Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Paolo Lipparini

Consequences of compactness properties for abstract logics

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **80** (1986), n.7-12, p. 501–503.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1986_8_80_7-12_501_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1986.

Logica matematica. — *Consequences of compactness properties for abstract logics.* Nota di PAOLO LIPPARINI, presentata (*) dal Socio G. ZAPPA.

RIASSUNTO. — Si determinano alcune restrizioni sulle possibili cardinalità dei modelli di teorie in logiche soddisfacenti alcune proprietà di compattezza. Si dà una caratterizzazione delle logiche $[\lambda, \mu]$-compatte generate da quantificatori di cardinalità. Si stabilisce che il primo cardinale k tale che una logica è (k, k)-compatta è debolmente inaccessibile e soddisfa la proprietà dell'albero. Dai risultati enunciati appare un raffronto assai particolareggiato fra i due concetti di (λ, μ)-compattezza e $[\lambda, \mu]$-compattezza.

For an Abstract Logic L, [4] introduced the notion of $[\lambda, \mu]$-compactness which is, in many respects, more natural than (λ, μ)-compactness (nevertheless, when one is interested in logics which are not fully compact, this new compactness property seems much stronger than the older one: as an example, $L_{\omega\omega}(Q_1)$ is (ω, ω) but not $[\omega, \omega]$–compact).

The aim of this paper is to announce some results about $[\lambda, \mu]$-compactness and, in most cases, alternative forms for (λ, μ)-compactness are given. At least, we show that the new notion is not only interesting in itself, but can also be used to suggest new theorems about (λ, μ)-compactness.

Unexplained notions and notations can be found in [1], [2], [4]; λ, μ, ν, k denotes *infinite* cardinals; if L is a logic, $F_\nu(L)$ is the class of all couples (D, V), D an ultrafilter and V filter such that $\prod_{D|V} \mathfrak{A} \equiv_L \mathfrak{A}$, for every model $\mathfrak{A}$ of cardinality $\leq \nu$.

(D, V) is (λ, μ)-regular iff in V there is a partition $(I_x)_{x \in S_\mu(\lambda)}$ such that $\bigcup_{x \supseteq \{\alpha\}} I_x \in D$, for every $\alpha \in \lambda$.

THEOREM 1. *If* L *is* $[\lambda, \omega]$*-compact and* $K = \{k \mid$ *there is* ν *such that:* $(D, V) \in F_\nu(L)$ *implies* $|\prod_{D|V} k| = k\}$, *then* $k \in K$ *implies that* $k^\lambda = k$. *Moreover, if* $[\mu, \mu^*] \cap K = \varnothing$, *then for every* ν *there is* $(D, V) \in F_\nu(L)$, (D, V) (λ, ω)-*regular, such that* $\inf\{K \cap [\mu^*, \infty)\} \geq |\prod_{D|V} \mu'| = |\prod_{D|V} \mu| > \mu^*$, *for every* $\mu' \in [\mu, \mu^*]$.

(*) Nella seduta del 29 novembre 1986.

COROLLARY 1. *There is no logic generated by monadic and equivalence quantifiers satisfying the Relativized Upward Lowenheim Skolem Property and properly extending* $L_{\omega\omega}$.

Corollary 1 strengthens [1, Ch. VI, Theorem 3.1.3] (but compare also with [3, p. 236]). The method of proof of Corollary 1 can be applied to many other kinds of quantifiers.

In view of Theorem 1, a $[\lambda, \omega]$-compact logic L either is rich in L-complete extensions of small cardinality, or satisfies many Lowenheim-Skolem-Tarsky properties. A counterpart for (λ, ω)-compactness is:

THEOREM 2. *Suppose that* L *is* (λ, ω)*-compact, and put* $K = \{k \mid$ *for every* L-theory T *with* $|T| \leq \lambda$, *and for every unary predicate* U, *if every finite subset of* T *has a model in which* $|U| \leq k$, *then* T *has a model in which* $|U| \leq k\}$. *Then, if* $|T| \leq \lambda$, $(U_\alpha)_{\alpha\in\lambda}$ *are unary predicates,* $[\mu, \mu'] \cap K = \varnothing$ *and every finite subset of* T *has a model in which* $|U_\alpha| \in [\mu, \mu']$ $(\alpha \in \lambda)$, *then* T *has a model in which* $\inf(K \cap (\mu', \infty)) \geq |U_\alpha| = |U_\beta| > \mu'$, *for every* α, $\beta \in \lambda$.

