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Energy of the harmonics in a vibrating string after the impact of a hammer

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Fisica matematica. — *Energy of the harmonics in a vibrating string after the impact of a hammer.* Nota di FRANCO RAMPAZZO, presentata (*) dal Corrisp. A. BRESSAN.

RIASSUNTO. — In questa Nota vengono usati alcuni risultati precedentemente ottenuti – v. [4] e [5] –, riguardanti l'urto di un martelletto rigido e di una corda elastica. Da essi possono dedursi le condizioni della corda – deformazione e atto di moto – all'istante in cui essa rimane libera dall'influenza del martelletto. È dunque possibile determinare mediante l'analisi di Fourier, i valori delle energie delle varie armoniche, i quali, com'è ben noto, determinano il timbro del suono emesso dalla corda (timbro del pianoforte).

1. INTRODUCTION

The oscillatory motion of a string after an impact with a rigid little hammer has been already studied by several authors in the past (see i.e. [8], [3], [6], and [2]), also because of the special interest of this subject for the acoustics of the piano. In fact, by means of the Fourier analysis, one can establish the amplitude, and thus the energy, of the harmonic modes that form the motion of the string: as is well known, the ratios between the energies of these harmonic modes determine what is called the " timbre " of the sound emitted by the string.

In the linear approximation the motion of the string is determined by the initial conditions, i.e. the velocity and displacement fields at a given initial instant t_0.

Some authors set out the problem of the initial conditions by considering the initial elongation to vanish everywhere and the initial speed equal to zero except on the little tract on which the hammer strikes the string. Others consider the impact as represented by a force acting on a little tract of the string for a while.

Both these approaches meet objections when more realistic considerations about the impact are made.

Particularly, the first approach neglects that there must be a time interval in which the hammer and the string are in contact. Instead, the second one considers the impelling force as a known datum, in spite of the fact that in many actual cases this force is unknown.

For example, in the piano, the trasmission between the key and the hammer is made in such a way that, in the last part of its running, the hammer goes on freely, out of the control of the player.

(*) Nella seduta dell'8 febbraio 1986.

In order to overcome these difficulties and in accordance with the description given by the acoustic physicist A. Benade (see [1]), another approach to the problem is considered here.

Precisely, after the impact, the hammer $\mathscr{X}$ and the string $\mathscr{S}$ are regarded as forming a single mechanical system $\mathscr{S} + \mathscr{X}$: $\mathscr{X}$ and $\mathscr{S}$ remain joined as long as the absolute value of the acceleration of the contact point $Q = \mathscr{S} \cap \mathscr{X}$ is higher than the gravity acceleration g (this is the case of the grand coda piano, in which the strings are hit from the bottom upwards; instead, in the case of the upright piano, the detatching takes place when the absolute value of the acceleration vanishes).

Assume that: (i) $\mathscr{X}$ consists of a rigid material segment O Q and a mass concentrated in Q; (ii) $\mathscr{X}$ can only turn around the fixed point O on a plane containing $\mathscr{S}$; (iii) O is very close to the string, so that OQ can be regarded, on the one hand, as parallel to the string at the impact, and on the other hand, as influencing the motion only through Q. Then it is trivial to show that the motions of $\mathscr{S} + \mathscr{X}$ with $\mathscr{S} \cap \mathscr{H} = Q$, under the action of the gravity forces and of the tensions at the end points, are substantially the motions of a system $\mathscr{S}_M$ formed by $\mathscr{S}$ with a mass $M = m_Q + m/3$ concentrated in Q, when the "acceleration of gravity" on Q is $\bar{g} = \alpha g$, $\alpha = (m_Q + m/2)/M$. Here m and m_Q are the mass of OQ and the concentrated mass of $\mathscr{X}$, respectively.

In two previous papers—see [4] and [5]—the transversal oscillatory motion of a system $\mathscr{S}_M$ was analysed in the linear approximation. In particular, in [5] the motion was determined in connection with the following initial conditions:

(A) The displacement field vanishes at $t = 0$.

(B) The speed of every point P of $\mathscr{S} + \mathscr{X}$, different from $Q = \mathscr{S} \cap \mathscr{H}$ vanishes at $t = 0$.

