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The equation $\bar{\partial} u = f$ the intersection of pseudoconvex domains

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RENDICONTI

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DELLA ACCADEMIA NAZIONALE DEI LINCEI

Classe di Scienze fisiche, matematiche e naturali

Ferie e Sedute del 29 novembre e del 13 dicembre 1986

SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Matematica. — *The equation $\bar{\partial}u = f$ on the intersection of pseudoconvex domains.* Nota di ALESSANDRO PEROTTI, presentata (*) dal Corrisp. E. VESENTINI.

RIASSUNTO. — Viene studiata l'equazione $\bar{\partial}u = f$ per le forme regolari sulla chiusura dell'intersezione di k domini pseudoconvessi. Si costruisce un operatore soluzione in forma integrale e sotto ipotesi opportune si ottengono stime della soluzione nelle norme C^k.

Let D be a bounded domain of $\mathbf{C}^n$, $n \geq 2$, obtained by intersecting k pseudoconvex domains $D_1, \ldots, D_k$ with smooth boundary. Assume that D_j $(j = 1, \ldots, k)$ is of (at least) one of the following types:

i) a strictly pseudoconvex domain;

ii) a (Euclidean) convex domain;

iii) a Levi-flat domain whose boundary is defined as the zero-set of a pluriharmonic function on $\mathbf{C}^n$.

Moreover, suppose that the domains D_j satisfy the condition of real transversality:

(*) Nella seduta del 29 novembre 1986.

if $1 \leq i_1 < \ldots < i_r \leq k$ and $\rho_{i_1}(z) = \ldots = \rho_{i_1^r}(z) = 0$, then $(d\rho_i \wedge \wedge \ldots \wedge d\rho_{i_r})(z) \neq 0$ (ρ_j is a defining function for D_j).

Let $C^m(\bar{D})$ denote the space of functions $g \in C^m(D)$ such that g and all its derivatives extend continuously to $\bar{D}$, and $C^m_{p,q}(\bar{D})$ the corresponding space of (p, q)-forms.

We construct an explicit integral solution operator for the equation $\bar{\partial} u = f$, where $f \in C^m_{p,q}(\bar{D})$ ($0 \leq p \leq n$, $1 \leq q \leq n$ and $1 \leq m \leq +\infty$) and $\bar{\partial} f = 0$ on D.

Similar integral operators were constructed by Range and Siu [13] and by Poljakov [9] on the intersection of k strictly pseudoconvex domains. They also gave estimates in the uniform norms.

Our construction is based on the existence of a sequence of forms linked, in a sense we shall see, to the Martinelli-Bochner-Koppelman Kernel $U_{p,q}$ (see [1] for definitions and properties about this Kernel).

Let $\rho_j : \mathbf{C}^n \to \mathbf{R}$ be a defining function for D_j ($j = 1, \ldots, k$). We can find an open neighbourhood W_j of $\partial D_j \times \bar{D}_j$ and a vector valued function $h^j = (h^i_1, \ldots, h^j_n)$ of class C^∞ on

$$U_j := \{(z, \zeta) \in W_j : \rho_j(z) > \rho_j(\zeta)\}$$

with the following properties:

a) $h^j(z, \zeta)$ is holomorphic in ζ for fixed z;

b) $\sum_{j=1}^{n} h^j_i(z, \zeta)(z_i - \zeta_i) = 1$ for $(z, \zeta) \in U_i$.

