Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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Some chain rules for certain derivatives of double tensors depending on other such tensors and some point variables. II. On the Lagrangian spatial derivative in relativity

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **80** (1986), n.4, p. 205–213.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1986_8_80_4_205_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1986.

Fisica matematica. — *Some chain rules for certain derivatives of double tensors depending on other such tensors and some point variables.* II. *On the Lagrangian spatial derivative in relativity.* Nota (*) del Corrisp. ALDO BRESSAN.

§ 6. ON THE MOTION $\mathscr{M}$ OF A MATERIAL BODY $\mathscr{C}$ IN THE RELATIVISTIC SPACE TIME S_4

Part II is substantially the extension of Part I (on $\tilde{T}^{\cdots}_{\cdots\,;R}$) to relativity theory, for which the Lagrangian spatial derivative $\tilde{T}^{\cdots}_{\cdots\,|R}$ is relevant. The description of this part is included in the introduction to Part I—see [3], § 1.

Now let us identify S_μ for $\mu = 4$ with a Riemannian space-time S_4 of special or general Relativity. For every event point $\mathscr{E} \in S_4$ the metric $ds^2 = - g_{\alpha\beta}\, dx^\alpha\, dx^\beta$ is assumed to be reducible, at $\mathscr{E}$, to the pseudo-Pitagorical form

$$(6.1) \qquad ds^2 = - \delta'_{\alpha\beta}\, dx^\alpha\, dx^\beta \qquad \text{with} \quad \delta'_{\alpha R} = \delta_{\alpha R} \qquad \text{and} \quad \delta'_{\alpha 0} = - \delta_{\alpha 0}$$

by a suitable choice of frame ϕ, i.e. (x); in this paper $\delta_{\alpha\beta}$ is Kronecker's delta and Greek [Latin] indices run from 0 to 3 [1 to 3].

Let $\mathscr{C}$ be a body moving in S_4 regularly enough to regard it as a set of material points—see § 52 in [2], p. 138—. Hence its world tube $W_{\mathscr{C}}$ is the union of the world lines W_{P*} of these points ($P^* \in \mathscr{C}$). Furthermore $\mathscr{C}$'s 4-velocity u^α and intrinsic acceleration $A^\alpha = Du^\alpha/Ds$ exist at every $\mathscr{E} \in W_{\mathscr{C}}$. Then the *spatial projector* $\overset{\perp}{g}_{\alpha\beta}$ and *spatial metric* $\overset{\perp}{ds^2}$ exist in $W_{\mathscr{C}}$:

$$(6.2) \qquad \overset{\perp}{ds^2} = \overset{\perp}{g}_{\alpha\beta}\, dx^\alpha\, dx^\beta \qquad \text{with} \quad \overset{\perp}{g}_{\alpha\beta} = g_{\alpha\beta} + u_\alpha\, u_\beta\,.$$

Fix a physically possible space time S_4^*, with metric tensor $g^*_{\alpha\beta}$, and an admissible frame (y) in it—see [2], p. 139— (1).

Consider the intersection of $\mathscr{C}$'s world tube $W^*_{\mathscr{C}}$ in S_4^* with the (space-like) hypersurface $y^0 = 0$; and endow it with the following Riemannian metric

(*) Presentata nella seduta dell'8 febbraio 1986.

(1) The frame (x) is *admissible* if x^0 increases towards future and the hypersurfaces $x^0 = \text{const}$ are space-like, i.e. $ds^2 > 0$ for dx^α tangent to them.

ds^{*2}, which is strictly positive definite:

$$
\begin{aligned}
(6.3) \qquad ds^{*2} = a^*_{LM}\, dy^L\, dy^M \qquad \text{with} \quad a^*_{rs} &= \hat{a}^*_{rs}(y^1, y^2, y^3) = \\
&= \overset{\perp}{g}{}^*_{rs}\,(0, y^1, y^2, y^3)\,,
\end{aligned}
$$

$\overset{\perp}{g}_{\alpha\beta}$ being the spatial projector in $W^*_{\mathscr{C}}$. Identify S^*_3 with the resulting Riemannian space. Furthermore for $P^* \in \mathscr{C}$ call the co-ordinate y^L of the intersection $\mathscr{E}^*$ of S^*_3 with P^*'s world line W^*_{P*} in S^*_3 *L-th material co-ordinate* of $\mathscr{E}^*$ or P^*. Thus a frame, or co-ordinate system has been determined on S^*_3 or $\mathscr{C}$; it will be denoted by (y) or ϕ^*. It is convenient to identify P^* with $\mathscr{E}^* = \phi^{*-1}(y)$.

As well as in [2] lower case [capital] indices refer to S_μ [S^*_3], i.e. are space-time [material] indices.

The equations or functions

$$(6.4) \qquad x^\alpha = \hat{x}^\alpha(t, y^1, y^2, y^3)\,, \qquad \text{or} \quad x = \hat{x}(t, y)\,, \qquad \text{with} \quad \partial\hat{x}^0/\partial t > 0$$

are said to *represent* $\mathscr{C}$'s motion $\mathscr{M}$ in S_μ if, for every $P^* \in \mathscr{C}$, the function $t \vdash \hat{x}^\alpha(t, y^1, y^2, y^3)$ with $(y^1, y^2, y^3) = \phi^*(P^*)$ describes W_{P*}.

