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Analysis of a discrete model for the contact problem between a membrane and an elastic obstacle


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**RIASSUNTO.** — In questo lavoro viene risolto il problema del contatto tra una membrana ed un suolo od ostacolo elastico con una approssimazione lineare a tratti della soluzione. Sono date alcune formulazioni equivalenti del problema discreto e se ne discutono le corrispondenti proprietà computazionali.

1. Let us denote by $\Omega \in \mathbb{R}^{1+1}$ [1] the open, bounded region of $\mathbb{R}^2$ occupied by a plane membrane. The membrane is fixed at the boundary points of $\Omega$, is transversely loaded by distributed forces $f \in L^2(\Omega)$ orthogonal to its plane and positive in the $x_3$-axis direction, and is uniformly stretched in its plane by a stress $t \in [0, +\infty]$. Furthermore, the membrane is stretched (or constrained) by an elastic body. We describe the body shape by a function $\varphi \in L^1(\Omega)$, and we assume its reactions on the membrane to be parallel to the $x_3$-axis (frictionless contact) and to have the form $-(h(u - \varphi)^+)$, where $h \in L^\infty(\Omega)$, $h \geq 0$ a.e. on $\Omega$, and the membrane's deflection, $u = u(x_1, x_2)$, is positive in the $x_3$-axis direction.

The problem is to find the membrane's equilibrium configuration, i.e. to solve

**PROBLEM 1:**

$$u \in H_0^1(\Omega) \cap H^2(\Omega) : Au + h(u - \varphi)^+ = f \quad \text{a.e. on } \Omega$$

where

$$A = -t \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right).$$

In [2], [3] the proof that Problem 1 has a unique solution $u$ is given. Now, we set

$$a(u, v) = t \int_\Omega \left( \frac{\partial u}{\partial x_1} \frac{\partial v}{\partial x_1} + \frac{\partial u}{\partial x_2} \frac{\partial v}{\partial x_2} \right) dx \quad \forall (u, v) \in (H_0^1(\Omega))^2.$$

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\begin{align*}
\langle F, v \rangle &= \int_{\Omega} f v \, dx & \forall v \in L^2(\Omega) \\
E_2(v) &= \frac{1}{2} \int_{\Omega} h [(v - \varphi)^+]^2 \, dx & \forall v \in L^2(\Omega).
\end{align*}

In [3] it is proven that Problem 1 is equivalent to the total energy minimum problem

**PROBLEM 2:**

\[
\begin{array}{rcl}
u \in H_0^1(\Omega) : & \frac{1}{2} a(u, u) - \langle F, u \rangle + E_2(u) & \leq \frac{1}{2} a(v, v) - \\
& - \langle F, v \rangle + E_2(v) & \forall v \in H_0^1(\Omega)
\end{array}
\]

and to the mixed type variational inequality

**PROBLEM 3:**

\[
\begin{array}{rcl}
u \in H_0^1(\Omega) : & a(u, v - u) + E_2(v) - E_2(u) - \langle F, v - u \rangle & \geq 0 & \forall v \in H_0^1(\Omega).
\end{array}
\]

Let us now recall a discrete model of the membrane contact problem, as given in [3].

Let \( n \in \mathbb{N} \). Let \( T_n \) be a finite family of closed triangles of \( \mathbb{R}^2 \) such that, \( \forall T \in T_n, T \subseteq \overline{\Omega} \) and meas \((T) > 0 \) and such that, \( \forall T_1, T_2 \in T_n, T_1 \cap T_2 \) is empty or equal to \{a\} where \( a \) is a vertex of \( T_1 \) and of \( T_2 \) or is an edge of \( T_1 \) and of \( T_2 \). Moreover, we let \( \Omega_n = \bigcup_{T \in T_n} T \), \( l_n = \sup_{T \in T_n} \text{diam}(T) \), \( s_n = \sup_{T \in T_n} \text{diam}(T)/\sup \{ \text{diam}(C) : C \text{ closed circle } \subseteq T \} \), \( I_n = \{ x \in \overline{\Omega} : x \text{ is a vertex of } T \in T_n \} \), \( \Omega_n' = \{ x \in \partial \Omega_n : x \in \partial \Omega_n \} \), \( T_n' = \{ T \in T_n' : a \text{ vertex of } T \} \) and suppose \( \lim_{n \to +\infty} l_n = 0 \), \( \exists C \in ]0, +\infty[ \) : \( \forall m \in \mathbb{N}, s_m \leq C \), \( \forall \text{ compact } W \subseteq \Omega \) \( \exists v \in \mathbb{N} : W \subseteq \Omega_m \forall m > n \). Let us introduce the space \( P_n \) given by not greater than 1st degree polynomials of \( \mathbb{R}^2 \) and let us denote with \( a_1, \ldots, a_m \) the elements of \( \Omega_n' \). Moreover, we denote \( \forall \iota \in \{1, \ldots, m_n \} \) with \( g_{ni} \) the element of \( C^0(\overline{\Omega}) \) such that \( \leq g_{ni}(a_i) = 1 \), \( g_{ni}(a) = 0 \) \( \forall a \in \Omega_n' - \{a_i\} \), \( g_{ni} \mid_T \in P_1 \forall T \in T_n \) and we introduce the subspace \( H_n \) of \( H_0^1(\Omega) \)

