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Analysis of a discrete model for the contact problem between a membrane and an elastic obstacle

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Meccanica dei solidi. — *Analysis of a discrete model for the contact problem between a membrane and an elastic obstacle* (*). Nota di ALDO MACERI (**) e FRANCO MACERI (***), presentata (****) dal Corrisp. E. GIANGRECO.

RIASSUNTO. — In questo lavoro viene risolto il problema del contatto tra una membrana ed un suolo od ostacolo elastico con una approssimazione lineare a tratti della soluzione. Sono date alcune formulazioni equivalenti del problema discreto e se ne discutono le corrispondenti proprietà computazionali.

1. Let us denote by $\Omega \in \mathscr{R}^{(1),1}$ [1] the open, bounded region of R^2 occupied by a plane membrane. The membrane is fixed at the boundary points of Ω, is transversely loaded by distributed forces $f \in L^2(\Omega)$ orthogonal to its plane and positive in the x_3-axis direction, and is uniformly stretched in its plane by a stress $t \in]0, +\infty[$. Furthermore, the membrane is stretched (or constrained) by an elastic body. We describe the body shape by a function $\varphi \in L^2(\Omega)$, and we assume its reactions on the membrane to be parallel to the x_3-axis (frictionless contact) and to have the form (1) $-h(u-\varphi)^+$, where $h \in L^\infty(\Omega)$, $h \geq 0$ a.e. on Ω, and the membrane's deflection, $u = u(x_1, x_2)$, is positive in the x_3-axis direction.

The problem is to find the membrane's equilibrium configuration, i.e. to solve

PROBLEM 1:

$$u \in H_0^1(\Omega) \cap H^2(\Omega) : Au + h(u-\varphi)^+ = f \qquad a.e.\ on \quad \Omega$$

where

$$A = -t\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right).$$

In [2], [3] the proof that Problem 1 has a unique solution u is given. Now, we set

$$a(u, v) = t\int_\Omega \left(\frac{\partial u}{\partial x_1}\frac{\partial v}{\partial x_1} + \frac{\partial u}{\partial x_2}\frac{\partial v}{\partial x_2}\right) dx \qquad \forall (u, v) \in (H_0^1(\Omega))^2$$

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(1) We let $v^+ = \max\{v, 0\}$.

$$\langle F , v \rangle = \int_\Omega fv \, dx \qquad \forall v \in L^2(\Omega)$$

$$E_2(v) = \frac{1}{2} \int_\Omega h \, [(v - \varphi)^+]^2 \, dx \qquad \forall v \in L^2(\Omega) .$$

In [3] it is proven that Problem 1 is equivalent to the total energy minimum problem

PROBLEM 2:

$$u \in H_0^1(\Omega) : \frac{1}{2} a(u , u) - \langle F , u \rangle + E_2(u) \leq \frac{1}{2} a(v , v) -$$

$$- \langle F , v \rangle + E_2(v) \qquad \forall v \in H_0^1(\Omega)$$

and to the mixed type variational inequality

PROBLEM 3:

$$u \in H_0^1(\Omega) : a(u , v - u) + E_2(v) - E_2(u) - \langle F , v - u \rangle \geq 0 \qquad \forall v \in H_0^1(\Omega).$$

Let us now recall a discrete model of the membrane contact problem, as given in [3].

Let $n \in N$. Let T_n be a finite family of closed triangles of R^2 such that, $\forall T \in T_n$, $T \subseteq \overline{\Omega}$ and meas $(T) > 0$ and such that, $\forall T_1 , T_2 \in T_n$, $T_1 \cap T_2$ is empty or equal to $\{ a \}$ where a is a vertex of T_1 and of T_2 or is an edge of T_1 and of T_2. Moreover, we let $\Omega_n = \bigcup_{T \in T_n} T$, $l_n = \sup_{T \in T_n} \operatorname{diam}(T)$, $s_n = \sup_{T \in T_n} \operatorname{diam}(T) / \sup \{ \operatorname{diam}(C) : C \text{ closed circle } \subseteq T \}$, $I_n = \{ x \in \overline{\Omega} : x$ is a vertex of $T \in T_n \}$, $\tilde{I}_n = \{ x \in I_n : x \notin \partial\Omega_n \}$, $T'_n = \{ T \in T_n :$ a vertex of $T \in \partial\Omega_n \}$, $\Omega'_n = \bigcup_{T \in T_n} T$ and suppose $\lim_{n \to +\infty} l_n = 0$, $\exists c_1 \in]0 , + \infty[$ $: \forall m \in N \; s_m \leq c_1$, $\forall$ compact $W \subseteq \Omega \; \exists \nu \in N : W \subseteq \Omega_m \; \forall m > \nu$. Let us introduce the space P_1 given by not greater than 1st degree polynomials of R^2 and let us denote with $a_1 , \ldots , a_{m_n}$ the elements of $\tilde{I}_n$. Moreover, we denote $\forall i \in \{ 1 , \ldots , m_n \}$ with g_{ni} the element of $C^0(\overline{\Omega})$ such that $\leq g_{ni}(a_i) = 1$, $g_{ni}(a) = 0 \; \forall a \in I_n - \{ a_i \}$, $g_{ni}|_T \in P_1 \; \forall T \in T_n$ and we introduce the subspace H_n of $H_0^1(\Omega)$