In the case when D_j is strictly pseudoconvex, the existence of functions h^j as above was proved by Henkin [3] and Øvrelid [8]. If D_j is elementary convex, then we can take

$$h^j_i(z, \zeta) = \frac{\partial \rho_j}{\partial z_i} \left(\sum_{m=1}^{n} \frac{\partial \rho_j(z)}{\partial z_m}(z_m - \zeta_m) \right)^{-1} \qquad (i = 1, \ldots n)$$

Finally, if ρ_j is a pluriharmonic function on $\mathbf{C}^n$, then there exists $\varphi \in O(\mathbf{C}^n)$ such that $\rho_j = \operatorname{Re} \varphi$. If $g_1, \ldots, g_n$ are holomorphic functions on $\mathbf{C}^n \times \mathbf{C}^n$ such that

$$\varphi(z) - \varphi(\zeta) = \sum_{i=1}^{n} g_i(z, \zeta)(z_i - \zeta_i) \text{ on } \mathbf{C}^n \times \mathbf{C}^n,$$

then we set $h^j_i(z, \zeta) = g_i(z, \zeta)(\varphi(z) - \varphi(\zeta))^{-1}$.

The existence of functions h^j with properties *a*) and *b*) permits to obtain the following lemma.

LEMMA. *For every* $1 \leq s \leq k$, $L = (l_1, \ldots, l_s)$ *with* $1 \leq l_m \leq k$ *and* $0 \leq p \leq n$, $0 \leq q \leq n - s$, *there exists a double form* $\tau^L_{p,q}(z, \zeta)$ *of type* (p, q) *in* ζ *and* $(n - p, n - q - s - 1)$ *in* z, *of class* C^∞ *on* $U_{l_1} \cap \ldots \cap U_{l_s}$, *such that when* $q \geq 1$

$$\bar{\partial}_z \tau^L_{p,q} + (-1)^{n-p+s} \bar{\partial}_\zeta \tau^L_{p,q-1} = \begin{cases} U_{p,q} & \text{for } s = 1 \\ (-1)^{n-p+s+1} \sum_{j=1}^{s} (-1)^j \tau^{L[j]}_{p,q} & \text{for } 2 \leq s \leq k . \end{cases}$$

■

Remark. For the case $s = 1$, this result is essentially Lemma 1.4 in [1].

Now we proceed to define the integral operators: let $0 \leq p \leq n$ and $0 \leq m \leq +\infty$ be fixed. Let D'_1 be an open, smoothly bounded neighbourhood of $\bar{D}_1$ such that $(D'_1 \setminus D_1) \times D'_1 \subset W_j$ for every $j = 1, \ldots, k$. For $1 \leq q \leq n$, we define $T_0 : C^m_{p,q}(\mathbf{C}^n) \to C^m_{p,q-1}(\mathbf{C}^n)$ by setting

$$T_0(g)(\zeta) = \int_{D'_1} g(z) \wedge U_{p,q-1}(z, \zeta) .$$

From Corollary 1.5 of [5] we have $T_0(g) \in C^m_{p,q-1}(\mathbf{C}^n)$.

For $1 \leq s \leq k$, $L = (l_1, \ldots, l_s)$ with $1 \leq l_1 < \ldots < l_s \leq k$ and $1 \leq q \leq n - s$, let $T_L : C^m_{p,q+1}(\mathbf{C}^n) \to C^\infty_{p,q-1}(D)$ be defined by

$$T_L(g)(\zeta) = \int_{S_L} g(z) \wedge \tau^L_{p,q-1}(z, \zeta) ,$$

where $S_{(1)} = D'_1 \setminus \bar{D}_1$ and $S_L = \left(\bigcap_{m=s}^{1} \partial D_{1_m} \cap \bigcap_{i=1}^{1_s - 1} D_i \right) \setminus D_{l_s}$ for $L \neq (1)$.

Here $\partial D_i \cap \partial D_j$ has the orientation induced by being the boundary of $\partial D_i \cup D_j$. The integrals are meaningful when $\zeta \in D$, and they define C^∞ forms on D, since $\tau^L_{p,q-1}$ is of class C^∞ on an open neighbourhood of $\bar{S}_L \times D$.

