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Aldo Bressan

Some chain rules for certain derivatives of double tensors depending on other such tensors and some point variables. I. On the pseudo-total derivative

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Fisica matematica. — *Some chain rules for certain derivatives of double tensors depending on other such tensors and some point variables. I. On the pseudo-total derivative.* Nota (*) del Corrisp. ALDO BRESSAN.

RIASSUNTO. — Si considerano due spazi S_μ e S^*_ν, Riemanniani e a metrica eventualmente indefinita, riferiti a sistemi di co-ordinate $\varnothing$ e $\varnothing^*_\nu$; e inoltre un doppio tensore $T^{\cdots}_{\cdots}$ associato ai punti $\varnothing^{-1}(x) \in S_\mu$ e $\varnothing^{*-1}(y) \in S^*_\nu$. Si pensa $T^{\cdots}_{\cdots}$ dato da una funzione $\tilde{T}^{\cdots}_{\cdots}$ di m altri tali doppi tensori e di variabili puntuali $x\,(\in \mathfrak{R}^\mu)$, $t \in \mathfrak{R}$ e $y\,(\in \mathfrak{R}^\nu)$; poi si considera la funzione composta

$$\underset{\sqcup}{T}^{\cdots}_{\cdots}(x, t, y) = \tilde{T}^{\cdots}_{\cdots}[\underset{1}{\breve{H}}^{\cdots}_{\cdots}(x, t, y), \ldots, \underset{m}{\breve{H}}(x, t, y), x, t, y].$$

Nella Parte I si scrivono due regole per eseguire la derivazione totale di questa, connessa con una mappa $\overset{\sqcap}{\mathscr{E}}\,(= \overset{\sqcap}{\mathscr{E}}_t)$ fra S^*_ν e S_μ; una è a termini generalmente non covarianti e l'altra a termini (sempre) covarianti. Si applicano queste regole per esprimere il risultante I^ρ degli sforzi in un corpo (iper-)elastico classico.

Nella Parte II si scrivono due regole analoghe per la derivata assoluta di $\underset{\sqcup}{T}^{\cdots}_{\cdots}$, e altre due per la derivata Lagrangiana spaziale (o trasversa) $\underset{\sqcup}{T}^{\cdots}_{\cdots}|_R$ di $\underset{\sqcup}{T}^{\cdots}_{\cdots}$. La $\underset{\sqcup}{T}^{\cdots}_{\cdots}|_R$ è utile in Relatività generale o ristretta; e si applicano le due regole riferentesi ad essa per scrivere due espressioni di I^ρ appunto nel caso di un corpo (iper-)elastico relativistico.

§ 1. INTRODUCTION (**)

The total derivative $\underset{\sqcup}{T}^{\cdots}_{\cdots};_R$ of a double tensor field $\underset{\sqcup}{T}^{\cdots}_{\cdots}(x, t, y)$ where x, t, and y are point variables—see (2.8) or [3]—is used also in classical physics, for instance, to treat continuous media in general co-ordinates—see e.g. [4]—. In general relativity, where everywhere (pseudo-) Euclidean co-ordinates are lacking, algorithms enabling us to use general co-ordinates are more important than in classical physics. However, in this theory a natural representation of the motion $\mathscr{M}$ of a continuous body $\mathscr{C}$ depends on an arbitrary function $\hat{t}$ (time parameter)—see N 6, or better § 52 in [2]. Therefore in [1]

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(**) The present work, performed in the sphere of activity of research group n. 3 of the Consiglio Nazionale delle Ricerche, in 1984 and 1985, is an improved and enriched version of some lessons given by the author in his course of Continuum Mechanics (Padua 1984-85) and in his CIME course of non-stationary relativistic thermodynamics (Ravello Sept. 1985).

$\overset{\sqcup}{\mathrm{T}}{}^{\cdots}_{\cdots}{}_{;\mathrm{R}}$ was replaced by the Lagrangian spatial (or transverse) derivative $\overset{\sqcup}{\mathrm{T}}{}^{\cdots}_{\cdots}{}_{|\mathrm{R}}$ which, besides being covariant, is also independent of $\check{t}$ (1).

For instance, let $\mathscr{C}'$ be a possibly non-homogeneous hyperelastic body within classical physics [special or general relativity]. Then the 1^{st} Piola Kirchhoff stress tensor $\mathrm{K}^{a\mathrm{B}}$ is expressed by a constitutive function $\tilde{\mathrm{K}}^{a\mathrm{B}}$ whose arguments are other double tensors, say $\underset{1}{\mathrm{H}}{}^{\cdots}_{\cdots}$ to $\underset{m}{\mathrm{H}}{}^{\cdots}_{\cdots}$ and some point variables, such as the set y of the reference co-ordinates y^1, y^2, y^3 of the typical matter point P^* of $\mathscr{C}'$—see (5.1), (10.1). Along the motion $\mathscr{M}$, represented by the equation $x = \hat{x}(t, y)$, we have $\underset{i}{\mathrm{H}}{}^{\cdots}_{\cdots} = \underset{i}{\hat{\mathrm{H}}}(t, y)$. Then, in order to calculate the spatial stress divergence, we have to calculate the pseudo-total Lagrangian spatial derivative of a compound function, whose form is included in the form

$$\overset{\sqcup}{\mathrm{T}}{}^{\cdots}_{\cdots}(x, t, y) = \tilde{\mathrm{T}}{}^{\cdots}_{\cdots}[\underset{1}{\overset{\sqcup}{\mathrm{H}}}{}^{\cdots}_{\cdots}(x, t, y), \ldots, \underset{m}{\overset{\sqcup}{\mathrm{H}}}{}^{\cdots}_{\cdots}(x, t, y), x, t, y], \tag{1.1}$$

where x is the set of co-ordinates for the actual position of P^* in the kinematic space being considered (in space time) (2).