\[
\left\{ \sum_{i=1}^{m_n} X_i g_{ni} : X_i \in \mathbb{R} \right\}.
\]

Furthermore, \( \forall v \in C^0(\overline{\Omega}) \), let us denote with \( r_n v \) the element of \( H_n \) such that \( r_n v(x) = v(x) \) \( \forall x \in \Omega_n \). Moreover, we let

\[
\begin{align*}
\varepsilon_n &= \max \{ l_n^{1/4}, \text{meas}^{1/4}(\Omega - \Omega_n), \text{meas}^{1/4}(\Omega_n') \}, \\
\varphi_n &= r_n \int_{\varepsilon_n} \varphi = \sum_{i=1}^{m_n} \varepsilon_{ni} g_{ni},
\end{align*}
\]
\[ E_{2n}(v_n) = \frac{1}{2} \int_{\Omega} \left[ \sum_{i=1}^{m_n} (V_{ni} - \varphi_{ni})^2 + g_{ni} \right] \, dx \quad \forall v_n = \sum_{i=1}^{m_n} V_{ni} g_{ni} \in H_n \]

and consider the mixed type variational inequality

**Problem 3 a:**

\[ u_n \in H_n : a(u_n, v_n - u_n) + E_{2n}(v_n) = E_{2n}(u_n) + \langle F, v_n - u_n \rangle \geq 0 \quad \forall v_n \in H_n. \]

Now, let us notice with \( u_n \) the unique solution \([3]\) of Problem 3 a. We have \([3]\)

\[ \lim_{n \to +\infty} \| u_n - u \|_{H(\Omega)} = 0. \]

2. Let \( n \in \mathbb{N} \). To compute \( u_n \), we give now some alternative formulations of Problem 3 a. We let

\[ \forall i, j \in \{1, \ldots, m_n\} \quad S_{ij} = \int_{\Omega} \left( \frac{\partial g_{ni}}{\partial x_1} \frac{\partial g_{nj}}{\partial x_1} + \frac{\partial g_{ni}}{\partial x_2} \frac{\partial g_{nj}}{\partial x_2} \right) \, dx \]

\[ \forall i, i \in \{1, \ldots, m_n\} \quad L_{ij} = \int_{\Omega} h g_{nj} g_{ni} \, dx, \quad D_i = \int_{\Omega} f g_{ni} \, dx \]

\[ S = [S_{ij}], \quad L = [L_{ij}], \quad D = [D_i], \]

and, \( \forall V = (V_1, \ldots, V_{m_n}) \in \mathbb{R}^{m_n} \), we still denote with \( V \) the column vector

\[
\begin{bmatrix}
V_1 \\
\vdots \\
V_{m_n}
\end{bmatrix}
\]

\( S \) and \( L \) are symmetric; moreover

\[ \forall V \in \mathbb{R}^{m_n} \to \{0\} \quad V^T SV > 0 \quad (1) \]

\[ \forall V \in \mathbb{R}^{m_n} \quad V^T LV \geq 0. \]

Now we consider the \((2)\)

\[ \forall W = (W_1, \ldots, W_n) \in \mathbb{R}^n \] we denote with \( W^+ \) the element \((W_1^+, \ldots, W_n^+)\) of \( \mathbb{R}^n \).
PROBLEM 3 b:

\[ U \in R^{mn} : (V - U)^T (SU - D) + \frac{1}{2} [(V - \varnothing)^+]^T L (V - \varnothing)^+ \]

\[ + \frac{1}{2} [(U - \varnothing)^+]^T L (U - \varnothing)^+ \geq 0 \quad \forall V \in R^{mn}. \]

Obviously, if \( U = (U_1, \ldots, U_{mn}) \) is solution of Problem 3 b, then \( u_n = \sum_{i=1}^{mn} U_i g_{ni} \) is solution of Problem 3 a and vice versa. As a consequence Problem 3 b allows a unique solution.