The schematization of the impact that is proposed here differs from the preceding schematizations just in that initial conditions (A) and (B) are considered for $\mathscr{S}_M$ and not for $\mathscr{S}$.

In [5] an equation was also found whose smallest positive solution is the first instant t_0 at which Q has the acceleration g: consequently it was possible to determine the configuration and velocity distributions for $\mathscr{S}_M$ at the instant t_0. This is all we need here, because now it is sufficient to regard the configuration and the velocity distribution of $\mathscr{S}_M$ at t_0 as initial conditions for the system $\mathscr{S}$ and to compute, by the Fourier analysis, the amplitudes of the harmonic modes. This is in fact what is done in NN.4—5 under a particular hypothesis on the physical constants of the problem.

Although the procedure holds for any set of values for the physical constants, in general it is very laborious to perform all necessary calculations. To avoid these drawbacks, a different method for the general case is illustrated in N. 6: this method is based on considerations made in [5] and concerning the propagation of singularities on the system $\mathscr{S}_M$.

2. SOME RESULTS ATTEINED IN [5]

Now let us recall some results of [5], that will be useful in the next sections.

Under the initial conditions (A) and (B)—see N. 1—the motion $y = u(x, t)$ of $\mathscr{S}_M$ is given, in the linear approximation, by

$$(2.1)\qquad u(x, t) = 1/K \left\{ \sum_{r=\nu}^{n} (-1)^{r+1} u_t \left(\bar{x}, t - x/a + ((-1)^r \bar{x} - 2[r/2] L)/a\right) + \sum_{s=1-\nu}^{m} (-1)^{s+1} u_t \left(\bar{x}, t + x/a + ((-1)^{s+1} \bar{x} - 2[(s+1)/2] L)/a\right) + f(n, m) v_0 \right\}$$

where: (i) $K = 2\rho_0 a/M$; (ii) $a = (T/\rho_0)^{1/2}$ (wave speed), T and ρ_0 being the (necessarily) constant tension and the (supposed) constant line-density of $\mathscr{S}_M$ respectively, in the (reference) configuration that $\mathscr{S}$ assumes when it is acted on by a couple of tractions at its extremes; (iii) $\bar{x}$ is the Lagrangean co-ordinate of Q; (iv) $2f(n, m) = (-1)^n + (-1)^m$; (v) $[\xi]$ denotes the integer part of ξ; (vi) $\nu = 0$ if $x \geq \bar{x}$ and $\nu = 1$ if $x < \bar{x}$; (vii) $(x, t) \in S_n \cap D_m$ $(\subseteq [0, L] \times [0, +\infty[)$ with

$$S_i = \left\{(x, y) \mid [i/2]\, 2L + x - (-1)^i (L - \bar{x}) \leq y \leq [(i+1)/2]\, 2L + x + (-1)^i (L - \bar{x})\right\} \cap \Big([0, L] \times [0, +\infty[\Big)$$

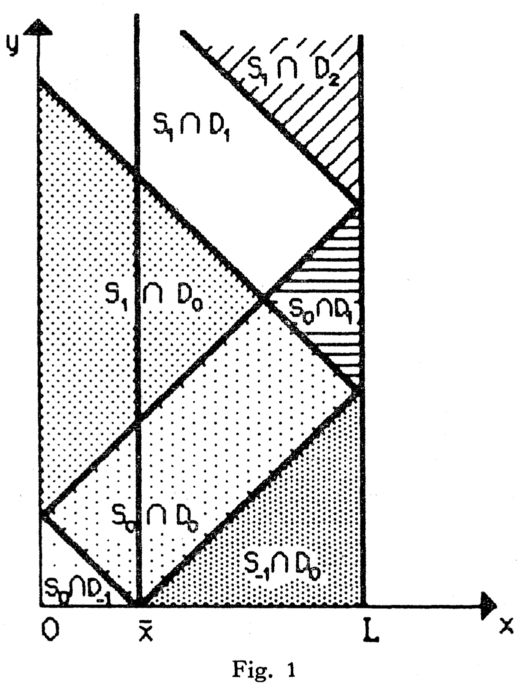

Fig. 1

$$D_i = \Big\{(x\,,y) \mid [(i+1)/2]\,2\,L - x + (-1)^i\,(L - \bar{x}) \leq y \leq (1 + [(i+$$

$$+1)/2])\,2\,L - x + (-1)^i\,(L - \bar{x})\Big\} \cap \Big([0\,,L] \times [0\,,+\infty[\Big)$$

