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Analysis of a discrete model for the contact problem between a membrane and an elastic obstacle

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Meccanica dei solidi. — *Analysis of a discrete model for the contact problem between a membrane and an elastic obstacle* (*). Nota di ALDO MACERI (***) e FRANCO MACERI (***)<***>, presentata (***) dal Corrisp. E. GIANGRECO.

**RIASSUNTO.** — In questo lavoro viene risolto il problema del contatto tra una membrana ed un suolo od ostacolo elastico con una approssimazione lineare a tratti della soluzione. Sono date alcune formulazioni equivalenti del problema discreto e se ne discutono le corrispondenti proprietà computazionali.

1. Let us denote by \( \Omega \in \mathbb{R}^{(1)-1} [1] \) the open, bounded region of \( \mathbb{R}^2 \) occupied by a plane membrane. The membrane is fixed at the boundary points of \( \Omega \), is transversely loaded by distributed forces \( f \in L^2(\Omega) \) orthogonal to its plane and positive in the \( x_3 \)-axis direction, and is uniformly stretched in its plane by a stress \( t \in [0, +\infty] \). Furthermore, the membrane is stretched (or constrained) by an elastic body. We describe the body shape by a function \( \varphi \in L^4(\Omega) \), and we assume its reactions on the membrane to be parallel to the \( x_3 \)-axis (frictionless contact) and to have the form \((1) - h (u - \varphi)^+\), where \( h \in L^\infty(\Omega) \), \( h \geq 0 \) a.e. on \( \Omega \), and the membrane's deflection, \( u = u(x_1, x_2) \), is positive in the \( x_3 \)-axis direction.

The problem is to find the membrane's equilibrium configuration, i.e. to solve

**PROBLEM 1:**

\[
\begin{align*}
\quad & \quad u \in H_0^1(\Omega) \cap H^2(\Omega) : Au + h(u - \varphi)^+ = f \quad \text{a.e. on} \quad \Omega \\
\end{align*}
\]

where

\[
A = -t \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right).
\]

In [2], [3] the proof that Problem 1 has a unique solution \( u \) is given. Now, we set

\[
a(u, v) = t \int_{\Omega} \left( \frac{\partial u}{\partial x_1} \frac{\partial v}{\partial x_1} + \frac{\partial u}{\partial x_2} \frac{\partial v}{\partial x_2} \right) dx \quad \forall \ (u, v) \in (H_0^1(\Omega))^2
\]

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(1) We let \( v^+ = \max \{v, 0\} \).
\[ \{ F, v \} = \int_{\Omega} fv \, dx \quad \forall v \in L^2(\Omega) \]

\[ E_2(v) = \frac{1}{2} \int_{\Omega} h \left[(v - \varphi)^+\right]^2 \, dx \quad \forall v \in L^2(\Omega). \]

In [3] it is proven that Problem 1 is equivalent to the total energy minimum problem

**PROBLEM 2:**

\[ u \in H_0^1(\Omega) : \frac{1}{2} (u^1, u) - \{ F, u \} + E_2(u) \leq \frac{1}{2} (v, v) - \{ F, v \} + E_2(v) \quad \forall v \in H_0^1(\Omega) \]

and to the mixed type variational inequality

**PROBLEM 3:**

\[ u \in H_0^1(\Omega) : (u, v - u) + E_2(v) - E_2(u) - \{ F, v - u \} \geq 0 \quad \forall v \in H_0^1(\Omega). \]

Let us now recall a discrete model of the membrane contact problem, as given in [3].

Let \( n \in \mathbb{N} \). Let \( T_n \) be a finite family of closed triangles of \( \mathbb{R}^2 \) such that, \( \forall T \in T_n, T \subseteq \overline{\Omega} \) and \( \text{meas} \,(T) > 0 \) and such that, \( \forall T_1, T_2 \in T_n, T_1 \cap T_2 \) is empty or equal to \( \{ a \} \) where \( a \) is a vertex of \( T_1 \) and of \( T_2 \) or is an edge of \( T_1 \) and of \( T_2 \). Moreover, we let \( \Omega_n = \bigcup_{T \in T_n} T, l_n = \sup_{T \in T_n} \text{diam}(T), \)

\[ s_n = \sup_{T \in T_n} \text{diam}(T)/\sup \{ \text{diam}(C) : C \text{ closed circle } \subseteq T \}, \]

\[ I_n = \{ x \in \overline{\Omega} : x \text{ is a vertex of } T \in T_n \}, \]

\[ \overline{I}_n = \{ x \in I_n : x \in \partial \Omega_n \}, \]

\[ T_n' = \{ T \in T_n : \text{ a vertex of } T \in \partial \Omega_n \}, \]

\[ \Omega_n' = \bigcup_{T \in T_n} T \text{ and suppose } \lim_{n \to +\infty} l_n = 0, \exists c_1 \in ]0, +\infty[ : \forall m \in \mathbb{N} s_m \leq c_1, \forall \text{ compact } W \subseteq \Omega \exists v \in \mathbb{N} : W \subseteq \Omega_m \forall m > v. \]