In order to calculate the derivatives $\overset{\sqcup}{\mathrm{T}}{}^{\cdots}_{\cdots}(x, t, y)_{;\mathrm{R}}$, $\overset{\sqcup}{\mathrm{T}}{}^{\cdots}_{\cdots}(x, t, y)_{,\mathrm{R}}$, and the absolute (relativistic) derivative $\mathrm{D}\overset{\sqcup}{\mathrm{T}}{}^{\cdots}_{\cdots}(x, t, y)/\mathrm{D}s$, chain rules are not strictly necessary; however, they are useful. Therefore, in this work two chain rules are stated for each of the three derivatives above, one with generally non-convariant terms and the other with only covariant terms—see (3.5), (4.8), (9.1), (9.3), (9.5), and (9.7). In the relativistic case the terms of the latter rule are also independent of the choice of $\hat{t}$. Furthermore a certain equality which in my opinion has some chances of being taken as a natural chain rule—see e.g. (3.6)—is shown to be generally false, unless both co-ordinate systems being used are locally geodesic.

As examples, the rules for $\overset{\sqcup}{\mathrm{T}}{}^{\cdots}_{\cdots}{}_{;\mathrm{R}}$ $[\overset{\sqcup}{\mathrm{T}}{}^{\cdots}_{\cdots}{}_{|\mathrm{R}}]$ are used to calculate the density I^{ρ} of the local internal forces for $\mathscr{C}'$ in classical physics [N 5] and relativity theory [N 10].

This work consists of two notes: Part 1 and Part 2. The former is devoted to $\overset{\sqcup}{\mathrm{T}}{}^{\cdots}_{\cdots}{}_{;\mathrm{R}}$ and classical physics whereas the latter is mainly concerned with $\mathrm{D}\overset{\sqcup}{\mathrm{T}}{}^{\cdots}_{\cdots}/\mathrm{D}s$, $\hat{\mathrm{T}}{}^{\cdots}_{\cdots}{}_{|\mathrm{R}}$, and relativity theory.

In the typical case the derivative $\tilde{\mathrm{T}}{}^{\cdots}_{\cdots}(\underset{1}{\mathrm{H}}{}^{\cdots}_{\cdots}, \ldots, \underset{m}{\mathrm{H}}{}^{\cdots}_{\cdots}, x, t, y)_{;\mathrm{R}}$ involves partial derivatives of $\tilde{\mathrm{T}}{}^{\cdots}_{\cdots}$ with respect to only a part of $\tilde{\mathrm{T}}{}^{\cdots}_{\cdots}$'s arguments. Therefore it is called *pseudo-total derivative*. Also the *pseudo-absolute derivative* $\mathrm{D}\tilde{\mathrm{T}}{}^{\cdots}_{\cdots}/\mathrm{D}^{\mathrm{P}}s$ of $\tilde{\mathrm{T}}{}^{\cdots}_{\cdots}$ is considered, i.e. the absolute derivative of $(x, t, y) \vdash$

(1) In [1] and [2] I called $\tilde{\mathrm{T}}{}^{\cdots}_{\cdots}{}_{|\mathrm{R}}$ *Lagrangian transverse derivative of* $\tilde{\mathrm{T}}{}^{\cdots}_{\cdots}$. However the qualification *spatial* seems to me now more appropriate than *transverse*.

(2) The constitutive function $\tilde{\mathrm{K}}^{a\mathrm{B}}$ for $\mathrm{K}^{a\mathrm{B}}$ must also have a time parameter t as an argument, in case C' is undergoing some chemical reactions independent of $\mathscr{M}$.

$\vdash \tilde{T}^{\cdots}_{\cdots}(\underset{1}{H}^{\cdots}_{\cdots}, \dots, \underset{m}{H}^{\cdots}_{\cdots}, x, t, y)$. For $m > 0$ it generally fails to be covariant as well as $\tilde{T}^{\cdots}_{\cdots\,;R}$ and $\tilde{T}^{\cdots}_{\cdots\,|R}$. Therefore the *stationary* (*or covariant partial*) *pseudo-total derivative* $\tilde{T}^{\cdots}_{\cdots\,S_t;R}$ of $\tilde{T}^{\cdots}_{\cdots}$ is introduced [§ 4] and the analogue is done with $D\tilde{T}^{\cdots}_{\cdots}/D^p s$ and $\tilde{T}^{\cdots}_{\cdots\,;R}$ [§ 9].

These stationary derivatives enter the chain rules, all of whose terms are covariant. The remaining chain rules involve *connectionless derivatives*—see (3.4), (48.), and (8, 1-2).