$$\left\{ \sum_{i=1}^{m_n} X_i g_{ni} \; : \; X_i \in R \right\} .$$

Furthermore, $\forall v \in C^0(\overline{\Omega})$, let us denote with $r_n v$ the element of H_n such that $r_n v(x) = v(x) \; \forall x \in \tilde{I}_n$. Moreover, we let

$$\epsilon_n = \max \{ l_n^{1/4} , \operatorname{meas}^{1/4} (\Omega - \Omega_n), \operatorname{meas}^{1/4} (\Omega'_n) \} ,$$

$$\varphi_n = r_n J_{\epsilon_n} * \varphi = \sum_{i=1}^{m_n} \varnothing_{ni} g_{ni} ,$$

$$E_{2n}(v_n) = \frac{1}{2} \int_\Omega h \left[\sum_{i=1}^{m_n} (V_{ni} - \varnothing_{ni})^+ g_{ni} \right]^2 dx \qquad \forall v_n = \sum_{i=1}^{m_n} V_{ni}\, g_{ni} \in H_n$$

and consider the mixed type variational inequality

PROBLEM 3 *a*:

$$u_n \in H_n : a(u_n, v_n - u_n) + E_{2n}(v_n) - E_{2n}(u_n) - \langle F, v_n - u_n \rangle \geq 0 \quad \forall v_n \in H_n .$$

Now, let us notice with u_n the unique solution [3] of Problem 3 *a*. We have [3]

$$\lim_{n \to +\infty} \| u_n - u \|_{H(\Omega)} = 0 .$$

2. Let $n \in N$. To compute u_n, we give now some alternative formulations of Problem 3 *a*. We let

$$\forall i, j \in \{1, \ldots, m_n\} \qquad S_{ij} = t \int_\Omega \left(\frac{\partial g_{ni}}{\partial x_1} \frac{\partial g_{nj}}{\partial x_1} + \frac{\partial g_{ni}}{\partial x_2} \frac{\partial g_{nj}}{\partial x_2} \right) dx$$

$$\forall i, i \in \{1, \ldots, m_n\} \qquad L_{ij} = \int_\Omega h g_{nj}\, g_{ni}\, dx , \qquad D_i = \int_\Omega f g_{ni}\, dx$$

$$S = [S_{ij}] , \qquad L = [L_{ij}] , \qquad D = [D_i] ,$$

and, $\forall V = (V_1, \ldots, V_{m_n}) \in R^{m_n}$, we still denote with V the column vector

$$\begin{bmatrix} V_1 \\ \cdots \\ V_{m_n} \end{bmatrix} .$$

S and L are symmetric; moreover

$$(1) \qquad \begin{aligned} &\forall V \in R^{m_n} - \{0\} && V^T S V > 0 \\ &\forall V \in R^{m_n} && V^T L V \geq 0 . \end{aligned}$$

Now we consider the [2]

(2) $\forall W = (W_1, \ldots, W_n) \in R^n$ we denote with W^+ the element $(W_1^+, \ldots, W_n^+)$ of R^n.

PROBLEM 3 *b*:

$$U \in R^{m_n} : (V - U)^T (SU - D) + \frac{1}{2} [(V - \varnothing)^+]^T L (V - \varnothing)^+ +$$

$$- \frac{1}{2} [(U - \varnothing)^+]^T L (U - \varnothing)^+ \geq 0 \qquad \forall V \in R^{m_n} .$$

Obviously, if $U = (U_1 , \ldots , U_{m_n})$ is solution of Problem 3 *b*, then $u_n = \sum_{i=1}^{m_n} U_i g_{ni}$ is solution of Problem 3 *a* and vice versa. As a consequence Problem 3 *b* allows a unique solution.