Let us now introduce, \( \forall V \in R^{mn} : \)

\[ J_1 (V) = \frac{1}{2} V^T SV - V^T D, \]

\[ J_2 (V) = \frac{1}{2} [(V - \varnothing)^+]^T L (V - \varnothing)^+ \]

\[ J (V) = J_1 (V) + J_2 (V). \]

The functional \( J_2 \) isn’t Gateaux differentiable, but it is convex \(^{\circ}\).

After that, we consider the minimum problem

PROBLEM 2 b:

\[ U \in R^{mn} : J (U) \leq J (V) \quad \forall V \in R^{mn}. \]

We have

THEOREM 1. The following statements are equivalent:

1) \( U \) is solution of Problem 3 b.

2) \( U \) is solution of Problem 2 b.

Proof. 1) \( \Rightarrow \) 2). Is sufficient to observe that, because of (1), it results

\[ V^T SU \leq \frac{1}{2} U^T SU + \frac{1}{2} V^T SV. \]

(3) For every \( U, V \in R^{mn} \) and \( \forall \varepsilon \in [0, 1] \)

\[ J_2 (\varepsilon U + (1 - \varepsilon) V) \leq \frac{1}{2} \int_{\Omega} h \left\{ e \left[ \sum_{i=1}^{mn} (U_{ni} - \varnothing_{ni})^+ g_{ni} \right] + (1 - e) \left[ \sum_{i=1}^{mn} (V_{ni} - \varnothing_{ni})^+ g_{ni} \right] \right\}^2 dx \leq \varepsilon J_2 (U) + \]

\[ + (1 - \varepsilon) J_2 (V). \]
2) $\Rightarrow$ 1). Let $U$ be a solution of Problem 2 b and $V \in \mathbb{R}^{m_n}$. We observe that

$$\forall \varepsilon \in ]0, 1[ \quad J(U) \leq J(\varepsilon U + (1 - \varepsilon) V).$$

We put

$$v_n = \sum_{i=1}^{m_n} V_i g_{ni}, \quad u_n = \sum_{i=1}^{m_n} U_i g_{ni},$$

and we obtain, taking into account that $J_2$ is convex

$$\forall \varepsilon \in ]0, 1[ \quad \varepsilon J_2(U) + (1 - \varepsilon) J_2(V) - J_2(U) + \frac{1}{2} \alpha (\varepsilon u_n + (1 - \varepsilon) v_n, \varepsilon u_n + (1 - \varepsilon) v_n) \geq 0$$

from which

$$\forall \varepsilon \in ]0, 1[ \quad J_2(V) - J_2(U) + (V - U)^T (SU - D) - J_2(V) - J_2(U) + \frac{1}{2} \alpha (u_n, v_n) - a(u_n, u_n) + \frac{1}{2} (1 - \varepsilon) a(u_n, u_n) + \alpha (u_n, v_n) - \{ F, v_n - u_n \} \geq 0.$$

As a consequence, we get

$$J_2(V) - J_2(U) + (V - U)^T (SU - D) = J_2(V) - J_2(U) - a(u_n, u_n) + \alpha (u_n, v_n) - \{ F, v_n - u_n \} \geq 0.$$

As is well known, some methods exist [4] to solve the convex minimum Problem 2 b. However, for the purpose of numerical computations, more suitable formulations can be given, involving continuously Gateaux differentiable functionals.

To this aim, let us observe that, if $h = 0$ a.e. on $\Omega_n$, we have

$$L_{ij} = 0 \quad \forall i, j \in \{ 1, \ldots, m_n \}$$

and then Problem 2 b is solved by classical algorithms.

Thus, in the following we suppose

$$\text{meas} \{ x \in \Omega_n : h(x) > 0 \} > 0.$$

Now, for every $i \in \{ 1, \ldots, m_n \}$ we put

$$T_{ni} = \{ T \in T_n : a_i \text{ is a vertex of } T \}, \quad Q_i = \bigcup_{T \in T_{ni}} T.$$
Moreover, let us denote with $i_1, \ldots, i_{M_n}$ the elements of $\{1, \ldots, m_n\}$ such that
\[ \forall a \in \{1, \ldots, M_n\} \quad \text{meas} \{x \in Q_a : h(x) > 0\} > 0. \]