THEOREM 3. *Suppose that* L *is single-sorted and* (λ, ω)*-compact, and satisfies the Craig Interpolation Property. Then either:* (i) L *contains a sentence of empty type not in* $L_{\omega\omega}$ *or* (ii) *if* $(T_\alpha)_{\alpha\in\lambda}$ *are* L*-theories,* $|T_\alpha| \leq \lambda$ $(\alpha \in \lambda)$, *each having an infinite model, then they have models of the same infinite power.*

In Theorem 3 we do not need the hypothesis that L is closed under relativization, which is essential in all the other theorems.

If N is a logic, let $N_{\lambda\mu}$ be the logic obtained by N, admitting conjunctions and disjunctions of less than λ sentences, and quantifications over less than μ constants. The following generalizes a result of [4].

THEOREM 4. *If* μ *is the first cardinal such that* N *is* $[\mu, \mu]$*-compact, then also* $N_{\mu\omega}$ *is* $[\mu, \mu]$*-compact, and* μ *is a measurable cardinal.*

Theorem 4 cannot be extended to $N_{\mu\mu}$: if μ is an uncountable measurable cardinal, $L_{\mu\omega}(Q^{cf\mu})$ is $[\mu, \mu]$-compact, but $L_{\mu\mu}(Q^{cf\mu})$ is not $[\mu, \mu]$-compact. Nevertheless, we have an analogue for (k, k)-compactness:

THEOREM 5. *If* k *is the first cardinal such that* N *is* (k, k)*-compact, then* $N_{k\omega}$ *is still* (k, k)*-compact, so that* k *is weakly inaccessible and has the tree property. If, in addition,* k *is strong limit, then* k *is weakly compact.*

THEOREM 6. *If* K *is any class of cardinals, then* $L_{\omega\omega}(Q_\alpha)_{\alpha\in K}$ *is* $[\lambda, \mu]$*-compact iff there exists a* (λ, μ)*-regular not* $(\mathrm{cf}\,\omega_\alpha, \mathrm{cf}\,\omega_\alpha)$*-regular* $(\alpha \in K)$ *ultrafilter* D *such that* $k < \omega_\alpha$ *implies* $|\prod_D k| < \omega_\alpha$ $(\alpha \in K)$.

Theorem 6 shows an influence of set-theoretical axioms (concerning the existence of non-regular ultrafilter) on the problem of compactness of cardinality logics. Set theory influences also the possible compactness spectrum of logics:

THEOREM 7. *If* I, J *are sets, and the* ν_j*'s are regular cardinals, then the following are equivalent:*

(i) *There exists a logic* $[\lambda_i, \mu_i]$*-compact* $(i \in I)$ *not* $[\nu_j, k_j]$*-compact* $(j \in J)$.

(ii) *There exists a logic as in* (i) *generated by a set of cardinality quantifiers.*

(iii) *For every* $j \in J$ *there exists* $\nu_j^*, k_j \leq \nu_j^* \leq \nu_j$, *such that for every* $i \in I$ *there is an ultrafilter which is* (λ_i, μ_i)*-regular but not* (ν_j^*, ν_j^*)*-regular, for* $j \in J$.

Theorem 7 improves [3, Lemma 6.4 (ii)].

In many particular cases, Theorem 6 can be used in order to give a more explicit characterization of $[\lambda, \mu]$-compact cardinality logics. An example is:

THEOREM 8. *If* k *is strongly compact* (*or just* $\sup(k, \omega_\alpha)$*-compact*) *and* $\lambda \geq k$ *is regular, then* $L_{\omega\omega}(Q_\alpha)$ *is* $[\lambda, k]$*-compact iff* $cf(\omega_\alpha) \notin [k, \lambda]$ *and* $\nu^\lambda < \omega_\alpha$, *for all* $\nu < \omega_\alpha$ *with* $cf\nu \geq k$.

BIBLIOGRAPHY

[1] J. BARWISE and S. FEFERMAN (*editors*) (1984) – *Model-theoretic logics*, Berlin.

[2] A. KANAMORI and MAGIDOR (1978) – *The evolution of large cardinal axioms in Set Theory* - Higher Set Theory, Lecture Notes in Mathematics 669, Berlin.

[3] J.A. MAKOWSKY and S. SHELAH (1979) – *The Theorems of Beth and Craig in abstract model theory. I: The Abstract Setting*, «Trans. Amer. Math. Soc.», *256*, 215-239.

[4] J.A. MAKOWSKY and S. SHELAH (1983) – *Positive results in abstract model theory*, «Ann. Pure Appl. Logic», *25*, 263-299.