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A note on paracomplete logic

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Logica matematica. — *A note on paracomplete logic.* Nota di NEWTON C.A. DA COSTA e DIEGO MARCONI, presentata (*) dal Socio G. ZAPPA.

RIASSUNTO. — In questa nota gli Autori descrivono nuovi sistemi di logica (detta « paracompleta ») connessi con la logica della vaghezza (« fuzzy logic ») e con le logiche paraconsistenti.

1. INTRODUCTION

In paracomplete logic two propositions A and $\neg$ A, the negation of A, can both be false. Intuitionistic logic and several systems of many-valued logic are paracomplete in this sense. The motivation for paracomplete logic is connected with the fact that the classical requirement that at least one of a proposition and its negation be true does not always fit out intuitions. For instance, if P is a vague predicate and a is a borderline individual we may feel that both P (a) and $\neg$ P (a) are false. Similarly, if p is a contingent proposition and F is the future operator, we may think that Fp and $\neg$ Fp are both false. Even in certain philosophical theories, such as Hegel's logic, a proposition and its negation are sometimes said to be both false (see e.g. Hegel's *Wissenschaft der Logik,* Miller transl., p. 67). In general, a paracomplete logic can be conceived as the underlying logic of an incomplete theory in the strong sense, i.e. of a theory according to which a proposition and its negation are both false.

In this note, we describe a hierarchy of paracomplete logics and mention the possibility of extending it to others which are, in a certain sense, " dual " of the hierarchies of da Costa (1963), (1964 *a*), (1964 *b*), (1964 *c*), (1964 *d*). It seems worthwhile to remark that our logical calculi are related to the earlier work of Arruda and Alves (1979 *a*), (1979 *b*), and to the ideas that lie at the basis of " fuzzy logic ".

We shall use, without any comments, the terminology, symbols, etc. of Kleene (1952), with obvious adaptations.

The complete development of the contents of this note will appear elsewhere.

(*) Nella seduta del 29 novembre 1986.

2. THE HIERARCHY P_n, $0 \leq n \leq \omega$, OF PROPOSITIONAL CALCULI

Let us begin with the description of P_1. Its language has 1) Propositional letters; 2) Connectives: $\supset$ (implication), $\vee$ (disjunction), $\in$ (conjunction), and $\neg$ (negation). The symbol $\sim$, for equivalence, is introduced as usual. The concepts of formula and subformula, the conventions used in the writing of formulas etc. are as in Kleene (1952), Ch. IV.

DEFINITION 1. $A^* =_{DF} A \vee \neg A$.

Postulates of P_1:

$\supset_1$) $(A \supset B) \supset ((A \supset (B \supset C)) \supset (A \supset C))$
$\supset_2$) $A \supset (B \supset A)$
$\supset_3$) $\dfrac{A \quad A \supset B}{B}$
$\supset_4$) $((A \supset B) \supset A) \supset A$

$\&_1$) $A \& B \supset A$
$\&_2$) $A \& B \supset B$
$\&_3$) $A \supset (B \supset A \& B))$

$\vee_1$) $A \supset A \vee B$
$\vee_2$) $B \supset A \vee B$
$\vee_3$) $(A \supset C) \supset ((B \supset C) \supset (A \vee B \supset C))$

$\neg_1$) $A^* \supset ((A \supset B) \supset ((A \supset \neg B) \supset \neg A))$
$\neg_2$) $A^* \& B^* \supset (A \supset B)^* \& (A \varepsilon B)^* \& (A \vee B)^* \& (\neg A)^*$
$\neg_3$) $\neg (A \& \neg A)$
$\neg_4$) $A \supset (\neg A \supset B)$
$\neg_5$) $A \supset \neg\neg A$.

The intuitive justification for the above postulates is analogous to the justification of the postulates for the calculus C_1 (cf. da Costa (1963), da Costa and Carnielli (1986), and Marconi (1979)).