Now, $u_t(\bar{x}\,,t)$ can be explicitly known in certain time intervals $[\tau_{i-1}\,,\tau_i]$ (with $i \in \mathbb{N}$ and $\lim_{i \to +\infty} \tau_i = +\infty$) in which the time semi-axis $[0 + \infty[$ is subdivided (see [5] for more details). In particular, for $t \in [0\,,\,4\,\bar{x}/a]$ we obtained

$$(2.2) \qquad u_t(\bar{x}\,,t) = \begin{cases} v_0\,e^{-Kt} & 0 \leq t \leq 2\,\bar{x}/a \\ v_0\,e^{-Kt}\,(1 - K e^{2K\bar{x}/a}\,(t - 2\,\bar{x}/a)) & 2\,\bar{x}/a \leq t \leq 4\,\bar{x}/a \end{cases}$$

As soon as $v(t) = u_t(\bar{x}\,,t)$ is known in $[0\,,+\infty[$, in the schematization considered in [5], the instant t_0 of detatching is the least positive solution of the equation

$$(2.3) \qquad \dot{v}(t) = -g\,.$$

In [5] it is shown that, if v_0 is greater than a reasonable value (for the piano hammer), then (2.3) can be satisfactorily replaced by

$$(2.4) \qquad \dot{v}(t) = 0\,.$$

Afterwards it was observed that (2.4) never has a solution in $[0\,,2\,\bar{x}/a]$—see (2.2)—and that t_0 is in $[2\,\bar{x}/a\,,4\,\bar{x}/a]$ if and only if the physical constants satisfy the condition $M < 4/1.28\,\rho_0\,\bar{x}$. In this case

$$(2.5) \qquad t_0 = 2\,\bar{x}/a + (1 + e^{2K\bar{x}/a})/(K e^{2K\bar{x}/a})\,.$$

3. Statement of the problem

Let us consider a finite, elastic, and perfectly flexible string $\mathscr{S}$ with fixed end points A and B, for which only adiabatic processes are considered, so that it can be regarded as purely mechanical. Let $\mathscr{C}^*$ be the equilibrium configuration of $\mathscr{S}$ when the external forces reduce to the reactions at A and B. These are two tractions $\mathbf{T}$ and $-\mathbf{T}$, and $\mathscr{C}^*$ is rectilinear with a tension T that equals $|\mathbf{T}|$ everywhere. We shall regard $\mathscr{C}^*$ as reference configuration and suppose that the line-density ρ_0 is constant in it.

If AB is horizontal we can introduce the orthonormal Euclidean frame $(O\,,x\,,y\,,z)$ for which A is in $(0\,,0\,,0)$ and B is in $(L\,,0\,,0)$ $(L > 0)$ and the gravity acceleration has the co-ordinate triple $(0\,,0\,,-g)$.

We want to consider small oscillations of $\mathcal{S}$ in the direction of the y-axis when $\mathcal{S}$ is acted on only by the gravity force (besides the reactions at A and B). In the linear approximation they are independent of the motion along the x- and z-directions and they are governed by the well-known differential equation

$$\mathrm{T} w_{xx}(x,t) - \rho_0 w_{tt}(x,t) = \rho_0 g \tag{3.1}$$

where the function $y = w(x,t)$, defined in $[0, \mathrm{L}] \times [0, +\infty[$ represents the y-component of the motion of $\mathcal{S}$.

Since the end points are fixed,

$$w(0,t) \equiv w(\mathrm{L},t) \equiv 0 \qquad \forall t \in [0 + \infty[\tag{3.2}$$

The equilibrium solution $(w_{tt}(x,t) \equiv 0)$ of (3, 1-2) is

$$\mathcal{E}(x) = \rho_0 g/(2\,\mathrm{T})\,(x^2 - \mathrm{L}x) \tag{3.3}$$

Let $w_1(x,t)$ be a solution of (3.1-2) and let us consider $w_2(x,t) = w_1(x,t) - \mathcal{E}(x)$. Then $w_2(x,t)$ is a solution of the homogeneous equation

$$\mathrm{T}\, w_{xa}(x,t) - \rho_0 w_{tt}(x,t) = 0 \tag{3.4}$$

and satisfies (3.2).