Clearly
\[(2) \quad \forall i, j \in \{1, \ldots, m_n\} \quad (i \in \{i_1, \ldots, i_{M_n}\}) \Rightarrow (L_{ij} = 0).\]

Thus, we put
\[ L_{ab} = L_{i_a i_b} \quad \forall a, b \in \{1, \ldots, M_n\}, \quad L = [L_{ab}]. \]

Obviously, $L$ is symmetric and we get
\[ \forall a \in \{1, \ldots, M_n\} \quad L_{aa} > 0. \]

As a consequence, we obtain

**Lemma 1.** For every $Y \in \mathbb{R}^{M_n} - \{0\}$
\[ Y^T L Y > 0. \]

**Proof.** First of all we observe that
\[ \forall Y \in \mathbb{R}^{m_n} \quad Y^T L Y = \int_{\Omega} h \left( \sum_{i=1}^{m_n} Y_i g_{n_{i_a}} \right)^2 dx \geq 0. \]

Let $Y \in \mathbb{R}^{M_n} - \{0\}$ such that $Y^T L Y = 0$. Then $\exists a \in \{1, \ldots, M_n\}$ such that $Y_a \neq 0$. Moreover
\[ \sum_{i=1}^{m_n} Y_i g_{n_{i_a}} = 0 \quad \text{a.e. on} \quad \{x \in Q_{i_a} : h(x) > 0\} \]
and from this condition, taking into account that $Y_a \neq 0$, the thesis follows. \hfill \blacksquare

Now, for every $V \in \mathbb{R}^{m_n}$ we put
\[ \bar{V} = (V_{i_1}, \ldots, V_{i_{M_n}}). \]

From (2) we get
\[ \forall Y \in \mathbb{R}^{m_n} \quad Y^T L Y = \sum_{i,j=1}^{m_n} Y_i L_{ij} Y_j = \sum_{a,b=1}^{M_n} Y_i \bar{L}_{ab} Y_b = \bar{Y}^T \bar{L} \bar{Y}. \]
Finally, \( \forall V \in \mathbb{R}^{m_n+M_n} \) we notice with \( V_1 \) the element of \( \mathbb{R}^{m_n} \) whose components are the first \( m_n \) components of \( V \) and with \( V_2 \) the element of \( \mathbb{R}^{M_n} \) whose components are the last \( M_n \) components of \( V \), and we let \( \forall V \in \mathbb{R}^{m_n+M_n} \)

\[
\bar{J}_2(V_2) = \frac{1}{2} V_2^T \bar{L} V_2,
\]

\[
\bar{J}(V) = \bar{J}_1(V_1) + \bar{J}_2(V_2),
\]

\[
K = \{ V \in \mathbb{R}^{m_n+M_n} : \bar{V}_1 - \bar{\delta} - \bar{V}_2 \leq 0, \ \bar{V}_2 \geq 0 \}.
\]

Obviously, \( K \) is a non-empty, closed and convex set, and the quadratic functional \( \bar{J} \) is strictly convex.

We consider the problem

**Problem 2 c:**

\[
\forall V \in K : \bar{J}(U) \leq \bar{J}(V)
\]

Clearly, if \( U_1 \) is a solution of Problem 2 b, then the element of \( \mathbb{R}^{m_n+M_n} \) whose first \( m_n \) components are \( U_1 \) and whose last \( M_n \) components are \( (\bar{U}_1 - \bar{\delta})^+ \) is a solution of Problem 2 c. Furthermore, if \( U \) is a solution of Problem 2 c, then \( U_1 \) is a solution of Problem 2 b and \( U_2 = (\bar{U}_1 - \bar{\delta})^+ \). Thus, to approximate the solution of Problem 1 it is sufficient to solve Problem 2 c. For this purpose, because of properties of \( K \) and \( \bar{J} \), many well-known quadratic programming algorithms for the computation of \( U \) apply [4], [5].

Let us now notice \( \forall V \in K \) with \( G_V \) the element of \( \mathbb{R}^{2M_n} \) whose first \( M_n \) components are \( \bar{V}_1 - \bar{\delta} - \bar{V}_2 \) and whose last \( M_n \) components are \( -\bar{V}_2 \). Moreover, we introduce the Lagrangian of Problem 2 c, i.e.