Let $g \in C^m(\bar{D})$. We observe that g can be considered as a function of class C^m on an open neighbourhood of $\bar{D}$. In fact it is possible to construct a continuous linear operator $E : C^m(\bar{D}) \to C^m(\mathbf{C}^n)$ such that $E g_{|D} = g$. This extension operator can be defined locally as the Seeley operator [11], making use of the fact that D is locally diffeomorphic to a intersection of halfspaces. Moreover E can be constructed in such a way that we have $\operatorname{supp}(E g) \subset\subset D'_1$ for every $g \in C^m(\bar{D})$.

Now we can state our main result.

THEOREM 1. *For* $0 \leq p \leq n$, $1 \leq q \leq n$ *and* $1 \leq m \leq +\infty$, *let* $T : C^m_{p,q}(\bar{D}) \to C^m_{p,q-1}(D)$ *be the linear operator defined by* $Tf = -T_0(Ef) + (-1)^{p+q} \sum_{s=1}^{\min(k,n-q)} \sum_{|L|=s}{}' (-1)^{(n-q)s} T_L(\bar{\partial} Ef)$. *Then, if* $\bar{\partial} f = 0$ *on* D, *we have* $\bar{\partial} T f = f$ *on* D. ■

Under the assumption that the domains D_j are all strictly pseudoconvex, we obtain a boundary regularity result for the given solution.

THEOREM 2. *Let* $D \subset\subset \mathbf{C}^n$ *be the intersection of* k *strictly pseudoconvex domains with smooth boundary that verify the condition of real transversality. For* $0 \leq p \leq n$, $1 \leq q \leq n$ *and* $1 \leq m \leq +\infty$, *let* $T : C^m_{p,q}(\bar{D}) \to C^m_{p,q-1}(D)$ *be the operator defined in Theorem* 1. *If* $f \in C^m_{p,q}(\bar{D})$ *and* $\bar{\partial} f = 0$ *on* D, *then*:

if $q = n$, $Tf \in C^m_{p,n-1}(\bar{D})$ *and* $\|Tf\|_{C^m_{p,n-1}(D)} \leq C \|f\|_{C^m_{p,n}(D)}$;

if $q < n$ *and* $m \geq 2 \min(k, n-q) - 1$, $Tf \in C^{m'}_{p,q-1}(\bar{D})$, *where* $m' = [(m+1)/2] - \min(k, n-q)$, *and*

$$\|Tf\|_{C^{m'}_{p,q-1}(D)} \leq C \|f\|_{C^{m''}_{p,q}(D)},$$

where $m'' = 2[(m-1)/2] + 1$, *and* $C > 0$ *does not depend on* f.

(*in particular, if* $f \in C^\infty_{p,q}(\bar{D})$ *and* $\bar{\partial} f = 0$, *then* $Tf \in C^\infty_{p,q-1}(\bar{D})$). ■

In the particular situation in which D is the intersection of k balls of $\mathbf{C}^n$, we can improve this result.

THEOREM 3. *Let* $D = D_1 \cap \ldots \cap D_k$, *where every* D_j *is a ball of* $\mathbf{C}^n$ *and the domains satisfy the condition of real transversality. Then if* $f \in C^m_{p,q}(\bar{D})$ $(0 \leq p \leq n, 1 \leq q \leq n, 1 \leq m \leq +\infty)$ *and* $\bar{\partial} f = 0$ *on* D.

$$Tf \in C^m_{p,q-1}(\bar{D}) \text{ and } \|Tf\|_{C^m_{p,q-1}(D)} \leq C \|f\|_{C^m_{p,q}(D)}$$

where C *is a positive constant not depending on* f. ■

As far as we know, C^m estimates were obtained only in the case $k = 1$, that is when D is a smoothly bounded strictly pseudoconvex domain (Aizenberg and Dautov [1] and Lieb and Range [6]).

We obtain a boundary regularity result also in the case when one of the domains D_j is not strictly pseudoconvex, or ∂D has two Levi-flat portions.