More precisely we know that <u>$w_1(x,t)$ is a solution of (3.1-2) that satisfies the initial conditions</u>

$$w(x,0) = \varphi(x)\,, \qquad w_t(x,0) = \psi(x)$$

<u>if and only if $w_2(x,t) = w_1(x,t) - \mathcal{E}(x)$ is a solution of (3.4,2) that satisfies the initial conditions</u>

$$w(x,0) = \varphi(x) - \mathcal{E}(x)\,, \qquad w_t(x,0) = \psi(x)\,.$$

However, aiming at applications to piano strings, we assume that T, ρ_0, and L have values for which $\mathcal{E}(x)$ can be considered to vanish identically. For example, if T, ρ_0, and L have values common in the case of the piano, such as $\mathrm{T} = 8 \cdot 10^2$ N, $\rho_0 = 10^{-2}$ Kg/m, and $\mathrm{L} = 1\,m$, then $\|\mathcal{E}\| = \max_{x \in [0,\mathrm{L}]} |\mathcal{E}(x)| = 1.56 \cdot 10^{-5}\,m$. Hence, in the following, we shall regard (3.4) as the differential equation to be solved by the function $y = w(x,t)$. Thus we neglect the inhomogeneous term of (3.1), whose only effect is to add a negligible time-independent function $y = \mathcal{E}(x)$ to the solutions of (3.4).

* * *

If sufficiently regular initial conditions

$$w(x, 0) = \varphi(x) \qquad w_t(x, 0) = \psi(x) \tag{3.5}$$

are given, then, as is well known, the (generalized) solution of (3.4), (3.2), (3.5) can be written in the form

$$w(x, t) = \sum_{n=1}^{\infty} (\mathrm{A}_n \cos(n\pi\, at/\mathrm{L}) + \mathrm{B}_n \sin(n\,\pi\, at/\mathrm{L})) \sin(n\,\pi\, x/\mathrm{L}) \tag{3.6}$$

with

$$\mathrm{A}_r = b_n^{\Psi} \quad \mathrm{B}_n = \mathrm{L}/(n\,\pi)\, b_n^{\psi} \quad \text{where } b_i^f = (2\,\mathrm{L})^{-1} \int_0^{\mathrm{L}} f(x) \sin(i\,\pi\, x/\mathrm{L})\, \mathrm{d}x \tag{3.7}$$

If E is the (constant) total energy of $\mathscr{S}$, then — see e. g. [1] —

$$\mathrm{E} = \sum_{n=1}^{\infty} \mathrm{E}_n \tag{3.8}$$

E_n being the energy of the n-th normal mode

$$w_n(x, t) = \Big(\mathrm{A}_n \cos(n\,\pi\, at/\mathrm{L}) + \mathrm{B}_n \sin(n\,\pi\, at/\mathrm{L})\Big) \sin(n\,\pi\, x/\mathrm{L})\,. \tag{3.9}$$

Actually, by denoting the kinetic and potential energies by E_n^k and E_n^p respectively,

$$\mathrm{E}_n^k(t) = 1/2 \int_0^{\mathrm{L}} \rho_0\, w_t^2(x, t)\, \mathrm{d}x = \tag{3.10}$$

$$= \rho_0\, a^2\, n^2\, \pi^2/(4\,\mathrm{L}) \Big(-\mathrm{A}_n \sin(n\,\pi\, at/\mathrm{L}) + \mathrm{B}_n \cos(n\,\pi\, at/\mathrm{L})\Big)^2$$

$$\mathrm{E}_n^p(t) = 1/2 \int_0^{\mathrm{L}} \mathrm{T}\, w_x^2(x, t)\, \mathrm{d}x = \tag{3.11}$$

$$= \rho_0\, a^2\, n^2\, \pi^2/(4\,\mathrm{L}) \Big(\mathrm{A}_n \cos(n\,\pi\, at/\mathrm{L}) + \mathrm{B}_n \sin(n\,\pi\, at/\mathrm{L})\Big)^2$$