Let us now introduce, $\forall V \in R^{m_n}$:

$$J_1 (V) = \frac{1}{2} V^T SV - V^T D ,$$

$$J_2 (V) = \frac{1}{2} [(V - \varnothing)^+]^T L (V - \varnothing)^+$$

$$J (V) = J_1 (V) + J_2 (V) .$$

The functional J_2 isn't Gateaux differentiable, but it is convex (3).
After that, we consider the minimum problem

PROBLEM 2 *b*:

$$U \in R^{m_n} : J (U) \leq J (V) \qquad \forall V \in R^{m_n} .$$

We have

THEOREM 1. *The following statements are equivalent:*

1) U *is solution of Problem* 3 *b*.
2) U *is solution of Problem* 2 *b*.

Proof. 1) $\Rightarrow$ 2). Is sufficient to observe that, because of (1), it results

$$V^T SU \leq \frac{1}{2} U^T SU + \frac{1}{2} V^T SV .$$

(3) For every $U , V \in R^{m_n}$ and $\forall \varepsilon \in [0 , 1]$ $J_2 (\varepsilon U + (1 - \varepsilon) V) \leq \leq \frac{1}{2} \int_\Omega h \{ \varepsilon [\sum_{i=1}^{m_n} (U_{ni} - \varnothing_{ni})^+ g_{ni}] + (1 - \varepsilon) [\sum_{i=1}^{m_n} (V_{ni} - \varnothing_{ni})^+ g_{ni}] \}^2 dx \leq \varepsilon J_2 (U) + + (1 - \varepsilon) J_2 (V)$.

2) $\Rightarrow$ 1). Let U be a solution of Problem 2 *b* and $V \in R^{m_n}$. We observe that

$$\forall \varepsilon \in \,]0\,,1[\qquad J(U) \leq J(\varepsilon U + (1-\varepsilon) V)\,.$$

We put

$$v_n = \sum_{i=1}^{m_n} V_i g_{ni} \qquad , \qquad u_n = \sum_{i=1}^{m_n} U_i g_{ni}$$

and we obtain, taking into account that J_2 is convex

$$\forall \varepsilon \in]\,0\,,1[\quad \varepsilon J_2(U) + (1-\varepsilon) J_2(V) - J_2(U) + \frac{1}{2} a(\varepsilon u_n + (1-\varepsilon) v_n\,, \varepsilon u_n + $$
$$+ (1-\varepsilon) v_n) - \frac{1}{2}\, a(u_n\,, u_n) - \langle F\,, \varepsilon u_n + (1-\varepsilon) v_n - u_n \rangle \geq 0$$

from which

$$\forall \varepsilon \in \,]0\,,1[\quad J_2(V) - J_2(U) + \frac{1}{2}(1-\varepsilon)\, a(v_n\,, v_n) - \frac{1}{2}(1+\varepsilon)\, a(u_n\,, u_n) + $$
$$+ \varepsilon\, a(u_n\,, v_n) - \langle F\,, v_n - u_n \rangle \geq 0\,.$$

As a consequence, we get

$$J_2(V) - J_2(U) + (V-U)^T (SU - D) = J_2(V) - J_2(U) - a(u_n\,, u_n) +$$
$$+ a(u_n\,, v_n) - \langle F\,, v_n - u_n \rangle \geq 0\,. \qquad \blacksquare$$

As is well known, some methods exist [4] to solve the convex minimum Problem 2 *b*. However, for the purpose of numerical computations, more suitable formulations can be given, involving continuously Gateaux differentiable functionals.

To this aim, let us observe that, if $h = 0$ a.e. on Ω_n, we have

$$L_{ij} = 0 \qquad \forall i\,, j \in \{1\,, \ldots\,, m_n\}$$

and then Problem 2 *b* is solved by classical algorithms.