Assume that $D_1, \ldots, D_{k-1}$ are strictly pseudoconvex, and the domain D_k verifies (at least) one of the following properties:

a) $D_k \subset\subset \mathbf{C}^n$ is convex, smoothly bounded, defined by $\rho_k : \mathbf{C}^n \to \mathbf{R}$ and there exist ε, $M > 0$ such that

$$|\Phi^k(z, \zeta)| \geq M\, \rho_k(z) \text{ for every } (z, \zeta) \in (\mathbf{C}^n \setminus D_k) \times \bar{D}_k \text{ with } |z - \zeta| < \varepsilon,$$

where $\Phi^k(z, \zeta) = \sum_{i=1}^{n} \frac{\partial \rho_k(z)}{\partial z_i}(z_i - \zeta_i)$;

b) $D_k = \{z \in C^n : \rho_k(z) < 0\}$, where ρ_k is a pluriharmonic function of class C^∞ on $\mathbf{C}^n$.

Under these assumptions and the hypothesis of real transversality of the domains, we get the following result.

THEOREM 4. *Let* $D = D_1 \cap \ldots \cap D_k$. *Then if* $f \in C^m_{p,q}(\bar{D})$ $(0 \leq p \leq n$, $1 \leq q \leq n$, $1 \leq m \leq +\infty)$ *and* $\bar\partial f = 0$ *on* D,

$$Tf \in C^{m'}_{p,q-1}(\bar{D}) \text{ and } \|Tf\|_{C^{m'}_{p,q-1}(D)} \leq C\, \|f\|_{C^m_{p,q}(D)}$$

where

(i) $m' = m$ *if* $q = n$;

(ii) $m' = [(m-1)/2] - n + q$ *if* D_k *verifies a*);

(iii) $m' = [(m-1)/2] - \min(k, n-q)$ *if* D_k *verifies b*);

(iv) $m' = m - n + q$ *if* $D_1, \ldots, D_{k-1}$ *are balls and* D_k *verifies a*);

(v) $m' = m - 2$ *if* $D_1, \ldots, D_{k-1}$ *are balls and* D_k *verifies b*). ■

A result in this direction was recently obtained by Michel [7] in the case when D is the half ball of $\mathbf{C}^n$. He showed that one can find a solution with the same regularity as the datum.

Finally, suppose that $D_1, \ldots, D_{k-2}$ are strictly pseudoconvex and D_{k-1} and D_k are defined by two pluriharmonic functions ρ_{k-1}, ρ_k. Assume that there exists a complex number w with $\mathrm{Re}\, w \neq 0$, $\mathrm{Im}\, w \geq 0$ such that $\varphi = w\,\rho_{k-1} + i\,\rho_k$ is holomorphic on $\mathbf{C}^n$.

For example, we can take $D_1 = B(0, 1)$, $\rho_2 = \mathrm{Re}\, z_n$ and $\rho_3 = \mathrm{Im}\, z_n - a\,\mathrm{Re}\, z_n$, where $a \geq 0$, and get a ball sector with an angle not greater than $\pi/2$ (take $w = 1 + i\,a$).

Under the transversality assumption, we obtain this boundary regularity result.

THEOREM 5. *Let* $D = D_1 \cap \ldots \cap D_k$. *Then if* $f \in C^m_{p,q}(\bar{D})$ $(0 \leq p \leq n$, $1 \leq q \leq n$, $1 \leq m \leq +\infty)$ *and* $\bar\partial f = 0$ *on* D,

$$Tf \in C^{m'}_{p,q-1}(\bar{D}) \text{ and } \|Tf\|_{C^{m'}_{p,q-1}(D)} \leq C\, \|f\|_{C^m_{p,q}(D)}$$

where $m' = [(m-1)/2] - \min(k, n-q)$. *If* $q = n$, *then we can take* $m' = m$; *if the domains* $D_1, \ldots, D_{k-2}$ *are balls, then* $m' = m - 3$. ■

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