In the sequel the functions $(6.4)_1$ are tacitly supposed to be regular in the sense that (i) they are one-to-one, (ii) of class $C^{(2)}$, (iii) for every $t \in \mathbf{R}$, the hypersurface $S_3(t)$ represented by $y \vdash \hat{x}(t, y)$ for $y \in \phi^*(S^*_3)$ is space-like, and (iv) in S_3 we have

$$(6.5) \qquad \frac{\partial(x^1, x^2, x^3)}{\partial(y^1, y^2, y^3)} \neq 0\,, \qquad \text{hence} \quad \frac{\partial(x^0, \ldots, x^3)}{\partial(t, y^1, y^2, y^3)} \neq 0\,.$$

By $(6.5)_2$ equation $(6.4)_1$ can be solved by

$$(6.6) \qquad t = \hat{t}(x) \quad , \quad y^L = y^L(x)\,.$$

The motion $\mathscr{M}$ determines and is determined by $(6.6)_2$; the function $(6.6)_1$, i.e. the *time parameter* $\hat{t}$, characterizes the arbitrary part of $\mathscr{M}$'s representation $(6.4)_1$; more in detail this representation is determined up to a change of the time parameter:

$$(6.7) \qquad t = t(\bar{t}, y) \qquad \text{with} \quad \partial\hat{t}/\partial\bar{t} > 0 \qquad [t = \hat{t}(x) \quad \bar{t} = \bar{t}(x)].$$

Having fixed $\mathscr{E} \in W_{\mathscr{C}}$ arbitrarily, we can choose $\hat{t}$ *time-orthogonal* at $\mathscr{E}$ in the sense that we have there

$$(6.8) \qquad u^\dagger_L = 0 \qquad \text{with} \quad u^\dagger_L \equiv u_\rho\, x^\rho_L \qquad \text{and} \quad x^\rho_L = \frac{\partial\hat{x}^\rho}{\partial y^L} \quad \text{—see (6.4)—}.$$

§ 7. Pseudo-absolute derivative and Lagrangian spatial derivative of a double tensor field such as (2.4)

Consider the double tensor field $\hat{T}^{\cdots}_{\cdots}$ —see (2.4)—. Now, within relativity, it is regarded to change with the frames ϕ and ϕ^* in the obvious way, and also with the time parameter $\hat{t}$: the determination $\bar{T}^{\cdots}_{\cdots}$ of $\tilde{T}^{\cdots}_{\cdots}$ for $\bar{\phi} = \phi$, $\bar{\phi}^* = \phi^*$, and $\bar{t}$ —see (6.7)— is given by

$$\bar{T}^{\cdots}_{\cdots}(\underset{1}{H}^{\cdots}_{\cdots}, \ldots, \underset{m}{H}^{\cdots}_{\cdots}, x, \bar{t}, y) = \tilde{T}^{\cdots}_{\cdots}[\underset{1}{H}^{\cdots}_{\cdots}, \ldots, \underset{m}{H}^{\cdots}_{\cdots}, x, t(\bar{t}, y), y]. \tag{7.1}$$

In order to give (2.4) an interpretation completely independent of ϕ, ϕ^*, and $\hat{t}$, let us remember that, by the representation (6.4) of $\mathscr{M}$, (t, y) determines the event point $\mathscr{E} = \phi^{-1}[x(t, y)]$. Of course the argument x in (7.1) is meant to refer to $\mathscr{E}$ again, for many practical purposes. However, for other such purposes, e.g. calculating partial derivatives with respect to x^ρ, t, and y^R, these variables have to be able to run independently of one another. Therefore it is useful to set $E = \phi^{-1}(x)$ and to regard (7.1) as an expression in ϕ, ϕ^*, and $\hat{t}$ of the double tensor field

$$\mathbf{T} = \tilde{\mathbf{T}}(\underset{1}{\mathbf{H}}, \ldots, \underset{m}{\mathbf{H}}, E, \mathscr{E}), \tag{7.2}$$

whose values are attached at E and $P^* \in \mathscr{C}$, where P^* is determined by the condition $\mathscr{E} \in W_{P^*}$.

The *pseudo-absolute derivative* of the field (2.4) (connected with $\mathscr{M}$) can be defined by

$$\frac{D}{D^P s}\tilde{T}^{\sigma_1 \ldots S_1 \ldots}_{\rho_1 \ldots R_1 \ldots} = \left(\tilde{T}^{\cdots}_{\cdots /\alpha}\frac{\partial \hat{x}^\alpha}{\partial t} + \frac{\partial \tilde{T}^{\cdots}_{\cdots}}{\partial t}\right)\frac{Dt}{Ds} \qquad \left(\frac{Ds}{Dt} = \left(-g_{\alpha\beta}\frac{\partial \hat{x}^\alpha}{\partial t}\frac{\partial \hat{x}^\beta}{\partial t}\right)^{-1/2}\right) \tag{7.3}$$

and is independent of the choice of $\hat{t}$. For $m = 0$ this derivative is simply denoted by $D\ldots/Ds$, is called *absolute derivative*, and is a double tensor.