Thus, in the following we suppose

$$\text{meas}\,\{x \in \Omega_n : h(x) > 0\} > 0\,.$$

Now, for every $i \in \{1\,, \ldots\,, m_n\}$ we put

$$T_{ni} = \{T \in T_n : a_i \text{ is a vertex of } T\}\,, \; Q_i = \bigcup_{T \in T_{ni}} T\,.$$

Moreover, let us denote with $i_1 \ldots, i_{M_n}$ the elements of $\{1, \ldots, m_n\}$ such that

$$\forall a \in \{1, \ldots, M_n\} \qquad \text{meas} \ \{x \in Q_{i_a} : h(x) > 0\} > 0 .$$

Clearly

$$(2) \qquad \forall i, j \in \{1, \ldots, m_n\} \qquad (i \notin \{i_1, \ldots, i_{M_n}\}) \Rightarrow (L_{ij} = 0) .$$

Thus, we put

$$\tilde{L}_{ab} = L_{i_a i_b} \qquad \forall a, b \in \{1, \ldots, M_n\}, \ \tilde{L} = [\tilde{L}_{ab}] .$$

Obviously, $\tilde{L}$ is symmetric and we get

$$\forall a \in \{1, \ldots, M_n\} \qquad \tilde{L}_{aa} > 0 .$$

As a consequence, we obtain

LEMMA 1. *For every* $Y \in R^{M_n} - \{0\}$

$$Y^T \tilde{L} Y > 0 .$$

Proof. First of all we observe that

$$\forall Y \in R^{m_n} \quad Y^T \tilde{L} Y = \int_\Omega h \left(\sum_{i=1}^{m_n} Y_a g_{ni_a} \right)^2 dx \geq 0 .$$

Let $Y \in R^{M_n} - \{0\}$ such that $Y^T \tilde{L} Y = 0$. Then $\exists a \in \{1, \ldots, M_n\}$ such that $Y_a \neq 0$. Moreover

$$\sum_{b=1}^{m_n} Y_b g_{ni_b} = 0 \qquad \text{a.e. on} \quad \{x \in Q_{i_a} : h(x) > 0\}$$

and from this condition, taking into account that $Y_a \neq 0$, the thesis follows. ■

Now, for every $V \in R^{m_n}$ we put

$$\tilde{V} = (V_{i_1}, \ldots, V_{i_{M_n}}) .$$

From (2) we get

$$\forall Y \in R^{m_n} \qquad Y^T L Y = \sum_{i,j=1}^{m_n} Y_i L_{ij} Y_j = \sum_{a,b=1}^{M_n} Y_{i_a} \tilde{L}_{ab} Y_{i_b} = \tilde{Y}^T \tilde{L} \tilde{Y} .$$

Finally, $\forall V \in R^{m_n+M_n}$ we notice with V_1 the element of R^{m_n} whose components are the first m_n components of V and with V_2 the element of R^{M_n} whose components are the last M_n components of V, and we let $\forall V \in R^{m_n+M_n}$

$$\tilde{J}_2(V_2) = \frac{1}{2} V_2^T \tilde{L} V_2 ,$$

$$\tilde{J}(V) = J_1(V_1) + \tilde{J}_2(V_2) ,$$

$$K = \{ V \in R^{m_n+M_n} : \tilde{V}_1 - \tilde{\varnothing} - V_2 \leq 0 , \quad V_2 \geq 0 \} .$$

Obviously, K is a non–empty, closed and convex set, and the quadratic functional $\tilde{J}$ is strictly convex.

We consider the problem

PROBLFM $2\,c$:

$$U \in K : \tilde{J}(U) \leq \tilde{J}(V) \qquad \forall V \in K .$$

Clearly, if U_1 is a solution of Problem $2\,b$, then the element of $R^{m_n+M_n}$ whose first m_n components are U_1 and whose last M_n components are $(\tilde{U}_1 - \tilde{\varnothing})^+$ is a solution of Problem $2\,c$. Furthermore, if U is a solution of Problem $2\,c$, then U_1 is a solution of Problem $2\,b$ and $U_2 = (\tilde{U}_1 - \tilde{\varnothing})^+$. Thus, to approximate the solution of Problem 1 it is sufficient to solve Problem $2\,c$. For this purpose, because of properties of K and $\tilde{J}$, many well-known quadratie programming algorithms for the computation of U apply [4], [5].