and thus

$$\mathrm{E}_n = \mathrm{E}_n^{\mathrm{K}}(t) + \mathrm{E}_n^p(t) = \rho_0\, n^2\, a^2\, \pi^2/(4\,\mathrm{L})\, (\mathrm{A}_n^2 + \mathrm{B}_n^2) \tag{3.12}$$

where $a = \sqrt{\mathrm{T}/\rho_0}$ is the (constant) wave speed of $\mathscr{S}$. Incidentally one can note that the convergence of (3.8) requires A_n and B_n to be $o\Big((1/n)^{3/2}\Big)$

4. CONDITIONS OF THE STRING AT THE END OF THE IMPACT WITH THE HAMMER

Now we are going to consider $\varphi(x)$ and $\psi(x)$ that represent the values of the displacement and velocity fields on the string at the instant t_0 at which the hammer leaves it.

As was said in the Introduction, in [5] a method was provided to find t_0, $\varphi(x)$ and $\psi(x)$; in a particular case ($M < 4/1.28\,\rho_0\,\bar{x} \Leftrightarrow 2\,\bar{x}/a \leq t_0 \leq 4\,\bar{x}/a$), t_0 is expressed by (2.5). Under the further hypothesis that $2\,\bar{x}/a \leq t_0 \leq 3\,\bar{x}/a$, equivalent to $M < 4/2.23\,\rho_0\,\bar{x}$ — see (2.5)—, let us now compute $\varphi(x) = u(x, t_0)$ and $\psi(x) = u_t(x, t_0)$. From (2.1-2) we obtain

$$
\text{(4.1)}\quad \varphi(x) = u(x, t_0) = \begin{cases}
v_0/K\, e^{-K(t_0-\bar{x}/a)}\,(e^{Kx/a} - e^{-Kx/a}) & 0 \leq x \leq 3\,\bar{x} - a\,t_0 \\[2mm]
v_0/K\, e^{-K(t_0-\bar{x}/a)}\,\{e^{K\bar{x}/a} - e^{-Kx/a} + e^{K/a(2\bar{x}-x)}\,(K\,x/a + & \\
\quad + K\,(t_0 - 3\,\bar{x}/a))\} & 3\,\bar{x} - a\,t_0 \leq x \leq \bar{x} \\[2mm]
v_0/K\, e^{-K(t_0-\bar{x}/a)}\,(1 - e^{-2K\bar{x}/a} + K\,(t_0 - \bar{x}/a) - & \\
\quad - K\,x/a)\, e^{Kx/a} & \bar{x} \leq x \leq a\,t_0 - \bar{x} \\[2mm]
v_0/K\,(1 - e^{-K(t_0+\bar{x}/a)}\, e^{Kx/a}) & a\,t_0 - \bar{x} \leq x \leq a\,t_0 + \bar{x} \\[2mm]
0 & a\,t_0 + \bar{x} \leq x \leq L
\end{cases}
$$

$$
\text{(4.2)}\quad \psi(x) = u_t(x, t_0) = \begin{cases}
-K\,\varphi(x) & 0 \leq x \leq 3\,\bar{x} - a\,t_0 \\[2mm]
-K\,\varphi(x) + v_0\, e^{-K(t_0-3\bar{x}/a)}\, e^{-Kx/a} & 3\,\bar{x} - a\,t_0 \leq x \leq \bar{x} \\[2mm]
-K\,\varphi(x) + v_0\, e^{-K(t_0-\bar{x}/a)}\, e^{Kx/a} & \bar{x} \leq x \leq a\,t_0 - \bar{x} \\[2mm]
-K\,\varphi(x) - v_0 & a\,t_0 - \bar{x} \leq x \leq a\,t_0 + \bar{x} \\[2mm]
0 & a\,t_0 + \bar{x} \leq x \leq L\,.
\end{cases}
$$

In wiew of the Fourier analysis it is important to observe that both $\varphi'(x)$ and $\psi(x)$ are continuous in $[0, L]$ except at the points $x_1 = 3\,\bar{x} - a\,t_0\ (< \bar{x})$ and $x_2 = \bar{x} + a\,t_0\ (> \bar{x})$: in both of these points the values of their discontinuities are v_0/a and v_0 respectively.