In order to deal with solid materials, the *Lagrangian spatial* (or *transverse*) *derivative*

$$\tilde{T}^{\sigma_1 \ldots S_1 \ldots}_{\rho_1 \ldots R_1 \ldots} = T^{\cdots}_{\cdots\, ;A} + \frac{DT^{\cdots}_{\cdots}}{D^P s}u^\dagger_A \qquad (u^\dagger_A = u_\rho x^\rho_A) \tag{7.4}$$

(connected with $\mathscr{M}$—see $(6.4)_1$ or $(6.6)_2$—) was introduced in [1] for $m = 0$ —see [2], p. 142. In (7.4) $\tilde{T}^{\cdots}_{\cdots|A}$ is determined in connection with the map $y \vdash \hat{x}(t, y)$—see $(6.4)_2$ and (2.8)—for any $m \geq 0$.

The usefulness of $\tilde{T}^{\cdots}_{\cdots|A}$ for $m = 0$ is due to its independence of the choice of $\hat{t}$, which generally fails to hold for each of the last two terms in (7.4) (2).

(2) In fact, on the one hand, the present definition (7.4) of $\tilde{T}^{\cdots}_{\cdots|A}$ yields the equality $\tilde{T}^{\cdots}_{\cdots\,|A} = \tilde{T}^{\cdots}_{\cdots\,;A}$ for $u^\dagger_A = 0$, which is the condition by which $T^{\cdots}_{\cdots\,|A}$ is defined in [2], p. 142 (for $m = 0$); and on the other hand, in [2], p. 143, $\tilde{T}^{\cdots}_{\cdots\,|A}$ is also proved to have the expression (7.4) for every choice of $\hat{t}$ ($m = 0$). The value of m is irrelevant as far as the independence of $\tilde{T}^{\cdots}_{\cdots\,|A}$ of $\hat{t}$ is concerned.

Note that for any $m \geq 0$, (α) $\tilde{T}^{\cdots}_{\cdots\,|A} = \tilde{T}^{\cdots}_{\cdots\,;A}$ *if and only if either* $u^{\dagger}_{A} = 0$ or $D\tilde{T}^{\cdots}_{\cdots}/D^{P}s = 0$; hence ($\beta$) $\tilde{T}^{\cdots}_{\cdots\,;A}$ *is independent of the choice of* $\hat{t}$ *if and only if* $D\tilde{T}^{\cdots}_{\cdots}/D^{P}s = 0$.

One also has—see (53.9) in [2], p. 143— [(3)]

$$\tilde{T}^{\cdots}_{\cdots\,|A} = \tilde{T}^{\cdots}_{\cdots\,/\rho}\,\alpha^{\rho}_{A} + \tilde{T}^{\cdots}_{\cdots\,/A} + \frac{\partial\tilde{T}^{\cdots}_{\cdots}}{\partial t}\,\frac{Dt}{Ds}\,u^{\dagger}_{A}\,, \tag{7.5}$$

where

$$\alpha^{\rho}_{A} = \overset{\perp}{g}{}^{\rho}_{\sigma}\,x^{\sigma}_{A} = x^{\rho}_{A} + u^{\rho}\,u^{\dagger}_{A} = \hat{x}^{\rho}_{A} + \frac{\partial\hat{x}^{\rho}}{\partial t}\,\frac{Dt}{Ds}\,u^{\dagger}_{A}\,. \tag{7.6}$$

§ 8. Some analogues for $D\overset{\sqcup}{T}{}^{\cdots}_{\cdots}/D^{P}s$ and $\overset{\sqcup}{T}{}^{\cdots}_{\cdots\,|R}$ of the connectionless and stationary derivatives introduced in §§ 3-4

In dealing with the fields $\tilde{T}^{\cdots}_{\cdots}$, $\underset{i}{\overset{\sqcup}{H}}{}^{\cdots}_{\cdots}$, and $\overset{\sqcup}{T}{}^{\cdots}_{\cdots}$ —see (2.4) or (7.2), and (3.1-2)—we identify $x = \hat{x}\,(t\,,y)$ with the representation $(6.4)_2$ of $\mathscr{M}$. In order to write natural analogues for $D\overset{\sqcup}{T}{}^{\cdots}_{\cdots}/D^{P}s$ and $\tilde{T}^{\cdots}_{\cdots\,|R}$, of the chain rules (3.5) and (4.8) for $\tilde{T}^{\cdots}_{\cdots\,;R}$, let us introduce the following analogues $\partial\tilde{T}^{\cdots}_{\cdots}/\partial^{P}s$ and $\tilde{T}^{\cdots}_{\cdots\,||R}$ of $\tilde{T}^{\cdots}_{\cdots\,;R}$ —see (3.4)—, and the subsequent analogues $D\tilde{T}^{\cdots}_{\cdots}/D^{St}s$ and $\tilde{T}^{\cdots}_{\cdots\,St|R}$ of $\tilde{T}^{\cdots}_{\cdots\,St;R}$—see (4.6-7)—: the *connectionless (pseudo-) absolute derivative*

$$\frac{\partial\tilde{T}^{\cdots}_{\cdots}}{\partial^{P}s} = \left(\frac{\partial\tilde{T}^{\cdots}_{\cdots}}{\partial x^{\rho}}\,\frac{\partial\hat{x}^{\rho}}{\partial t} + \frac{\partial\tilde{T}^{\cdots}_{\cdots}}{\partial t}\right)\frac{Dt}{Ds} = \tilde{T}^{\cdots}_{\cdots\,,\rho}\,u^{\rho} + \frac{\partial\tilde{T}^{\cdots}_{\cdots}}{\partial t}\,\frac{Dt}{Ds}\,; \tag{8.1}$$

