sum_{x' \in G^*_\beta} |\partial^2 u (x')| \right)^{1/p} \leq C_{11} \left[ |y^*_1|^3 \left( \int_{G^*_\beta} \sum_{j=1}^{2} |f_j| \right)^{1/p} + \left( \int_{G^*_\beta} |u| \right)^{1/p} \right]
\]

where the constant \( C_{11} \) does not depend on \( y^*_1 \). It follows from the imbedding theorems that for \( x' \in G^*_\beta/2 \)

\[
\left( \int_{G^*_\beta/2} \sum_{x' \in G^*_\beta/2} |\partial^2 u (x')| \right)^{1/p} \leq C_{19} \left[ |y^*_1|^3 \left( \int_{G^*_\beta} \sum_{j=1}^{2} |f_j| \right)^{1/p} + \left( \int_{G^*_\beta} |u| \right)^{1/p} \right].
\]

We write the inequality (8) in the variables \( x \) and get

\[
|\partial^2 u (x)| \leq C_{19} \left[ |y^*_1|^{1-(\alpha/p)} \left( \int_{\Omega} \sum_{j=1}^{2} |f_j| \right)^{1/p} + \left( \int_{\Omega} |u| \right)^{1/p} \right],
\]

\[x \in \left\{ x : |x - y^*_1| < \frac{\beta |y^*_1|}{2} \right\}.
\]

Using (6) to estimate the last integral, we obtain
\[ \left| \frac{\partial^2 u (x)}{\partial x_i \partial x_j} \right| \leq C_{13} |y_*|^{-1/2} \left( \int_{\Omega} \sum_{j=1}^{2} |f_j|^p \, dx \right)^{1/p}, \quad p > 2 , \quad i, j = 1, 2. \]

Since \( |y - y_*| < \frac{\beta}{4} |y_*|, \quad |y| < \frac{4}{3} |y_*| \), we have from (9)

\[ \left| \frac{\partial^2 u (y)}{\partial x_i \partial x_j} \right| \leq C_{14} |y|^{-1/2} \left( \int_{\Omega} \sum_{j=1}^{2} |f_j|^p \, dx \right)^{1/p}. \]

This means that the estimate (5) is valid. The theorem is proved.

REFERENCES