Let us introduce the space \( P_1 \) given by not greater than 1st degree polynomials of \( \mathbb{R}^2 \) and let us denote with \( a_1, \ldots, a_{m_n} \) the elements of \( I_n \). Moreover, we denote \( \forall i \in \{ 1, \ldots, m_n \} \) with \( g_{ni} \) the element of \( C^0(\overline{\Omega}) \) such that \( \leq g_{ni}(a_i) = 1, g_{ni}(a) = 0 \forall a \in I_n - \{ a_i \}, g_{ni}|_T \in P_1 \forall T \in T_n \) and we introduce the subspace \( H_n \) of \( H_0^1(\Omega) \)

\[ \left\{ \sum_{i=1}^{m_n} X_i g_{ni} : X_i \in \mathbb{R} \right\}. \]

Furthermore, \( \forall v \in C^0(\overline{\Omega}) \), let us denote with \( r_n v \) the element of \( H_n \) such that \( r_n v(x) = v(x) \forall x \in I_n \). Moreover, we let

\[ \varepsilon_n = \max \{ l_n^{1/4}, \text{meas}^{1/4}(\Omega - \Omega_n), \text{meas}^{1/4}(\Omega_n') \}, \]

\[ \varphi_n = r_n \int_{\varepsilon_n} \varphi = \sum_{i=1}^{m_n} \varepsilon_{ni} g_{ni}, \]

\[ \varepsilon_n = \max \{ l_n^{1/4}, \text{meas}^{1/4}(\Omega - \Omega_n), \text{meas}^{1/4}(\Omega_n') \}, \]

\[ \varphi_n = r_n \int_{\varepsilon_n} \varphi = \sum_{i=1}^{m_n} \varepsilon_{ni} g_{ni}, \]

\[ \varepsilon_n = \max \{ l_n^{1/4}, \text{meas}^{1/4}(\Omega - \Omega_n), \text{meas}^{1/4}(\Omega_n') \}, \]

\[ \varphi_n = r_n \int_{\varepsilon_n} \varphi = \sum_{i=1}^{m_n} \varepsilon_{ni} g_{ni}, \]
\[ E_{2n}(v_n) = \frac{1}{2} \int_\Omega \left[ \sum_{i=1}^{m_n} (V_{ni} - \phi_n)^+ g_{ni} \right]^2 \, dx \quad \forall v_n = \sum_{i=1}^{m_n} V_{ni} g_{ni} \in H_n \]

and consider the mixed type variational inequality

**Problem 3 a:**

\( u_n \in H_n : a(u_n, v_n - u_n) + E_{2n}(v_n) - E_{2n}(u_n) - (F, v_n - u_n) \geq 0 \quad \forall v_n \in H_n. \)

Now, let us notice with \( u_n \) the unique solution [3] of Problem 3 a. We have [3]

\[
\lim_{n \to +\infty} \| u_n - u \|_{H(\Omega)} = 0.
\]

2. Let \( n \in \mathbb{N} \). To compute \( u_n \), we give now some alternative formulations of Problem 3 a. We let

\[
\forall i, j \in \{1, \ldots, m_n\} \quad S_{ij} = \int_\Omega \left( \frac{\partial g_{ni}}{\partial x_i} \frac{\partial g_{nj}}{\partial x_j} + \frac{\partial g_{ni}}{\partial x_2} \frac{\partial g_{nj}}{\partial x_1} \right) \, dx
\]

\[
\forall i, i \in \{1, \ldots, m_n\} \quad L_{ij} = \int_\Omega h g_{nj} g_{ni} \, dx, \quad D_i = \int_\Omega f g_{ni} \, dx
\]

\[
S = [S_{ij}], \quad L = [L_{ij}], \quad D = [D_i],
\]

and, \( \forall V = (V_1, \ldots, V_{m_n}) \in \mathbb{R}^{m_n} \), we still denote with \( V \) the column vector

\[
\begin{bmatrix}
V_1 \\
\vdots \\
V_{m_n}
\end{bmatrix}
\]