the *connectionless Lagrangian spatial derivative*

$$\tilde{T}^{\cdots}_{\cdots\,||R} = \tilde{T}^{\cdots}_{\cdots\,;;R} + \frac{\partial\tilde{T}^{\cdots}_{\cdots}}{\partial^{P}s}\,u^{\dagger}_{R} \qquad \text{—see (7.4);} \tag{8.2}$$

the *stationary* (or *covariant partial*) *absolute derivative*

$$\frac{D\tilde{T}^{\cdots}_{\cdots}}{D^{St}s}\left(\underset{i}{H}{}^{\cdots}_{\cdots}\,,\dots,\underset{m}{H}{}^{\cdots}_{\cdots}\,,x\,,t\,,y\right) = \frac{D\overset{\sqcup}{T}{}^{\cdots}_{\cdots}}{Ds} \quad \text{for } \underset{i}{\overset{\sqcup}{H}}{}^{\cdots}_{\cdots} = \underset{i}{\breve{H}}{}^{\cdots}_{\cdots} \tag{8.3}$$

$$\text{—see (3.2)}\,, \qquad (i = 1\,,\dots,m)\,,$$

(3) Fix the index ρ and set $l\,(x) = x^{\rho}$. Then $x^{\rho}_{A} = l\,(x)_{;A}$ and $\alpha^{\rho}_{A} = l\,(x)_{|A}$. This iustifies writing $x^{\rho}_{A} = x^{\rho}{}_{;A}$ and $\alpha^{\rho}_{A} = x^{\rho}{}_{|A}$, and shows that α^{ρ}_{A} is independent of the choice of $\hat{t}$. However that x^{ρ}_{A}, and hence α^{ρ}_{A} are double tensors of covariant [controvariant] order (0 , 1) [(1 , 0)] has to, and can easily be proved directly–see [2], (A 2.5) on p. 260 and (53.8) on p. 143.

where the fields $\underset{i}{\breve{\mathrm{H}}}{}^{\cdots}_{\cdots}$ satisfy the local conditions $(4.2)_1$ and

$$\frac{\mathrm{D}}{\mathrm{D}s}\underset{i}{\breve{\mathrm{H}}}{}^{\cdots}_{\cdots}(x,t,y)=0\;; \tag{8.4}$$

and lastly the *stationary* (or *covariant partial*) *Lagrangian spatial derivative*

$$\tilde{\mathrm{T}}{}^{\cdots}_{\cdots}(\underset{i}{\mathrm{H}}{}^{\cdots}_{\cdots},\ldots,\underset{m}{\mathrm{H}}{}^{\cdots}_{\cdots},x,t,y)_{\mathrm{S}t|\mathrm{R}}=\hat{\mathrm{T}}{}^{\cdots}_{\cdots}{}_{|\mathrm{R}} \text{ for } \underset{i}{\breve{\mathrm{H}}}{}^{\cdots}_{\cdots}=\underset{i}{\breve{\mathrm{H}}}{}^{\cdots}_{\cdots} \quad (i=1,\ldots,m)\,, \tag{8.5}$$

where the fields $\underset{i}{\breve{\mathrm{H}}}{}^{\cdots}_{\cdots}$ satisfy the local conditions (4.2) and (8.4), or at least $(4.2)_1$ and $\underset{i}{\breve{\mathrm{H}}}{}^{\cdots}_{\cdots}{}_{|\mathrm{R}}=0$ $(i=1,\ldots,m)$.

Note that (8.1-2), (3.4), and (7.6) imply the equality

$$\tilde{\mathrm{T}}{}^{\cdots}_{\cdots}{}_{||\mathrm{R}}=\tilde{\mathrm{T}}{}^{\cdots}_{\cdots}{}_{,\rho}\,\alpha^{\rho}_{\mathrm{R}}\,\tilde{\mathrm{T}}{}^{\cdots}_{\cdots}{}_{,\mathrm{R}}+\frac{\partial\tilde{\mathrm{T}}{}^{\cdots}_{\cdots}}{\partial t}\frac{\mathrm{D}t}{\mathrm{D}s}u^{\dagger}_{\mathrm{R}}\,, \tag{8.6}$$

which is to (7.5) as (8.2) is to (7.4).