§ 2. Double tensors and total derivatives

Let S_μ and S^*_ν be Riemannian spaces of respective dimensions μ and ν. Their metric tensors $g_{\alpha\beta}$ and a^*_{LM} may fail to be defined > 0 (strictly positive) or < 0 (strictly negative), and may also be everywhere Euclidean or everywhere pseudo-Euclidean.

Let $\{\alpha\beta, \gamma\}$ and $\{\alpha\,{}^\rho\beta\} = \{\alpha\beta, \gamma\}\, g^{\gamma\rho}$, where $(g^{\gamma\rho}) = (g_{\gamma\rho})^{-1}$, be the Christoffel symbols for S_μ and let $\{A\,B, C\}$ and $\{{}^{R}_{AB}\}$ be their analogues for S^*_ν.

Consider the points $\mathscr{E} \in S_\mu$ and $P^* \in S^*_\nu$; and let $\phi\,[\phi^*]$ be a (regular) frame, or co-ordinate system, for $S_\mu\,[S^*_\nu]$, i.e. a bijection of $S_\mu\,[S^*_\nu]$ onto an open subset of $\mathbf{R}^\mu\,[\mathbf{R}^\nu]$ (3), e.g.

$$(2.1)\qquad (x^1, \dots, x^\mu) = \phi(\mathscr{E}) = (\phi^1(\mathscr{E}), \dots, \phi^\mu(\mathscr{E}))\ (\forall \mathscr{E} \in S_\mu)\ ;$$
$$(y^1, \dots, y^\nu) = \phi^*(P^*)\ (\forall P^* \in S^*_\nu)\,.$$

Frame $\phi\,[\phi^*]$ can also be denoted by $(x)\,[(y)]$. Now consider the set of $\mu^{a+c}\,.\,\nu^{b+d}$ scalars

$$(2.2)\qquad \{T s_1 \dots s_a{}^{\sigma_1 \cdots \sigma_b}{}_{R_1 \dots R_c}{}^{S_1 \dots S_d}\} = \mathbf{T}(\phi, \phi^*)\,,$$

where Greek [Latin] indices run over a set of $\mu\,[\nu]$ elements—e.g. from 1 to $\mu\,[\nu]$. Let it depend on ϕ and ϕ^* in such a way that, whenever also $\bar\phi\,[\bar\phi^*]$ is a frame for $S_\mu\,[S^*_\nu]$, we have

$$(2.3)\qquad \bar{T}_{\alpha_1 \dots A_1 \dots}{}^{\beta_1 \dots B_1 \dots} = T_{\rho_1 \dots R_1 \dots}{}^{\sigma_1 \dots S_1 \dots} \frac{\partial x^{\rho_1}}{\partial \bar{x}^{\alpha}} \cdots \frac{\partial \bar{x}^{\beta_1}}{\partial x^{\sigma_1}} \cdots \frac{\partial y^{R_1}}{\partial \bar{y}^{A_1}} \cdots \frac{\partial \bar{y}^{B_1}}{\partial y^{S_1}} \cdots,$$

where (i) $\{\bar{T}_{\alpha_1 \dots A_1 \dots}{}^{\beta_1 \dots B_1 \dots}\} = T(\bar\phi, \bar\phi^*)$, (ii) $\partial\bar{x}^\beta/\partial x^\rho$ are the partial derivatives of the function $\bar{x}^\beta = \phi^\beta[\phi^{-1}(x^1, \dots, x^\mu)]$ evaluated at the point $\phi(\mathscr{E})$ of R^μ, and (iii) the analogues hold for $\partial x^\rho/\partial\bar{x}^\alpha$, $\partial y^R/\partial\bar{y}^A$, and $\partial\bar{y}^B/\partial y^S$. Then $\mathbf{T}$ is said to be a (double) tensor of *covariant order* (a, c) and *controvariant order*

(3) If one likes to consider an atlas for e.g. S_μ, let ϕ be a bijection of an open subset of S_μ that includes $\mathscr{E}$, into an open subset of $\mathfrak{R}^\mu$.

(b, d), attached to the point $\mathscr{E}$ of S_μ through its first $a+b$ indices, and to the point P^* of S^*_ν through its last $c+d$ indices. The scalars $T^{\cdots}_{\cdots} = T^{\sigma_1 \cdots}{}_{\rho_1 \cdots}{}^{S_1 \cdots}{}_{R_1 \cdots}$ — see (2.2)—are called the components of $\mathbf{T}$ in frames ϕ and ϕ^*.

Now regard the above double tensor $\mathbf{T}$ as a function $\tilde{\mathbf{T}}$ whose arguments are m (≥ 0) other double tensors $\underset{1}{\mathbf{H}}$ to $\underset{m}{\mathbf{H}}$ also attached to $\mathscr{E}$ and P^*, the point variables $\mathscr{E}$ and P^*, and (possibly) a real parameter t. Let all arguments of $\hat{\mathbf{T}}$ range over some open subsets of some suitable spaces.

Let us remark that the above parameter t is used throughout Part 1 mainly for purposes reached in Part 2. Readers interested in (pseudo-) total derivatives but not in (pseudo-) absolute or Lagrangian spatial derivatives, can cross out t everywhere in §§ 2-4.