Let us now notice $\forall V \in K$ with G_V the element of R^{2M_n} whose first M_n components are $\tilde{V}_1 - \tilde{\varnothing} - V_2$ and whose last M_n components are $-V_2$. Moreover, we introduce the Lagrangian of Problem $2\,c$, i.e.

$$\forall (U, X) \in R^{m_n+M_n} \times [0, +\infty[^{2M_n} \qquad \mathscr{L}(U, X) = \tilde{J}(U) + X^T G_U$$

and we consider the saddle point problem

PROBLEM $2\,d$:

$$(U, X) \in R^{m_n+M_n} \times [0, +\infty[^{2M_n} : \mathscr{L}(U, Y) \leq$$
$$\leq \mathscr{L}(U, X) \leq \mathscr{L}(V, X) \qquad \forall (V, Y) \in R^{m_n+M_n} \times [0, +\infty[^{2M_n}.$$

Kuhn-Tucker's Theorem ensures that if (U, X) is the solution of Problem $2d$ then U is the solution of Problem $2\,c$. Moreover, if U is the solution of Problem $2\,c$, then $X \in [0, +\infty[^{2M_n}$ exists such that (U, X) is the solution of Problem $2\,d$.

Furthermore, Problem 2 *d* is equivalent to the complementarity problem

PROBLEM 2 *e*:

$$(U, X) \in R^{m_n+M_n} \times [0, +\infty[^{2M_n}: \begin{cases} \dfrac{\partial \mathscr{L}}{\partial U_i}(U, X) = 0 & \forall i \in \{1, \ldots, m_n + M_n\} \\ \dfrac{\partial \mathscr{L}}{\partial X_i}(U, X) \leq 0 & \forall i \in \{1, \ldots, 2M_n\} \\ X_i \dfrac{\partial \mathscr{L}}{\partial X_i}(U, X) = 0 & \forall i \in \{1, \ldots, 2M_n\}. \end{cases}$$

Now, for every $X \in R^{2M_n}$ we notice with X_1 the element of R^{M_n} whose components are the first M_n components of X and with X_2 the element of R^{M_n} whose components are the last M_n components of X. Moreover, $\forall H \in R^{M_n}$ we notice with H the element of R^{m_n} whose components are

$$\forall i \in \{1, \ldots, m_n\} \qquad \hat{H}_i = \begin{cases} 0 \text{ if } i \notin \{i_1, \ldots, i_{M_n}\} \\ H_i \text{ if } i \in \{i_1, \ldots, i_{M_n}\}. \end{cases}$$

Because of the following expression of the Lagrangian

$$\forall (U, X) \in R^{m_n+M_n} \times [0, +\infty[^{2M_n} \qquad \mathscr{L}(U, X) = \frac{1}{2} U_1^T S U_1 - U_1^T D + \\ + \frac{1}{2} U_2^T \tilde{L} U_2 + X_1^T (\tilde{U}_1 - \tilde{\varnothing} - U_2) - X_1^T U_2$$

Problem 2 *e* can be written as

PROBLEM 2 *e*:

$$(U, X) \in R^{m_n+M_n} \times [0, +\infty[^{2M_n}: \begin{cases} SU_1 - D + \hat{X}_1 = 0 \\ \tilde{L} U_2 - X_1 - X_2 = 0 \\ \tilde{U}_1 - \tilde{\varnothing} - U_2 \leq 0 \\ U_2 \geq 0 \\ X_1^T (\tilde{U}_1 - \tilde{\varnothing} - U_2) = 0 \\ X_2^T U_2 = 0. \end{cases}$$

Finally, we consider the following reduced form of Problem 2 *e*, in which $U \in R^{m_n}$ only is involved

PROBLEM $2f$:

$$U \in R^{m_n}: \quad \begin{cases} SU - D \leq 0 \\ SU - D + L(U - \varnothing)^+ \geq 0 \\ [(U - \varnothing)^+]^T (SU - D + L(U - \varnothing)^+) = 0 \\ (SU - D)^T [(U - \varnothing)^+ - (U - \varnothing)] = 0 . \end{cases}$$

Taking into account that

$$\forall V \in R^{m_n} \qquad \tilde{L}\tilde{V} = \widetilde{LV}$$

and that if U is solution of Problem $2c$ then $U_2 = (\tilde{U}_1 - \tilde{\varnothing})^+$, it is easy to prove that Problem $2f$ and Problem $2b$ are equivalent (4).

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(4) We notice that if U is solution of Problem $2f$ then

$$\forall i \in \{1, \ldots, m_n\} - \{i_1, \ldots, i_{M_n}\} \qquad (SU - D)_i = 0 .$$

In fact, if $i \in \{1, \ldots, m_n\} - \{i_1, \ldots, i_{M_n}\}$, because of (2) it results $[L(U - \varnothing)^+]_i = 0$; consequently in the case $U_i - \varnothing_i = 0$ and in the case $U_i - \varnothing_i > 0$ we have $(SU - D)_i = 0$. The case $U_i - \varnothing_i < 0$ is obvious.