By (8.1), (7.3), and $(2.5)_1$,

$$\frac{\mathrm{D}}{\mathrm{D}^{\mathrm{P}}s}\tilde{\mathrm{T}}{}^{\cdots}_{\cdots}=\frac{\partial\tilde{\mathrm{T}}{}^{\cdots}_{\cdots}}{\partial^{\mathrm{P}}s}-\frac{\mathrm{D}t}{\mathrm{D}s}\frac{\partial\hat{x}^{\rho}}{\partial t}\,\mathrm{S}t_{,\rho}\,\mathrm{T}{}^{\cdots}_{\cdots}\,, \tag{8.7}$$

while by (8.6), (7.5), and (2.5)

$$\tilde{\mathrm{T}}{}^{\cdots}_{\cdots}{}_{|\mathrm{R}}=\tilde{\mathrm{T}}{}^{\cdots}_{\cdots}{}_{||\mathrm{R}}-\mathrm{S}t_{||\mathrm{R}}\,\mathrm{T}{}^{\cdots}_{\cdots} \tag{8.8}$$

with

$$\mathrm{S}t_{||\mathrm{R}}\,\mathrm{T}{}^{\cdots}_{\cdots}=(\mathrm{S}t_{,\rho}\,\mathrm{T}{}^{\cdots}_{\cdots})\,\alpha^{\rho}_{\mathrm{R}}+\mathrm{S}t_{,\mathrm{R}}\,\mathrm{T}{}^{\cdots}_{\cdots} \qquad \text{—see (2.6).} \tag{8.9}$$

Remark that, by (8.8), *for* ϕ *and* ϕ^* *fixed,* $\breve{\mathrm{T}}{}^{\cdots}_{\cdots}{}_{||\mathrm{R}}$ *is independent of the choice of* $\hat{t}$. (In fact this obviously holds for $\mathrm{S}t_{||\mathrm{R}}\,\mathrm{T}{}^{\cdots}_{\cdots}$ and $\breve{\mathrm{T}}{}^{\cdots}_{\cdots}{}_{|\mathrm{R}}$,).

§ 9. Four chain rules for pseudo-absolute derivatives and Lagrangian spatial derivatives. Explicit expression for $\mathrm{D}\tilde{\mathrm{T}}{}^{\cdots}_{\cdots}/\mathrm{D}^{\mathrm{S}t}s$ and $\tilde{\mathrm{T}}{}^{\cdots}_{\cdots}{}_{\mathrm{S}t|\mathrm{R}}$

By (3.2), (7.3), and (8.1) we have *the chain rule*

$$\frac{\mathrm{D}\breve{\mathrm{T}}{}^{\cdots}_{\cdots}}{\mathrm{D}s}=\sum_{i=1}^{m}\frac{\partial\tilde{\mathrm{T}}{}^{\cdots}_{\cdots}}{\partial\underset{i}{\mathrm{H}}{}^{\cdots}_{\cdots}}\frac{\partial\underset{}{\breve{\mathrm{H}}}{}^{\cdots}_{\cdots}}{\partial^{\mathrm{P}}s}+\frac{\mathrm{D}\tilde{\mathrm{T}}{}^{\cdots}_{\cdots}}{\mathrm{D}^{\mathrm{P}}s} \qquad \left(\frac{\mathrm{D}\breve{\mathrm{T}}{}^{\cdots}_{\cdots}}{\mathrm{D}s}=\frac{\mathrm{D}\breve{\mathrm{T}}{}^{\cdots}_{\cdots}}{\mathrm{D}^{\mathrm{P}}s}\right) \tag{9.1}$$

for the compound function (3.2) *with generally noncovariant terms.*

By applying relation (8.7) (in $\tilde{T}^{\cdots}_{\cdots}$) to $\underset{i}{\overset{\sqcup}{H}}{}^{\cdots}_{\cdots}$, one turns (9.1) into

$$(9.1') \qquad \frac{D\overset{\sqcup}{T}{}^{\cdots}_{\cdots}}{Ds} = \sum_{i=1}^{m} \frac{\partial \tilde{T}^{\cdots}_{\cdots}}{\partial \underset{i}{H}{}^{\cdots}_{\cdots}} \left(\frac{D\underset{i}{\overset{\sqcup}{H}}{}^{\cdots}_{\cdots}}{D^{P}s} + \frac{Dt}{Ds}\frac{\partial \hat{x}^{\rho}}{\partial t} St_{,\rho}\, \underset{i}{H}{}^{\cdots}_{\cdots} \right) + \frac{DT^{\cdots}_{\cdots}}{D^{P}s} \, .$$

Now, first, identify $\underset{i}{\overset{\sqcup}{H}}{}^{\cdots}_{\cdots}$ with a field $\underset{i}{\breve{H}}{}^{\cdots}_{\cdots}$ that satisfies $(4.2)_1$, and (8.4) ($i = 1, \dots, m$). Then by (9.1′) one can *render* (8.3) *explicit*:

$$(9.2) \qquad \frac{D\tilde{T}^{\cdots}_{\cdots}}{D^{St}s} = \frac{D\tilde{T}^{\cdots}_{\cdots}}{D^{P}s} + \frac{Dt}{Ds}\frac{\partial x^{\rho}}{\partial t} \sum_{i=1}^{m} \frac{\partial \tilde{T}^{\cdots}_{\cdots}}{\partial \underset{i}{H}{}^{\cdots}_{\cdots}} St_{,\rho}\, \underset{i}{H}{}^{\cdots}_{\cdots} \, .$$