Field $\tilde{T}^{\cdots}_{\cdots}$ is represented in the above frames ϕ and ϕ^* by the component functions

$$\text{(2.4)} \qquad T_{\alpha_1 \ldots}{}^{\beta_1 \ldots}{}_{A_1 \ldots}{}^{B_1 \ldots} = \tilde{T}^{\cdots}_{\cdots} \big(\underset{1}{H}{}_{\lambda \ldots}{}^{\mu \ldots}{}_{L \ldots}{}^{M \ldots}, \ldots, \underset{m}{H}{}^{\cdots}_{\cdots}, x, t, y\big).$$

Let us now define the *pseudo-covariant partial derivatives* (of $\tilde{\mathbf{T}}$) in S_μ and S^*_ν, by

$$\text{(2.5)} \qquad \tilde{T}^{\cdots}_{\cdots/\rho} = \frac{\partial \tilde{T}^{\cdots}_{\cdots}}{\partial x^\rho} - St,_\rho T^{\cdots}_{\cdots} \quad , \quad \tilde{T}^{\cdots}_{\cdots/R} = \frac{\partial \tilde{T}^{\cdots}_{\cdots}}{\partial y^R} - St,_R T^{\cdots}_{\cdots} ,$$

where (the linear operators) $St,_\rho$ and $St,_R$ are given by

$$\text{(2.6)} \qquad \begin{cases} St,_\rho T^{\beta_1 \ldots B_1 \ldots}_{\alpha_1 \ldots A_1 \ldots} = \left\{ {}^{\;\sigma}_{\rho\,\alpha_1} \right\} T^{\beta_1 \ldots \;\; B_1 \ldots}_{\sigma\alpha_2 \ldots A_1 \ldots} + \ldots - \left\{ {}^{\beta_1}_{\rho\,\sigma} \right\} T^{\sigma\beta_2 \ldots \; B_1 \ldots}_{\alpha_1 \ldots \; A_1 \ldots} - \ldots \\ St,_R T^{\beta_1 \ldots B_1 \ldots}_{\alpha_1 \ldots A_1 \ldots} = \left\{ {}^{S}_{RA_1} \right\} T^{\beta_1 \ldots B_1 \ldots}_{\alpha_1 \ldots SA_2 \ldots} + \ldots - \left\{ {}^{B_1}_{RS} \right\} T^{\beta_1 \ldots SB_2 \ldots}_{\alpha_1 \ldots A_1 \ldots} - \ldots \end{cases}$$

The symbols thus introduced are justified by simple stationarity properties—see below (4.5).

For $m = 0$ each of the scalar systems (2.5) (which depend on ϕ and ϕ^*) turns out to be a double tensor attached to $\mathscr{E} \in S_\mu$ and $P^* \in S^*_\nu$. Hence one speaks of *covariant partial derivatives* of the double tensor field $T^{\cdots}_{\cdots} = \tilde{T}^{\cdots}_{\cdots}(x, t, y)$ (regarded, if preferred, as a function of x and y).

Consider a $C^{(1)}$-homeomorphism $\mathscr{E} = \overleftarrow{\mathscr{E}}(P^*)$ of S^*_ν into S_μ, possibly depending on t ($\overleftarrow{\mathscr{E}} = \overleftarrow{\mathscr{E}}_t$) and represented, in frames ϕ and ϕ^*, by

$$\text{(2.7)} \qquad x^\rho = \hat{x}^\rho(t, y) \text{ — or precisely } x^\rho = \overleftarrow{x}^\rho(y) \equiv \hat{x}^\rho(t, y) \;\; (\overleftarrow{\mathscr{E}} = \overleftarrow{\mathscr{E}}_t, \hat{x}^\rho \in C^{(1)}).$$

For any $m \geq 0$, the *pseudo-total derivative* of the field $\tilde{T}^{\cdots}_{\cdots}$ — see (2.4) — connected with this map, is defined by

$$\text{(2.8)} \qquad \tilde{T}{}^{\beta_1 \ldots B_1 \ldots}_{\alpha_1 \ldots A_1 \ldots ;R} = \tilde{T}^{\cdots}_{\cdots/\rho} x^\rho_R + \tilde{T}^{\cdots}_{\cdots/R} \qquad (x^\rho_R = \partial \hat{x}^\rho / \partial y^R).$$

For $m = 0$, it is also a double tensor attached to $\mathscr{E}$ and P^*, but dependent on only P^* (and t). Some of its properties can be found in [3]—see also [2], p. 234.

§ 3. Compound functions whose arguments involve double tensors and point variables. A chain rule for the total derivative of them, with generally non covariant terms

Besides field (2.4), consider the m double tensor fields

$$\underset{i}{H}{}^{\mu\ldots M\ldots}_{\lambda\ldots L\ldots} = \underset{i}{\overset{\sqcup}{H}}{}^{\cdots}_{\cdots}\,(x\,,t\,,y) \qquad (i = 1\,,\ldots\,,m)\,; \tag{3.1}$$

and remembering (2.7), let us set

$$\begin{aligned} T^{\beta_1\ldots\,B_1\ldots}_{\alpha_1\ldots A_1\ldots} = \overset{\sqcup}{T}{}^{\cdots}_{\cdots}\,(x\,,t\,,y) = \tilde{T}{}^{\cdots}_{\cdots}\,[\underset{1}{\overset{\sqcup}{H}}{}^{\cdots}_{\cdots}\,(x\,,t\,,y)\,,\ldots\,, \\ \underset{m}{\overset{\sqcup}{H}}{}^{\cdots}_{\cdots}\,(x\,,t\,,y)\,,x\,,t\,,y]\,. \end{aligned} \tag{3.2}$$