We can prove the following results:

THEOREM 1. *In P_1 all the theorems and rules of the classical positive calculus are valid.*

THEOREM 2. $\Gamma \vdash A$ *in* P_1 *iff* $\Gamma, A_1^*, A_2^*, \ldots, A_n^* \vdash A$ *in the classical propositional calculus, where* $A_1, A_2, \ldots, A_n$ *are the prime components of the formulas of* Γ *and of* A.

THEOREM 3. *In* P_1 *the following schemes are not valid:* $A \vee \neg A$, $\neg (A \vee B) \sim \neg A \& \neg B$, $\neg (A \& B) \sim \neg A \vee \neg B$, $\neg\neg A \sim A$, $\neg\neg A \supset A$, $(A \supset B) \supset (\neg B \supset \neg A)$, A^{**}.

THEOREM 4. *In* P_1 *we have*:

$\vdash A \& \neg A \supset B$, $\vdash A \vee (A \supset B)$, $\vdash A^* \supset (\neg\neg A \sim A)$, $\vdash A^* \supset ((A \supset B) \& (A \supset \neg B) \supset \neg A)$, $\vdash A \& \neg A \sim B \& \neg B$.

THEOREM 5. *P_1 is not decidable by finite logical matrices.*

A semantics of valuations (cf. Loparic and da Costa (1984)) can be developed for P_1.

The notions of set of formulas, maximal consistent set of formulas, etc. are defined as usual. Let $v : F \twoheadrightarrow \{0, 1\}$ be a function whose domain F is the set of all formulas of P_1. v is said to be a *valuation* (of P_1) if v is the characteristic function of a maximal consistent set of formulas. We say that v is a *model* of Γ if $v(A) = 1$ for all formulas in Γ. When Γ is $\{A\}$ for some formula A, we say that v is a model of A. If v is a model of Γ (A), we write $v \models \Gamma$ ($v \models A$). When every model of Γ is also a model of A, we say that A is a semantic consequence of Γ, and write $\Gamma \models A$; if $\Gamma = \varnothing$, we write $\models A$ instead of $\varnothing \models A$.

THEOREM 6. (A. Loparic) $v : F \twoheadrightarrow \{0, 1\}$ *is a valuation of* P_1 *iff*:

1) $v(A) = 1$ *entails* $v(\neg A) = 0$,

2) $v(A) \neq v(\neg A)$ *entails* $v(\neg A) \neq v(\neg\neg A)$,

3) *if* $v(A) \neq v(\neg A)$ *and* $v(B) \neq v(\neg B)$, *then* $v(A \supset B) \neq v(\neg(A \supset B))$, $v(A \& B) \neq v(\neg(A \& B))$, *and* $v(A \vee B) \neq v(\neg(A \vee B))$,

4) $v \neg(A \in \neg A) = 1$.

COROLLARY. P_1 *is decidable (by the method of valuations) (see Loparic and da Costa* (1984)).

THEOREM 7. P_1 *has a tableaux semantics and is decidable by this method* (see Marconi (1980)).

In what follows, the relevant algebraic notions will be used as in Rasiowa and Sikorski (1970).

DEFINITION 2. $A \leq B$ iff $\vdash_{P_1} A \supset B$.

DEFINITION 3. $A \approx B$ iff $A \leq B$ *and* $B \leq A$.

It is easy to show that $\leq$ is a quasi-ordering; therefore $\approx$ is an equivalence relation. Let us call $C(P_1)$ the quotient algebra $F/\approx$, where F is the class of all formulas of P_1. $C(P_1)$ may be called the Curry algebra of P_1. $\| A \|$ will denote the equivalence-class of A.

THEOREM 8. *$C(P_1)$ is a relatively pseudo-complemented lattice. $\approx$ is a congruence with respect to $\vee$, &, and $\supset$. However, $\approx$ is not a congruence with respect to $\neg$. An element $\| A \|$ is the unit of $C(P_1)$ iff* $\vdash_{P_1} A$.