5. Evaluation of E

Now we have to expand $\varphi(x)$ and $\psi(x)$ in series of only sines. We shall show that, aiming at an approximation of E_n up to $O(1/n^4)$, we can obtain this expansion by taking only the discontinuities of $\varphi'(x)$ and $\psi(x)$ into consideration (this will lead to the next generalization). In fact, let $x_1, \ldots, x_N$ be the discontinuity-points of $\varphi'(x)$ and $\psi(x)$; denoting by $a_n^{(i)(f)}$ and $b_n^{(i)(f)}$ the n-th Fourier coefficients of the i-th derivative of a function $f(x)$ defined in $[0, L]$, one obtains—see e.g. [7]—

$$(5.1)\qquad b_n^{(\psi)} = -1/(2n\pi)\sum_{i=1}^{N}\psi\,|_{x_i^-}^{x_i^+}\cos(n\pi x_i/L) + L/(n\pi)\,a_n^{(1)(\psi)} =$$

$$= -1/(2n\pi)\sum_{i=1}^{N}\psi\,|_{x_i^-}^{x_i^+}\cos(n\pi x_i/L) + O(1/n^3)$$

and

$$(5.2)\qquad b_n^{(\varphi)} = L/(n\pi)\,a_n^{(1)(\varphi)} = -L/(2n^2\pi^2)\sum_{i=1}^{N}\varphi'\,|_{x_i^-}^{x_i^+}\sin(n\pi x_i/L) -$$

$$- L^2\,b_n^{(2)(\varphi)}/(n^2\pi^2) = -L/(2n^2\pi^2)\sum_{i=1}^{N}\varphi'\,|_{x_i^-}^{x_i^+}\sin(n\pi x_i/L) + O(1/n^3)$$

Thus, from (3.7) we obtain

$$(5.3)\qquad A_n = b_n^{(\varphi)} = -L/(2n^2\pi^2)\sum_{i=1}^{N}\varphi'\,|_{x_i^-}^{x_i^+}\sin(n\pi x_i/L) + O(1/n^3)$$

$$(5.4)\qquad B_n = L/(n\pi)\,b_n^{(\psi)} = -L/(2n^2\pi^2 a)\sum_{i=1}^{N}\psi\,|_{x_i^-}^{x_i^+}\cos(n\pi x_i/L) + O(1/n^3)$$

Therefore both A_n and B_n are $O(1/n^2)$ and, from (3.12), we have

$$(5.5)\qquad E_n = \rho_0 a^2 L/(16\pi^2 n^2)\Big(\sum_{i,j=1}^{N}\varphi'\,|_{x_i^-}^{x_i^+}\varphi'\,|_{x_j^-}^{x_j^+}\sin(n\pi x_i/L)\sin(n\pi x_j/L) +$$

$$+ 1/a^2\sum_{i,j=1}^{N}\psi\,|_{x_i^-}^{x_i^+}\psi\,|_{x_j^-}^{x_j^+}\cos(n\pi x_i/L)\cos(n\pi x_j/L)\Big) + O(1/n^4)\,.$$

(By (3.10) and (3.11) we can give an analogous expression for $E_n^K(t)$ and $E_n^P(t)$).

Actually within this approximation, the computation of every E_n is reduced to finding the discontinuity points of $\varphi'(x)$ and $\psi(x)$ and to evaluating the corresponding jumps. In the case considered in N. 4 (5.5) yields

$$E_n = \rho_0 v_0^2 L/(4 \pi n)^2 \left[\left(\sin (n \pi (3 \overline{x} - a t_0)/L) + \sin (n \pi (\overline{x} + a t_0)/L) \right)^2 + \right.$$

$$(5.6) \quad \left. + \left(\cos (n \pi (3 \overline{x} - a t_0)/L) + \cos (n \pi (\overline{x} + a t_0)/L) \right)^2 \right] + O(1/n^4) =$$

$$= \rho_0 L v_0^2/(8 \pi^2 n^2) \left[1 + \cos (2 n \pi (a t_0 - \overline{x})/L) \right] + O(1/n^4)$$

Hence

$$(5.7) \qquad E_n = \rho_0 L v_0^2 \cos^2 (n \pi (a t_0 - \overline{x})/L)/(2 \pi n)^2 + O(1/n^4) .$$

6. GENERALIZATION

The result (5.7) may be extended to the case in which the detatching instant t_0 is $\geq 3 \overline{x}/a$.