Then

$$\overset{\sqcup}{T}{}^{\cdots}_{\cdots}\,{}_{;;R} = \sum_{i=1}^{m} \frac{\partial\tilde{T}{}^{\cdots}_{\cdots}}{\partial \underset{i}{H}{}^{\cdots}_{\cdots}}\,\underset{i}{\overset{\sqcup}{H}}{}^{\cdots}_{\cdots}\,{}_{;;R} + \tilde{T}{}^{\cdots}_{\cdots}\,{}_{;;R} \tag{3.3}$$

where, for an arbitrary choice of the field $\tilde{T}{}^{\cdots}_{\cdots}$ ($m \geq 0$), its *connectionless pseudo-total derivative* $\tilde{T}{}^{\cdots}_{\cdots}\,{}_{;;R}$ is defined by —see $(2.8)_2$—

$$\tilde{T}{}^{\cdots}_{\cdots}\,{}_{;;R} = \frac{\partial\tilde{T}{}^{\cdots}_{\cdots}}{\partial x^{\rho}}\,x^{\rho}_{R} + \frac{\partial\tilde{T}{}^{\cdots}_{\cdots}}{\partial y^{R}}\,, \tag{3.4}$$

hence

$$\tilde{T}{}^{\cdots}_{\cdots}\,{}_{;;R} = \tilde{T}{}^{\cdots}_{\cdots}\,{}_{;R} \text{ for } g_{\alpha\beta,\gamma} = 0 = a^{*}_{AB,C}\left(f_{,\rho} = \frac{\partial f}{\partial x^{\rho}}, f_{R} = \frac{\partial f}{\partial y^{R}}\right). \tag{3.4'}$$

Expression (3.4) of $\tilde{T}{}^{\cdots}_{\cdots}\,{}_{;;R}$ can be obtained from the one of $T{}^{\cdots}_{\cdots}\,{}_{;R}$—see (2.8) and (2.5-6)—by crossing out the terms in the connections $\{^{\gamma}_{\alpha\beta}\}$ and $\{^{C}_{AB}\}^*$, which justifies the name for $\tilde{T}{}^{\cdots}_{\cdots}\,{}_{;;R}$. An analogous use of a redoubled derivation sign will be made in Part 2—see (8.2).

By (2.8), (2.5-6), and (3.3) we easily deduce the *chain rule*

$$\overset{\sqcup}{T}{}^{\cdots}_{\cdots}\,(x\,,t\,,y)_{;R} = \sum_{i=1}^{m} \frac{\partial\tilde{T}{}^{\cdots}_{\cdots}}{\partial \underset{i}{H}{}^{\cdots}_{\cdots}}\,\underset{i}{\overset{\sqcup}{H}}{}^{\cdots}_{\cdots}\,{}_{;;R} + \tilde{T}{}^{\cdots}_{\cdots}\,{}_{;R} \tag{3.5}$$

for compound functions such as (3.2). *For $m > 0$ its last two terms are generally* *non-covariant.* In fact $\hat{T}^{\cdots}_{\cdots}(x, t, y)_{;R}$ and $\partial\tilde{T}/\partial \underset{i}{H}^{\cdots}_{\cdots}$ are covariant ($i = 1, \ldots, m$), while $\underset{i}{\overset{\sqcup}{H}}{}^{\cdots}_{\cdots}{}_{;;R}$ generally fails to be so. Let us remark explicitly that therefore $\tilde{T}^{\cdots}_{\cdots}{}_{;R}$ *generally fails to be covariant* when $m > 0$.

By (3.5) the inequality

$$(3.6) \qquad \overset{\sqcup}{T}{}^{\cdots}_{\cdots}(x, t, y)_{;R} \neq \sum_{i=1}^{m} \frac{\partial \tilde{T}^{\cdots}_{\cdots}}{\partial \underset{i}{H}^{\cdots}_{\cdots}} \underset{i}{\overset{\sqcup}{H}}{}^{\cdots}_{\cdots}{}_{;R} + \tilde{T}^{\cdots}_{\cdots}{}_{;R} \qquad \text{—see (3.2)—},$$

obviously holds in the typical case. I note this because it seems to me relatively natural to assert the equality of the two sides of (3.6), in that the noncovariant character of $\tilde{T}^{\cdots}_{\cdots}{}_{;R}$ may be overlooked. This equality is acceptable after the replacement of its last term by a suitable co-variant one—see (4.8).