\( S \) and \( L \) are symmetric; moreover

\[
\forall V \in \mathbb{R}^{m_n} \quad \{0\} \quad V^T SV > 0
\]

\[
\forall V \in \mathbb{R}^{m_n} \quad V^T LV \geq 0.
\]

Now we consider the (2)

\[
(2) \ \forall W = (W_1, \ldots, W_n) \in \mathbb{R}^n \text{ we denote with } W^+ \text{ the element } (W_1^+, \ldots, W_n^+) \text{ of } \mathbb{R}^n.
\]
Problem 3 b:

\[ U \in R^{mn} : (V - U)^T (SU - D) + \frac{1}{2} [(V - \varnothing)^+]^T L (V - \varnothing)^+ \]

\[-\frac{1}{2} [(U - \varnothing)^+]^T L (U - \varnothing)^+ \geq 0 \quad \forall V \in R^{mn}.\]

Obviously, if \( U = \{U_1, \ldots, U_m\} \) is solution of Problem 3 b, then \( u_n = \sum_{i=1}^{m} U_i g_{ni} \) is solution of Problem 3 a and vice versa. As a consequence Problem 3 b allows a unique solution.

Let us now introduce, \( \forall V \in R^{mn} : \)

\[ J_1 (V) = \frac{1}{2} V^T SV - V^T D, \]

\[ J_2 (V) = \frac{1}{2} [(V - \varnothing)^+]^T L (V - \varnothing)^+ \]

\[ J (V) = J_1 (V) + J_2 (V). \]

The functional \( J_2 \) isn't Gateaux differentiable, but it is convex \(^3\). After that, we consider the minimum problem

Problem 2 b:

\[ U \in R^{mn} : J (U) \leq J (V) \quad \forall V \in R^{mn}. \]

We have

Theorem 1. The following statements are equivalent:

1) \( U \) is solution of Problem 3 b.

2) \( U \) is solution of Problem 2 b.

Proof. 1) \( \Rightarrow \) 2). Is sufficient to observe that, because of (1), it results \( V^T SU \leq \frac{1}{2} U^T SU + \frac{1}{2} V^T SV \).

(3) For every \( U, V \in R^{mn} \) and \( \forall \varepsilon \in [0, 1] \)

\[ J_2 (\varepsilon U + (1 - \varepsilon) V) \leq \frac{1}{2} \int_\Omega \{ \varepsilon \left[ \sum_{i=1}^{m_n} (U_{ni} - \varnothing_{ni})^+ g_{ni} \right] + (1 - \varepsilon) \left[ \sum_{i=1}^{m_n} (V_{ni} - \varnothing_{ni})^+ g_{ni} \right] \}^2 \, dx \leq \varepsilon J_2 (U) + + (1 - \varepsilon) J_2 (V). \]
2) $\Rightarrow$ 1). Let $U$ be a solution of Problem $2b$ and $V \in \mathbb{R}^{m_n}$. We observe that

$$\forall \varepsilon \in ]0, 1[, \quad J(U) \leq J(\varepsilon U + (1 - \varepsilon) V).$$

We put

$$v_n = \sum_{i=1}^{m_n} V_i g_{ni}, \quad u_n = \sum_{i=1}^{m_n} U_i g_{ni}$$

and we obtain, taking into account that $J_2$ is convex

$$\forall \varepsilon \in ]0, 1[, \quad J_2(U) + (1 - \varepsilon) J_2(V) - J_2(U) + \frac{1}{2} a(\varepsilon u_n, (1 - \varepsilon) v_n, \varepsilon u_n + (1 - \varepsilon) v_n) \geq 0$$

from which

$$\forall \varepsilon \in ]0, 1[, \quad J_2(V) - J_2(U) + (V - U)^T (SU - D) = J_2(V) - J_2(U) - a(u_n, u_n) + \frac{1}{2} (1 - \varepsilon) a(u_n, v_n) - (F, v_n - u_n) \geq 0.$$

As a consequence, we get

$$J_2(V) - J_2(U) + (V - U)^T (SU - D) = J_2(V) - J_2(U) - a(u_n, u_n) + a(u_n, v_n) - (F, v_n - u_n) \geq 0.$$"
Moreover, let us denote with $i_1, \ldots, i_{M_n}$ the elements of $\{1, \ldots, m_n\}$ such that

$$\forall a \in \{1, \ldots, M_n\} \quad \text{meas } \{x \in Q_{i_a} : h(x) > 0\} > 0.$$  

Clearly

(2) $\forall i, j \in \{1, \ldots, m_n\}$ ($i \in \{i_1, \ldots, i_{M_n}\}$) $\Rightarrow (L_{ij} = 0)$.