Furthermore, by considering $\underset{i}{\overset{\sqcup}{H}}{}^{\cdots}_{\cdots}$ to $\underset{m}{\overset{\sqcup}{H}}{}^{\cdots}_{\cdots}$ as arbitrary, from (9.1′) and (9.2) one deduces the *chain rule*

$$(9.3) \qquad \frac{D\hat{T}^{\cdots}_{\cdots}}{Ds} = \sum_{i=1}^{m} \frac{\partial \tilde{T}^{\cdots}_{\cdots}}{\partial \underset{i}{H}{}^{\cdots}_{\cdots}} \frac{D\underset{i}{\overset{\sqcup}{H}}{}^{\cdots}_{\cdots}}{Ds} + \frac{D\tilde{T}^{\cdots}_{\cdots}}{D^{St}s} \qquad \left(\frac{D\underset{i}{\overset{\sqcup}{H}}{}^{\cdots}_{\cdots}}{Ds} = \frac{D\underset{i}{\overset{\sqcup}{H}}{}^{\cdots}_{\cdots}}{D^{P}s} \right)$$

for compound function (3.2), *all of whose terms are covariant.*

From (3.2) and (8.6) we easily obtain the equality

$$(9.4) \qquad \overset{\sqcup}{T}{}^{\cdots}_{\cdots\, ||R} = \sum_{i=1}^{m} \frac{\partial \tilde{T}^{\cdots}_{\cdots}}{\partial \underset{i}{H}{}^{\cdots}_{\cdots}} \underset{i}{\overset{\sqcup}{H}}{}^{\cdots}_{\cdots\, ||R} + \tilde{T}^{\cdots}_{\cdots\, ||R}$$

which, by (8.8), yields the *chain rule*

$$(9.5) \qquad \breve{T}^{\cdots}_{\cdots\, |R} = \sum_{i=1}^{m} \frac{\partial \tilde{T}^{\cdots}_{\cdots}}{\partial \underset{i}{H}{}^{\cdots}_{\cdots}} \underset{i}{\breve{H}}{}^{\cdots}_{\cdots\, ||R} + \tilde{T}^{\cdots}_{\cdots\, |R}$$

for the compound function (3.2), *with generally noncovariant terms.* By (8.8), (9.5) is equivalent to

$$(9.5') \qquad \breve{T}^{\cdots}_{\cdots\, |R} = \sum_{i=1}^{m} \frac{\partial \tilde{T}^{\cdots}_{\cdots}}{\partial \underset{i}{H}{}^{\cdots}_{\cdots}} \left(\underset{i}{\breve{H}}{}^{\cdots}_{\cdots\, |R} + St_{||R}\, \underset{i}{H}{}^{\cdots}_{\cdots} \right) + \tilde{T}^{\cdots}_{\cdots\, |R} \, .$$

First identify $\underset{i}{\overset{\sqcup}{\mathrm{H}}}{}^{\cdots}_{\cdots}$ with a field $\underset{i}{\breve{\mathrm{H}}}{}^{\cdots}_{\cdots}$ that satisfies $(4.2)_1$ and the condition $\underset{i}{\breve{\mathrm{H}}}{}^{\cdots}_{\cdots}{}_{|\mathrm{R}} = 0$ locally $(i = 1, \dots, m)$. Then by (9.5') we can ***render*** (8.5) ***explicit***:

$$(9.6) \qquad \tilde{\mathrm{T}}^{\cdots}_{\cdots}(\underset{i}{\mathrm{H}}{}^{\cdots}_{\cdots}, \dots, \underset{m}{\mathrm{H}}{}^{\cdots}_{\cdots}, x, t, y)_{\mathrm{S}t;\mathrm{R}} = \tilde{\mathrm{T}}^{\cdots}_{\cdots}{}_{|\mathrm{R}} + \sum_{i=1}^{m} \frac{\partial \tilde{\mathrm{T}}^{\cdots}_{\cdots}}{\partial \underset{i}{\mathrm{H}}{}^{\cdots}_{\cdots}} \mathrm{S}t_{||\mathrm{R}}\, \underset{i}{\mathrm{H}}{}^{\cdots}_{\cdots}$$

—see (8.9).

Furthermore, regarding $\underset{1}{\overset{\sqcup}{\mathrm{H}}}{}^{\cdots}_{\cdots}$ to $\underset{m}{\overset{\sqcup}{\mathrm{H}}}{}^{\cdots}_{\cdots}$ as arbitrary, $(9.5')$ and (9.6) yield *the chain rule*

$$(9.7) \qquad \overset{\sqcup}{\mathrm{T}}{}^{\cdots}_{\cdots}{}_{|\mathrm{R}} = \sum_{i=1}^{m} \frac{\partial \tilde{\mathrm{T}}^{\cdots}_{\cdots}}{\partial \underset{i}{\mathrm{H}}{}^{\cdots}_{\cdots}} \underset{i}{\overset{\sqcup}{\mathrm{H}}}{}^{\cdots}_{\cdots}{}_{|\mathrm{R}} + \tilde{\mathrm{T}}^{\cdots}_{\cdots}{}_{\mathrm{S}t|\mathrm{R}}$$

for the compound function (3.2), *all of whose terms are covariant and independent of the choice of* $\hat{t}$.

Of course by replacing $\tilde{\mathrm{T}}^{\cdots}_{\cdots}{}_{\mathrm{S}t|\mathrm{R}}$ with $\tilde{\mathrm{T}}^{\cdots}_{\cdots}{}_{|\mathrm{R}}$ in (9.7) we turn (9.7) into a generally false equality (like (3.6)).