§ 4. STATIONARY PSEUDO-TOTAL DERIVATIVES FOR FUNCTIONS SUCH AS (2.4). A CHAIN RULE FOR THE PRECEDING COMPOUNDS FUNCTIONS, ALL OF WHOSE TERMS ARE COVARIANT

Let us first assume that spaces S_μ and S^*_ν are (pseudo-) Euclidean, so that some choices of ϕ and ϕ^* render $(3.4')_{2,3}$ true everywhere. Then, if the component functions $\underset{1}{\overset{\sqcup}{H}}{}^{\cdots}_{\cdots}$ to $\underset{m}{\overset{\sqcup}{H}}{}^{\cdots}_{\cdots}$ are constant, they represent constant double tensor fields. Thus the tensors $T^{\cdots}_{\cdots}$ and $\underset{1}{H}^{\cdots}_{\cdots}$ to $\underset{m}{H}^{\cdots}_{\cdots}$ in (2.4) can be regarded as attached simply to S_μ and S^*_ν. Hence for $\underset{1}{\mathbf{H}}$ to $\underset{m}{\mathbf{H}}$ fixed, $T^{\cdots}_{\cdots}$ can be regarded as a double tensor of S_μ and S^*_ν depending on x and y (and t). For the resulting field $\overset{\sqcup}{T}{}^{\cdots}_{\cdots}(x, y, t)$ we have

$$(4.1) \qquad \tilde{T}^{\cdots}_{\cdots}{}_{;R} = \overset{\sqcup}{T}{}^{\cdots}_{\cdots}(x, t, y)_{;R} \qquad \text{for} \quad g_{\alpha\beta,\gamma} = 0 = a^*_{AB,C} .$$

Now let S_μ and S^*_ν be arbitrary Riemannian spaces, so that constant double tensor fields of many orders fail to exist in them. Therefore (4.1) can be considered only locally, by choosing $\underset{i}{\overset{\sqcup}{H}}{}^{\cdots}_{\cdots}$ locally stationary:

$$\underset{i}{\overset{\sqcup}{H}}{}^{\cdots}_{\cdots}{}_{/\rho} = 0 = \underset{i}{\overset{\sqcup}{H}}{}^{\cdots}_{\cdots}{}_{/R}, \text{ or at least } \underset{i}{\overset{\sqcup}{H}}{}^{\cdots}_{\cdots}{}_{;R} = 0 \qquad (i = 1, \ldots, m) .$$

With a view to writing the chain rule hinted at in the title, for arbitrary choices of ϕ and ϕ^*, let us continue the considerations about (4.1) as follows. Fix arbitrary local values for the arguments $\underset{1}{H}^{\cdots}_{\cdots}$ to $\underset{m}{H}^{\cdots}_{\cdots}$ of function (2.4), attached to $P^* = \phi^{*-1}(y) \in S^*_\nu$ and $\mathscr{E} = \overset{\frown}{\mathscr{E}}(P^*) = \phi^{-1}[\hat{x}(t, y)] \in S_\mu$ —see

(2 7)—. Furthermore consider arbitrary tensor fields $\underset{i}{\breve{\mathrm{H}}}{}^{\cdot\cdot\cdot}_{\cdot\cdot\cdot}(x', t, y')$ attached to y' and $x(t, y')$ $(i = 1, \ldots, m)$, that at y (and $\hat{x}(t, y)$) (i) assume the locally fixed values and (ii) are pseudo-totally stationary:

$$\underset{i}{\breve{\mathrm{H}}}{}^{\cdot\cdot\cdot}_{\cdot\cdot\cdot}(x, t, y) = \underset{i}{\mathrm{H}}{}^{\cdot\cdot\cdot}_{\cdot\cdot\cdot} \quad , \quad \underset{i}{\breve{\mathrm{H}}}(x, t, y)_{;\mathrm{R}} = 0 \qquad (i = 1, \ldots, m). \tag{4.2}$$

Such tensor fields certainly exist, even with $\partial\underset{i}{\breve{\mathrm{H}}}{}^{\cdot\cdot\cdot}_{\cdot\cdot\cdot}/\partial x^{\rho} \equiv 0 \equiv \partial\underset{i}{\breve{\mathrm{H}}}{}^{\cdot\cdot\cdot}_{\cdot\cdot\cdot}/\partial t$. In fact

$$\overset{\sqcup}{\mathrm{H}}{}^{\cdot\cdot\cdot}_{\cdot\cdot\cdot}{}_{;;\mathrm{R}} = \overset{\sqcup}{\mathrm{H}}{}^{\cdot\cdot\cdot}_{\cdot\cdot\cdot}{}_{;\mathrm{R}} + \mathrm{S}t_{;\mathrm{R}}\,\mathrm{H}^{\cdot\cdot\cdot}_{\cdot\cdot\cdot} \qquad \text{for} \quad \mathrm{H}^{\cdot\cdot\cdot}_{\cdot\cdot\cdot} = \overset{\sqcup}{\mathrm{H}}{}^{\cdot\cdot\cdot}_{\cdot\cdot\cdot}, [\hat{x}(t, y), t, y] \tag{4.3}$$

where

$$\mathrm{S}t_{;\mathrm{R}}\,\mathrm{H}^{\cdot\cdot\cdot}_{\cdot\cdot\cdot} = (\mathrm{S}t_{\rho}, \mathrm{H}^{\cdot\cdot\cdot}_{\cdot\cdot\cdot})\, x^{\rho}_{\mathrm{R}} + \mathrm{S}t_{,\mathrm{R}}\,\mathrm{H}^{\cdot\cdot\cdot}_{\cdot\cdot\cdot} \qquad \text{—see (2.6).} \tag{4.4}$$