\[
\forall (U, X) \in \mathbb{R}^{m_n+M_n} \times [0, + \infty [^{2M_n} \quad \mathcal{L}(U, X) = \bar{J}(U) + X^T G_U
\]

and we consider the saddle point problem

**Problem 2 d:**

\[
(U, X) \in \mathbb{R}^{m_n+M_n} \times [0, + \infty [^{2M_n} : \mathcal{L}(U, Y) \leq \mathcal{L}(U, X) \leq \mathcal{L}(V, X) \quad \forall (V, Y) \in \mathbb{R}^{m_n+M_n} \times [0, + \infty [^{2M_n}.
\]

Kuhn-Tucker's Theorem ensures that if \( (U, X) \) is the solution of Problem 2d then \( U \) is the solution of Problem 2 c. Moreover, if \( U \) is the solution of Problem 2 c, then \( X \in [0, + \infty [^{2M_n} \) exists such that \( (U, X) \) is the solution of Problem 2 d.
Furthermore, Problem 2.d is equivalent to the complementarity problem

**Problem 2.e:**

\[
\begin{align*}
\frac{\partial \mathcal{L}}{\partial U_i} (U, X) &= 0 \quad \forall i \in \{1, \ldots, m_a + M_n\} \\
(U, X) \in \mathbb{R}^{m_n+M_n} 	imes [0, + \infty[^{2M_n}:
\frac{\partial \mathcal{L}}{\partial X_i} (U, X) &\leq 0 \quad \forall i \in \{1, \ldots, 2 M_n\} \\
X_i \frac{\partial \mathcal{L}}{\partial X_i} (U, X) &= 0 \quad \forall i \in \{1, \ldots, 2 M_n\}.
\end{align*}
\]

Now, for every \(X \in \mathbb{R}^{2M_n}\) we notice with \(X_1\) the element of \(\mathbb{R}^{M_n}\) whose components are the first \(M_n\) components of \(X\) and with \(X_2\) the element of \(\mathbb{R}^{M_n}\) whose components are the last \(M_n\) components of \(X\). Moreover, \(\forall H \in \mathbb{R}^{M_n}\) we notice with \(H\) the element of \(\mathbb{R}^{M_n}\) whose components are

\[
\forall i \in \{1, \ldots, m_n\} \quad \dot{H}_i = \begin{cases} 0 & \text{if } i \notin \{i_1, \ldots, i_{M_n}\} \\
H_i & \text{if } i \in \{i_1, \ldots, i_{M_n}\} \end{cases}.
\]

Because of the following expression of the Lagrangian

\[
\mathcal{L} (U, X) = \frac{1}{2} U_1^T S U_1 - U_1^T D + \frac{1}{2} U_2^T \bar{L} U_2 + X_1^T (\bar{U}_1 - \bar{\sigma} - U_2) - X_1^T U_2
\]

Problem 2.e can be written as

**Problem 2.e:**

\[
\begin{align*}
S U_1 - D + \dot{X}_1 &= 0 \\
\bar{L} U_2 - X_1 - X_2 &= 0 \\
\dot{U}_1 - \bar{\sigma} - U_2 &\leq 0 \\
U_2 &\geq 0 \\
X_1^T (\bar{U}_1 - \bar{\sigma} - U_2) &= 0 \\
X_2^T U_2 &= 0.
\end{align*}
\]

Finally, we consider the following reduced form of Problem 2.e, in which \(U \in \mathbb{R}^{m_n}\) only is involved
**Problem 2f:**

\[ SU - D \leq 0 \]

\[ SU - D + L(U - \varnothing)^+ \geq 0 \]

\[ [(U - \varnothing)^+]^T (SU - D + L(U - \varnothing)^+) = 0 \]

\[ (SU - D)^T [(U - \varnothing)^+ - (U - \varnothing)] = 0 . \]

Taking into account that

\[ \forall V \in \mathbb{R}^m, \quad \tilde{L}V = \tilde{L}V \]

and that if \( U \) is solution of Problem 2c then \( U_2 = (\tilde{U}_1 - \tilde{\varnothing})^+ \), it is easy to prove that Problem 2f and Problem 2b are equivalent (4).

**References**


(4) We notice that if \( U \) is solution of Problem 2f then

\[ \forall i \in \{1, \ldots, m_n\} - \{i_1, \ldots, i_{m_n}\} \quad (SU - D)_i = 0 . \]

In fact, if \( i \in \{1, \ldots, m_n\} - \{i_1, \ldots, i_{m_n}\} \), because of (2) it results \( [L(U - \varnothing)]_i = 0 \); consequently in the case \( U_i - \varnothing_i = 0 \) and in the case \( U_i - \varnothing_i > 0 \) we have \( (SU - D)_i = 0 \). The case \( U_i - \varnothing_i < 0 \) is obvious.