DEFINITION 4. $\neg{}^* A = A \supset (B \,\&\, \neg B)$, *where* B *is a fixed formula.*

THEOREM 9. *$\langle C(P_1), v, \&, \supset, \neg^* \rangle$ is a Boolean algebra. $\neg^*$ has all the properties of Boolean complementation.*

The algebraic study of P_1 can thus be undertaken, and the relevant representation theorem proved.

We now proceed to the introduction of a hierarchy of paracomplete propositional calculi, along the lines of da Costa (1963).

DEFINITION 5. $A^1 =_{DF} = A^*$

$$A^n =_{DF} A^* \,\&\, A^{**} \,\&\, \ldots \,\&\, A^{**\cdots *},$$

where the symbol * occurs n times ($n > 1$).

DEFINITION 6. $A^{(n)} =_{Df} A^1 \,\&\, A^2 \ldots \&\, A^n$.

The calculus P_n, $1 < n < \omega$, has the same language as that of P_1, and the same postulates, with the exception that axiom schemes $\neg_1$ and $\neg_2$ are replaced respectively by

$\neg_{1,n}$) $A^{(n)} \supset ((A \supset B) \supset ((A \supset \neg B) \supset \neg A))$

$\neg_{2,n}$) $A^{(n)} \,\&\, B^{(n)} \supset (A \supset B)^{(n)} \,\&\, (A \in B)^{(n)} \,\&\, (A \vee B)^{(n)} \,\&\, (\neg A)^{(n)}$.

The calculus P_ω is obtained from P_1 by dropping the axiom schemes $\neg_1$ and $\neg_2$.

Most of the preceding results concerning P_1 can be extended to P_n, $1 < < n \leq \omega$.

DEFINITION 7. *In P_n, $1 \leq n \leq \omega$, we introduce a "strong negation" $\neg^*$ as follows:*

$$\neg{}^* A =_{Df} A \supset (K \,\&\, \neg K),$$

where K *is a fixed formula.*

THEOREM 10. In P_n, $1 \leq n \leq \omega$, *the strong negation has all the properties of classical negation. For example, the following schemes are valid:* $(A \supset B) \supset ((A \supset \neg{}^* B) \supset \neg{}^* A)$, $A \supset (\neg{}^* A \supset B)$, $A \vee \neg{}^* A$.

THEOREM 11. *If* P_0 *denotes the classical propositional calculus* (Kleene (1952), Ch. VI), *then every calculus of the sequence* $P_0, P_1, P_2, \ldots, P_n, \ldots, P_\omega$ *is strictly stronger than the following ones.*

3. SOME EXTENSIONS OF THE CALCULI P_n, $1 \leq n \leq \omega$.

Starting with the hierarchy P_n, $0 \leq n \leq \omega$, we can construct corresponding hierarchies of first-order predicate calculi P_n^*, $0 \leq n \leq \omega$, of first-order predicate calculi with equality $P_n^=$, $0 \leq n \leq \omega$, of calcul iof descriptions P_n^D, $0 \leq n \leq \omega$, and of set theories NF'_n, $0 \leq n \leq \omega$, which are paracomplete. This is done similarly to the way in which one of the authors has built several hierarchies of paraconsistent calculi (see e.g. da Costa (1974) and Marconi (1979)). In particular, in the set theories that are so constructed there exist "paracomplete" (or "partially defined") sets, i.e. sets x for which $\forall y (y \in x \vee y \notin x)$ is not true.

Hierarchies of logics which are simultaneously paraconsistent and paracomplete can also be constructed. These logics can also be employed to deal with vague concepts (see e.g. Rolf (1981)), and are related to fuzzy logic and mathematics (see Zadeh (1975)); more generally, they can be conceived as the underlying logics of theories which are both inconsistent (but not trivial) and incomplete in the strong sense, such as certain dialectical theories.

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