Therefore $(4.2)_2$ is equivalent to

$$\underset{i}{\breve{\mathrm{H}}}{}^{\cdot\cdot\cdot}_{\cdot\cdot\cdot}{}_{;;\mathrm{R}} = \mathrm{S}t_{;\mathrm{R}}\,\underset{i}{\mathrm{H}}{}^{\cdot\cdot\cdot}_{\cdot\cdot\cdot} \qquad (\underset{i}{\breve{\mathrm{H}}}{}^{\cdot\cdot\cdot}_{\cdot\cdot\cdot}{}_{;;\mathrm{R}} = \underset{i}{\breve{\mathrm{H}}}{}^{\cdot\cdot\cdot}_{\cdot\cdot\cdot}{}_{,\mathrm{R}} \text{ for } \underset{i}{\breve{\mathrm{H}}}{}^{\cdot\cdot\cdot}_{\cdot\cdot\cdot}{}_{,\rho} = 0). \tag{4.5}$$

Thus $\mathrm{S}t_{;\mathrm{R}}\,\mathrm{H}^{\cdot\cdot\cdot}_{\cdot\cdot\cdot}$ is *the connectionless pseudo-total derivative of a stationary field of local value* $\mathrm{H}^{\cdot\cdot\cdot}_{\cdot\cdot\cdot}$. Incidentally $\mathrm{S}t_{,\rho}\,\mathrm{H}^{\cdot\cdot\cdot}_{\cdot\cdot\cdot}$ $[\mathrm{S}t_{,\mathrm{R}}\,\mathrm{H}^{\cdot\cdot\cdot}_{\cdot\cdot\cdot}]$ is its analogue for the partial pseudo-covariant derivative in S_{μ} $[\mathrm{S}^{*}_{\nu}]$.

Remembering (3.2) and (4.2) one can now define the *stationary*, or *covariant partial, pseudo-total derivative* $\tilde{\mathrm{T}}{}^{\cdot\cdot\cdot}_{\cdot\cdot\cdot}\,\mathrm{S}t_{;\mathrm{R}}$ of $\tilde{\mathrm{T}}{}^{\cdot\cdot\cdot}_{\cdot\cdot\cdot}$ (connected with the map $\overset{\sqcap}{\mathscr{E}}$):

$$\tilde{\mathrm{T}}{}^{\cdot\cdot\cdot}_{\cdot\cdot\cdot}{}_{\mathrm{S}t;\mathrm{R}} =_{\mathrm{D}} \overset{\sqcup}{\mathrm{T}}{}^{\cdot\cdot\cdot}_{\cdot\cdot\cdot}(x, t, y)_{;\mathrm{R}} \quad \text{for} \quad \underset{i}{\overset{\sqcup}{\mathrm{H}}}{}^{\cdot\cdot\cdot}_{\cdot\cdot\cdot} = \underset{i}{\breve{\mathrm{H}}}{}^{\cdot\cdot\cdot}_{\cdot\cdot\cdot} \qquad (i = 1, \ldots, m). \tag{4.6}$$

Then (4.5) and (3.5) yield the explicit expression

$$\tilde{\mathrm{T}}{}^{\cdot\cdot\cdot}_{\cdot\cdot\cdot}{}_{\mathrm{S}t;\mathrm{R}}(\underset{1}{\mathrm{H}}{}^{\cdot\cdot\cdot}_{\cdot\cdot\cdot}, \ldots, \underset{m}{\mathrm{H}}{}^{\cdot\cdot\cdot}_{\cdot\cdot\cdot}, x, t, y) = \sum_{i=1}^{m} \frac{\partial\tilde{\mathrm{T}}{}^{\cdot\cdot\cdot}_{\cdot\cdot\cdot}}{\partial\underset{i}{\mathrm{H}}{}^{\cdot\cdot\cdot}_{\cdot\cdot\cdot}}\,\mathrm{S}t_{;\mathrm{R}}\,\underset{i}{\mathrm{H}}{}^{\cdot\cdot\cdot}_{\cdot\cdot\cdot} + \tilde{\mathrm{T}}{}^{\cdot\cdot\cdot}_{\cdot\cdot\cdot}{}_{;\mathrm{R}}\,. \tag{4.7}$$

By (4.7) and (4.3), (3.5) yields the chain rule

$$\overset{\sqcup}{\mathrm{T}}{}^{\cdot\cdot\cdot}_{\cdot\cdot\cdot}(x, t, y)_{;\mathrm{R}} = \sum_{i=1}^{m} \frac{\partial\tilde{\mathrm{T}}{}^{\cdot\cdot\cdot}_{\cdot\cdot\cdot}}{\partial\underset{i}{\mathrm{H}}{}^{\mu\ldots\mathrm{M}\ldots}_{\lambda\ldots\mathrm{L}\ldots}}\,\overset{\sqcup}{\mathrm{H}}{}^{\mu\ldots\mathrm{M}\ldots}_{\lambda\ldots\mathrm{L}\ldots}{}_{;\mathrm{R}} + \tilde{\mathrm{T}}{}^{\cdot\cdot\cdot}_{\cdot\cdot\cdot}{}_{\mathrm{S}t;\mathrm{R}} \tag{4.8}$$

for compound functions such as (3.2), all of whose terms are covariant.