§ 10. Expressions for the stress divergence of a relativistic hyperelastic body afforded by two among the chain rules above

Consider the hyper-elastic body $\mathscr{C}'$, introduced in § 5, within (general) relativity, mutatis mutandis (according to [2]). Then the first Piola-Kirchhoff stress tensor $\mathrm{K}^{\alpha\mathrm{B}}$ is spatial ($u_\alpha \mathrm{K}^{\alpha\mathrm{B}} \equiv 0$); and it is given by a constitutive function of the form

$$(10.1) \qquad \mathrm{K}^{\alpha\mathrm{B}} = \tilde{\mathrm{K}}^{\alpha\mathrm{B}}(y, \alpha^{\mu}_{\mathrm{A}}, g_{\lambda\mu}, u^{\lambda}, \phi^*) \qquad (g_{\lambda\mu} u^{\lambda} u^{\mu} = -1),$$

where $g_{\lambda\mu}$ can be the metric tensor at some $\mathscr{E} \in \mathrm{S}_4$ in admissible coordinates [(4)]

Assume that $\mathrm{X}^{\rho\sigma}$ is the Euler spatial stress tensor and $\mathscr{D}$ is the actual proper volume per unit reference proper volume. Then, by the dynamic equation

(4) $\tilde{\mathrm{K}}^{\alpha\mathrm{B}}$ behaves under changes of ϕ^* in the obvious way; and it is determined by the function induced by it for any particular choice $\overline{\phi^*}$ of ϕ^*, and for $g_{\lambda\mu} = \delta'_{\lambda\mu}$—see $(6.1)_{2-3}$—and $u^{\lambda} = \delta_0{}^{\lambda}$. More in detail express $\overline{\phi^*} \circ \phi^{*-1}$ by $\bar{y} = \bar{y}(y)$, consider any matrix $\overset{\circ}{x}{}^{\rho}_{\mathrm{L}}$ for which $g_{\lambda\mu} = \delta'_{\rho\sigma}\, \overset{\circ}{x}{}^{\rho}_{\lambda}\, \overset{\circ}{x}{}^{\sigma}_{\mu}$ and $u^{\lambda} = x^{\lambda}_{0}$, where $(x^{\lambda}_{\rho}) = (\overset{\circ}{x}{}^{\rho}_{\lambda})^{-1}$. Then (10.1) holds if and only if

$$\mathrm{K}^{\alpha\mathrm{B}} = x^{\alpha}_{\rho} \frac{\delta y^{\mathrm{B}}}{\delta \bar{y}^{\mathrm{S}}} \tilde{\mathrm{K}}^{\rho\mathrm{S}}(\bar{y}, \bar{\alpha}^{\lambda}_{\mathrm{R}}, \delta'_{\lambda\mu}, u^{\lambda}, \overline{\phi^*}) \qquad \text{where} \quad \bar{\alpha}^{\lambda}_{\mathrm{R}} = \bar{\alpha}^{\mu}_{\mathrm{A}}\, x^{\lambda}_{\mu} \frac{\delta y^{\mathrm{A}}}{\delta \bar{y}^{\mathrm{R}}}.$$

In fact, as far as the cronotopic indices are concerned the equivalence above follows from the objectivity principle—see [2] p. 209—or from a relativistic principle of 1st order physical indistinguishability of locally natural (i.e. geodesic and pseudo- Euclidean) frames. The latter principle is related to the physical indistinguishability of inertial frames.

(24.8) for $q^\alpha = 0$ and formula (60.9) in [2], the resultant I^ρ of the contact forces per unit reference proper volume has the expressions

$$
(10.2) \qquad I^\rho = -\mathscr{D} \overset{\perp}{g}{}^\rho_\alpha \, X^{\alpha\beta}{}_{/\gamma} \, \overset{\perp}{g}{}^\gamma_\beta = -\overset{\perp}{g}{}^\rho_\alpha \, K^{\alpha B}{}_{|B} \quad [X^{\alpha B} = X^{\alpha B}(x)\,,\ K^{\alpha B} = K^{\alpha B}(t\,,y)]\,.
$$

Since ϕ^*, ϕ, and the representation (6.4) of $\mathscr{M}$ have to be regarded as given, this holds also for the functions $\alpha^\mu_A = \alpha^\mu_A(t\,,y)$ —see (7.6)—, $g_{\lambda\mu} = g_{\lambda\mu}(x)$, and $u^\lambda = u^\lambda(t\,,y) = [\partial\hat{x}(t\,,y)/\partial t]\,Dt/Ds$. Hence by (8.6)

$$
(10.3) \qquad \alpha^\mu_{A||B} = \alpha^\mu_{A,B} + \frac{Dt}{Ds}\frac{\partial\alpha^\mu_A}{\partial t} u^\dagger_B\,, \qquad g_{\lambda\mu||B} = g_{\lambda\mu,\sigma}\,\alpha^\rho_B\,,
$$

$$
u^\lambda{}_{||B} = u^\lambda{}_{,B} + \frac{Dt}{Ds}\frac{\partial u^\lambda}{\partial t} u^\dagger_B\,.
$$