In [5] a procedure was described to calculate t_0 in the general case and to find the displacement and velocity fields of the string at this instant, but, as was said in the Introduction, generally this method requires a lot of calculations. On the other hand, we have just seen that, in order to approximate E_n up to $O(1/n^4)$, we do not need to find $\varphi(x) = u(x, t_0)$ and $\psi(x) = u_t(x, t_0)$, but we have only to determine the discontinuities of $\varphi'(x) = u_x(x, t_0)$ and $\psi(x) = u_t(x, t_0)$.

On the bases of these considerations, a formula will now be stated that replaces (5.7) in the general case: $t_0 \in [0, +\infty[$. For this purpose we shall refer to N. 7 in [5], which enables us to find and to evaluate the discontinuities of $\varphi'(x) = u_x(x, t_0)$ and $\psi(x) = u_t(x, t)$ as soon as t_0 has been found, without determining $u_x(x, t_0)$ and $u_t(x, t_0)$ explicitly.

Then, let us consider the system $\mathscr{S}_M$ with the initial conditions (A) and (B) (representing the impact)—see N. 1. From what we stated in [5], we know that, after the initial instant two discontinuities start propagating from the mass M, both involving $x \longmapsto u_x(x, t)$ as well as $x \longmapsto u_t(x, t)$, one towards the left-hand side and the other towards the right-hand side. These discontinuities are successively reflected at the end points of the string, then again by the mass, and so on. The value of the jump of $x \longmapsto u_t(x, t)$ is unchanged at every reflection; instead, the jump of $x \longmapsto u_x(x, t)$ is reversed at each reflection. Considering that discontinuities propagate at the velocity a, it is easy to find the discontinuity points x_1 and x_2 $(x_1 < \overline{x} < x_2)$ at every instant $t \in [0, +\infty[$. These discontinuities have the value v_0 for $x \longmapsto u_t$ and the value v_0/a for $x \longmapsto u_x$ —see N. 7 in [5]—. Moreover, for each $t > 0$ the function $x \longmapsto u_t(x, t)$ cannot have a discontinuity at the point $\overline{x}$ where the mass is attached, as one can obtain from (7.3) in [5]. Lastly, since t_0 is the first instant at which

$u_{tt}(x, t) = 0$, from the condition $M\, u_{tt}(\bar{x}, t) = T\, u_x \mid_{(x^-, t)}^{(\bar{x}^+, t)}$ it follows that $y = u_x(x, t_0)$ will be continuous at $\bar{x}$. Therefore, if t_0 is known, we are able to determine all discontinuities of $u_x(x, t_0)$ and $u_t(x, t_0)$. Hence we can provide an expression of (5.5) more general than (5.6), regardless of particular conditions on the detatching instant (i.e. disregarding the values of the physical constants of the problem):

$$
\begin{aligned}
E_n &= \rho_0 v_0^2 L/(4\pi n)^2 \left[\left(\sin(n\pi x_1/L) \pm \sin(n\pi x_2/L) \right)^2 + \right. \\
&\quad + \left. \left(\cos(n\pi x_1/L) \pm \cos(n\pi x_2/L) \right)^2 \right] + O(1/n^4) = \\
&= \rho_0 v_0^2 L/(8\pi^2 n^2) \left[1 + \cos(n\pi(x_1 \mp x_2)/L) \right] + O(1/n^4) = \\
&\boxed{= \rho_0 v_0^2 L/(4\pi^2 n^2) \cos^2(n\pi(x_1 \mp x_2)/L) + O(1/n^4)}
\end{aligned}
\tag{6.1}
$$

We can conclude that, in general, E_n is $O(1/n^2)$. Moreover, the calculation of its value up to $O(1/n^4)$ essentially depends on the computation of t_0, allowed by the results stated in [5].

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