§ 5. THE STRESS DIVERGENCE FOR A HYPERELASTIC BODY AND THE ABOVE CHAIN RULES

Identify S_2 and S_3^* with a same inertial space. Furthermore assume that (i) $\mathscr{C}'$ is a hyperelastic—i.e. a purely mechanical elastic-body, (ii) C^* is a reference configuration for it, regarded as belonging to S_3^*, (iii) any motion $\mathscr{M}$, possible for $\mathscr{C}'$, is represented by an equation such as $(2.7)_1$ with $\hat{x}^\rho \in C^{(2)}$, so that $\mathscr{C}'$ can be thought of as a set of material points, and (iv) P^* is a typical one among these points.

Then (using the above frames ϕ and ϕ^*), at any instant t, the first Piola-Kirchhoff stress tensor $K^{\alpha B}$ at P^* in $\mathscr{M}$, is a double tensor attached to P^* and $\overline{\mathscr{E}}\,(P^*)$ through the indices α and B respectively; furthermore it is given by a constitutive equation of the form [4]

$$(5.1)\qquad K^{\alpha B} = \tilde{K}^{\alpha B}(y\,,x_A^\mu\,,g_{\lambda\mu}\,,\phi^*) \qquad \text{where} \quad y=(y^1\,,y^2\,,y^3)=\phi^*(P^*)\,.$$

Along $\mathscr{M}$ (5.1) induces a function $K^{\alpha B} = \hat{K}^{\alpha B}(t\,,y)$ in the well known way.

The dynamic equations for $\mathscr{C}$ involve the resultant $I^\rho = -\hat{K}^{\rho B}{}_{;B}$ of the internal forces acting on $\mathscr{C}$ at P^*, per unit reference volume. By rules (3.5) and (4.8) with $\underset{1}{H}{}_A^\mu = x_A^\mu(t\,,y)$ and $\underset{2}{H}{}_{\lambda\mu} = g_{\lambda\mu}(x)$ I^α *has the expressions*

$$(5.2)\qquad I^\alpha = -\hat{K}^{\alpha B}{}_{;B} = -\frac{\partial\tilde{K}^{\alpha B}}{\partial x_L^\mu}\,x^\mu{}_{,LB} - \frac{\partial\tilde{K}^{\;B}}{\partial g_{\lambda\mu}}\,g_{\lambda\mu,\rho}\,x_B^\rho - \tilde{K}^{\alpha B}{}_{,B}\,(x_{,L}^\mu = x_L^\mu)$$

and

$$(5.3)\qquad I^\alpha = -\frac{\partial\tilde{K}^{\alpha B}}{\partial\,x_L^\mu}\,x^\mu{}_{;LB} - \tilde{K}^{\alpha B}{}_{St;B} \qquad \text{—see (4.6)—} \qquad (x^\mu{}_{;L} = x_L^\mu)$$

respectively, where by (4.7) and (4.4)

$$(5.4)\qquad \tilde{K}^{\alpha B}{}_{St;B} = \tilde{K}^{\alpha B}{}_{;B} - \frac{\partial K^{\alpha B}}{\partial\,x_L^\mu}\left(\left\{{}^{\mu}_{\lambda\sigma}\right\}x_L^\sigma\,x_B^\lambda - \left\{{}^{S}_{LB}\right\}^*x_S^\mu\right).$$

Incidentally, since $S_3 = S_3^*$, one can choose $\phi = \phi^*$. However also in this case $\{\ \}$ and $\{\ \}^*$ are generally unrelated, because they are calculated at different points.

(4) $\tilde{K}^{\alpha B}$ behaves in the obvious way under changes of $\varnothing^*$, and it is determined by the function induced by it for any particular choice $\overline{\varnothing}^*$ of $\varnothing^*$ and for $g_{\alpha\mu} = \delta_{\alpha\mu}$. In more detail, let $\bar{y} = \bar{y}(y)$ be $\overline{\varnothing}^* \circ \varnothing^{*-1}$, let $(\bar{x}_\lambda^\delta)$ be any matrix for which $g_{\lambda\mu} = \delta_{\gamma\delta}\,\bar{x}_\lambda^\varphi\,\bar{x}_\mu^\delta$, and set $(x_s^\lambda) = (\bar{x}_\lambda^\rho)^{-1}$. Then (10.1) holds if and only if

$$K^{\alpha B} = x_\rho^\alpha\,(\partial y^B/\partial\bar{y}S)\,\tilde{K}^{\rho S}(\bar{y},\bar{\alpha}_S^\delta\,,g_{\lambda\mu}\,,\overline{\varnothing}^*) \qquad \text{with} \quad \bar{\alpha}_R^\lambda = \alpha_A^\mu\,\bar{x}_\mu^\lambda\,\partial y^A/\partial\bar{y}^R\,.$$

Let us add that in accordance with inequality (3.6), by (5.3-4) *the equality*

$$\mathrm{I}^{\alpha} = -\frac{\partial \mathrm{K}^{\alpha\mathrm{B}}}{\partial x^{\mu}_{\mathrm{L}}} x^{\mu}{}_{;\mathrm{LB}} - \tilde{\mathrm{K}}^{\alpha\mathrm{B}}{}_{;\mathrm{B}} \tag{5.5}$$

is generally false. It is true and coincides with both (5.2) and (5.3) in locally geodesic co-ordinates ($g_{\alpha\beta,\lambda} = 0 = a^{*}_{\mathrm{AB,C}}$).

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