Rules (9.5) and (9.7) can be applied to function (10.1) for $\underset{1}{\breve{H}}{}^\mu_A(t\,,y) = \alpha^\mu_A$, $\underset{2}{\breve{H}}_{\lambda\mu}(x) = g_{\lambda\mu}$, and $\underset{3}{\breve{H}}{}^\lambda(t\,,y) = u^\lambda$. Furthermore, by $(10.1)_2$, $\partial\tilde{K}^{\alpha B}/\partial u^\lambda = \xi\,u_\lambda$ for some $\xi \in \mathbf{R}$. Hence—see $(10.3)_3$—

$$
(10.4) \qquad (\partial\tilde{K}^{\alpha B}/\partial u^\lambda)\,u^\lambda{}_{||B} = 0 \quad , \quad (\partial\tilde{K}^{\alpha B}/\partial u^\lambda)\,u^\lambda{}_{|B} = 0\,.
$$

Now call $K^{\alpha B} = \hat{K}^{\alpha B}(t\,,y)$ the field induced by $\tilde{K}^{\alpha B}$ along $\mathscr{M}$. Then, on the one hand, by (9.5) and $(10.4)_1$

$$
(10.5) \qquad \hat{K}^{\alpha B}{}_{|B} = \frac{\partial\tilde{K}^{\alpha B}}{\partial\alpha^\mu_A}\,\alpha^\mu_{A||B} + \frac{\partial\tilde{K}^{\alpha B}}{\partial g_{\lambda\mu}}\,g_{\lambda\mu||B} + \tilde{K}^{\alpha B}{}_{|B} \; (\hat{K}^{\alpha B}{}_{|B} = \breve{K}^{\alpha B}{}_{|B})\,.
$$

Furthermore by (10.1) and (7.4) $\tilde{K}^{\alpha B}{}_{|B} = \tilde{K}^{\alpha B}{}_{/B}$. Hence, by $(10.3)_{1-2}$, (10.2) *affords the expression*

$$
(10.6) \qquad I^\rho = -\overset{\perp}{g}{}^\rho_\alpha \left[\frac{\partial\tilde{K}^{\alpha B}}{\partial\alpha^\mu_A}\left(\alpha^\mu_{A,B} + \frac{Ds}{Dt}\frac{\partial\alpha^\mu_A}{\partial t} u^\dagger_B\right) + \frac{\partial\tilde{K}^{\alpha B}}{\partial g_{\lambda\mu}}\,g_{\lambda\mu,\sigma}\,\alpha^\sigma_B + \tilde{K}^{\alpha B}{}_{/B}\right]
$$

for I^ρ. Thus $-\,I^\rho$ appears as the spatialization of a sum whose terms are generally noncovariant *but* (for ϕ and ϕ^* fixed) *are independent of the choice of* $\hat{t}$.

On the other hand, (9.7) and $(10.4)_2$ yield the 1st of the equalities

$$
(10.7) \qquad \hat{K}^{\alpha B}{}_{|B} = \frac{\partial\tilde{K}^{\alpha B}}{\partial\alpha^\mu_A}\,\alpha^\mu_{A|B} + \frac{\partial\tilde{K}^{\alpha B}}{\partial g_{\lambda\mu}}\,g_{\lambda\mu|B} + \tilde{K}^{\alpha B}{}_{St|B}\,,\; g_{\lambda\mu|B} = 0\,,
$$

while $(10.7)_2$ follows from (7.4). Furthermore (9.6), (8.9), and (2.6) easily yield, by $(10.7)_2$,

$$(10.8)\qquad \tilde{K}^{\alpha B}{}_{St|B} = \tilde{K}^{\alpha B}{}_{/B} + \frac{\partial \tilde{K}^{\alpha B}}{\partial \alpha^{\mu}_{A}} \left(\left\{ {S \atop AB} \right\}^{*} \alpha^{\mu}_{S} - \left\{ {\mu \atop \rho\sigma} \right\} \alpha^{\sigma}_{A} \alpha^{\rho}_{B} \right) + \frac{\partial \tilde{K}^{\alpha B}}{\partial g_{\lambda\mu}} g_{\lambda\mu,\rho}\, \alpha^{\rho}_{B} - \frac{\partial \tilde{K}^{\alpha B}}{\partial u^{\lambda}} \left\{ {\lambda \atop \rho\sigma} \right\} u^{\sigma} \alpha^{\rho}_{B} \,;$$

in fact by $(10.7)_2$ the last but one term in (10.8) equals the term in $\partial\tilde{K}^{\alpha B}/\partial g_{\lambda\mu}$ and in the Christoffel's symbols $\{\ \}$ that arises directly by (9.6), (8.9), and (2.6). It has been preferred, because it is simpler than the latter, and also this contains partial derivatives of $g_{\lambda\mu}$ through $\{\ \}$. Instead Christoffel's symbols have been left in other terms of (10.8), because thus the use of $\alpha^{\mu}_{S,B}$ or $u^{\mu}{}_{,B}$ is avoided.

By (10.7) and (10.2), *we have the simple expression*

$$(10.9)\qquad I^{\rho} = -\overset{\perp}{g}{}^{\rho}_{\alpha} \left[\frac{\partial \tilde{K}^{\alpha B}}{\partial \alpha^{\mu}_{A}} \alpha^{\mu}_{A|B} + \tilde{K}^{\alpha B}{}_{St|B} \right] \qquad \text{—see (10.8)—}$$

for I^{ρ}, *by which* $-I^{\rho}$ *appears as the spatialization of a sum of two terms that are both independent of* $\hat{t}$ *and covariant.*

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