Thus, we put

$$L_{ab} = L_{i_a i_b} \quad \forall a, b \in \{1, \ldots, M_n\}, \quad L = [L_{ab}].$$

Obviously, $L$ is symmetric and we get

$$\forall a \in \{1, \ldots, M_n\} \quad L_{aa} > 0.$$  

As a consequence, we obtain

**Lemma 1.** For every $Y \in \mathbb{R}^{M_n} - \{0\}$

$$YTLY > 0.$$  

**Proof.** First of all we observe that

$$\forall Y \in \mathbb{R}^{m_n} \quad YTLY = \int_{\Omega} h \left( \sum_{i=1}^{m_n} Y_a g_{n_i} \right)^2 \, dx \geq 0.$$  

Let $Y \in \mathbb{R}^{M_n} - \{0\}$ such that $YTLY = 0$. Then $\exists a \in \{1, \ldots, M_n\}$ such that $Y_a \neq 0$. Moreover

$$\sum_{i=1}^{m_n} Y_b g_{n_i} = 0 \quad \text{a.e. on } \{x \in Q_{i_a} : h(x) > 0\}$$

and from this condition, taking into account that $Y_a \neq 0$, the thesis follows.

Now, for every $V \in \mathbb{R}^{m_n}$ we put

$$\bar{V} = (V_{i_1}, \ldots, V_{i_{M_n}}).$$  

From (2) we get

$$\forall Y \in \mathbb{R}^{m_n} \quad YTLY = \sum_{i,j=1}^{m_n} Y_i L_{ij} Y_j = \sum_{a,b=1}^{M_n} Y_{i_a} L_{ab} Y_{i_b} = \bar{Y}^T \bar{L} \bar{Y}.$$
Finally, \( \forall V \in \mathbb{R}^{m_n+M_n} \) we notice with \( V_1 \) the element of \( \mathbb{R}^{m_n} \) whose components are the first \( m_n \) components of \( V \) and with \( V_2 \) the element of \( \mathbb{R}^{M_n} \) whose components are the last \( M_n \) components of \( V \), and we let \( \forall V \in \mathbb{R}^{m_n+M_n} \):

\[
\begin{align*}
\tilde{J}_2(V_2) &= \frac{1}{2} V_2^T \tilde{L} V_2, \\
\tilde{J}(V) &= \tilde{J}_1(V_1) + \tilde{J}_2(V_2), \\
K &= \{ V \in \mathbb{R}^{m_n+M_n} : \tilde{V}_1 - \tilde{\sigma} - V_2 \leq 0, \ V_2 \geq 0 \}.
\end{align*}
\]

Obviously, \( K \) is a non-empty, closed and convex set, and the quadratic functional \( \tilde{J} \) is strictly convex.

We consider the problem

**Problem 2 c:**

\[
U \in K : \tilde{J}(U) \leq \tilde{J}(V) \quad \forall V \in K.
\]

Clearly, if \( U_1 \) is a solution of Problem 2 b, then the element of \( \mathbb{R}^{m_n+M_n} \) whose first \( m_n \) components are \( U_1 \) and whose last \( M_n \) components are \( (\tilde{U}_1 - \tilde{\sigma})^+ \) is a solution of Problem 2 c. Furthermore, if \( U \) is a solution of Problem 2 c, then \( U_1 \) is a solution of Problem 2 b and \( U_2 = (\tilde{U}_1 - \tilde{\sigma})^+ \). Thus, to approximate the solution of Problem 1 it is sufficient to solve Problem 2 c. For this purpose, because of properties of \( K \) and \( \tilde{J} \), many well-known quadratic programming algorithms for the computation of \( U \) apply [4], [5].

Let us now notice \( \forall V \in K \) with \( G_V \) the element of \( \mathbb{R}^{2M_n} \) whose first \( M_n \) components are \( \tilde{V}_1 - \tilde{\sigma} - V_2 \) and whose last \( M_n \) components are \(-V_2\). Moreover, we introduce the Lagrangian of Problem 2 c, i.e.

\[
\forall (U, X) \in \mathbb{R}^{m_n+M_n} \times [0, + \infty]^{2M_n} \quad \mathcal{L}(U, X) = \tilde{J}(U) + X^T G_U
\]

and we consider the saddle point problem

**Problem 2 d:**

\[
(U, X) \in \mathbb{R}^{m_n+M_n} \times [0, + \infty]^{2M_n} : \mathcal{L}(U, Y) \leq \mathcal{L}(U, X) \quad \forall (V, Y) \in \mathbb{R}^{m_n+M_n} \times [0, + \infty]^{2M_n}.
\]

Kuhn-Tucker’s Theorem ensures that if \( (U, X) \) is the solution of Problem 2d then \( U \) is the solution of Problem 2 c. Moreover, if \( U \) is the solution of Problem 2 c, then \( X \in [0, + \infty]^{2M_n} \) exists such that \( (U, X) \) is the solution of Problem 2 d.

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Furthermore, Problem 2.d is equivalent to the complementarity problem

PROBLEM 2.e:

\[
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial U_i} (U, X) &= 0 & \forall i \in \{1, \ldots, m_n + M_n\} \\
\frac{\partial \mathcal{L}}{\partial X_i} (U, X) &\leq 0 & \forall i \in \{1, \ldots, 2 M_n\} \\
X_i \frac{\partial \mathcal{L}}{\partial X_i} (U, X) &= 0 & \forall i \in \{1, \ldots, 2 M_n\}.
\end{aligned}
\]

(U, X) \in \mathbb{R}^{m_n + M_n} \times [0, + \infty)^{2M_n}:

Now, for every X \in \mathbb{R}^{2M_n} we notice with X_1 the element of \(\mathbb{R}^{M_n}\) whose components are the first \(M_n\) components of X and with X_2 the element of \(\mathbb{R}^{M_n}\) whose components are the last \(M_n\) components of X. Moreover, \(\forall H \in \mathbb{R}^{M_n}\) we notice with H the element of \(\mathbb{R}^{M_n}\) whose components are

\[\forall i \in \{1, \ldots, m_n\} \quad \hat{H}_i = \begin{cases} 0 & \text{if } i \in \{i_1, \ldots, i_{M_n}\} \\
H_i & \text{if } i \in \{i_1, \ldots, i_{M_n}\}. \end{cases}\]

Because of the following expression of the Lagrangian

\[\forall (U, X) \in \mathbb{R}^{m_n + M_n} \times [0, + \infty)^{2M_n} \quad \mathcal{L} (U, X) = \frac{1}{2} U_i^T S U_i - U_i^T D + \frac{1}{2} U_i^T \bar{L} U_i + X_i^T (\bar{U}_1 - \bar{\xi} - U_2) - X_i^T U_2 \]

Problem 2.e can be written as

PROBLEM 2.e:

\[
\begin{aligned}
SU_i - D + \dot{X}_i &= 0 \\
\bar{L} U_i - X_i - X_2 &= 0 \\
\bar{U}_1 - \bar{\xi} - U_2 &\leq 0 \\
U_2 &\geq 0 \\
X_1^T (\bar{U}_1 - \bar{\xi} - U_2) &= 0 \\
X_2^T U_2 &= 0.
\end{aligned}
\]

Finally, we consider the following reduced form of Problem 2.e, in which \(U \in \mathbb{R}^{m_n}\) only is involved
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PROBLEM 2f:

\[ SU - D \leq 0 \]
\[ SU - D + L(U - \varnothing)^+ \geq 0 \]
\[ [(U - \varnothing)^+]^T (SU - D + L(U - \varnothing)^+) = 0 \]
\[ (SU - D)^T [(U - \varnothing)^+ -(U - \varnothing)] = 0 . \]

Taking into account that

\[ \forall V \in \mathbb{R}^{m_n} \quad \tilde{L} \tilde{V} = \tilde{L} \tilde{V} \]

and that if \( U \) is solution of Problem 2c then \( U \varepsilon = (\tilde{U}_1 - \tilde{U})^+ \), it is easy to prove that Problem 2f and Problem 2b are equivalent (4).

REFERENCES


(4) We notice that if \( U \) is solution of Problem 2f then

\[ \forall i \in \{1, \ldots, m_n \} - \{i_1, \ldots, i_{M_n}\} \quad (SU - D)_i = 0 . \]

In fact, if \( i \in \{1, \ldots, m_n \} - \{i_1, \ldots, i_{M_n}\} \), because of (2) it results \([L(U - \varnothing)^+]_i = 0\); consequently in the case \( U_i - \varnothing_i = 0 \) and in the case \( U_i - \varnothing_i > 0 \) we have \((SU - D)_i = 0\). The case \( U_i - \varnothing_i < 0 \) is obvious.