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Marco Abate

Boundary behaviour of invariant distances and complex geodesics

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Geometria. — *Boundary behaviour of invariant distances and complex geodesics.* Nota di MARCO ABATE, presentata (*) dal Corrisp. E. VESENTINI.

RIASSUNTO. — In questa Nota viene studiato il comportamento al bordo delle distanze di Carathéodory e Kobayashi in domini fortemente pseudoconvessi di classe C^2. Come applicazione si dimostra che ogni geodetica complessa in tali domini è estendibile al bordo di classe $C^{0,1/2}$.

0. In this paper we first investigate the boundary behaviour of the Carathéodory and Kobayashi invariant distances in strongly pseudoconvex C^2 domains in $\mathbf{C}^n$. Let $D \subset\subset \mathbf{C}^n$ be a relatively compact domain in $\mathbf{C}^n$: c_D (k_D) will be the Carathéodory (resp. Kobayashi) invariant distance on D; also, $\| \ \|$ will denote the euclidean norm, and $d(z, \partial D)$ the euclidean distance of z from ∂D. Finally, the sup-norm of a continuos function f on D will be denoted by $\| f \|_D$.

Our results are summarized in

THEOREM 0.1 *Let* $D \subset\subset \mathbf{C}^n$ *be a strongly pseudoconvex* C^2 *domain. Then*

$$\forall z_0 \in D \quad \lim_{z \to \partial D} \frac{k_D(z_0, z)}{-\log d(z, \partial D)} = \lim_{z \to \partial D} \frac{c_D(z_0, z)}{-\log d(z, \partial D)} = \frac{1}{2} .$$

This theorem follows estimating the invariant distances by means of $d(z, \partial D)$: see Propositions 1.2 and 1.3. In some sense, this is the integrated form of the results on the invariant differential metrics obtained by Graham in [2].

The second part of this paper deals with the boundary behaviour of complex geodesics. Let $D \subset\subset \mathbf{C}^n$ be a domain: a *complex geodesic* (as introduced in [4]) for c_D (k_D) is a map from the open unit disk Δ of $\mathbf{C}$ to D, $f : \Delta \to D$, such that

$$\forall \zeta_1, \zeta_2 \in \Delta \quad c_D(f(\zeta_1), f(\zeta_2)) = \omega(\zeta_1, \zeta_2)$$
$$(k_D(f(\zeta_1), f(\zeta_2)) = \omega(\zeta_1, \zeta_2)),$$

where $\omega = c_\Delta = k_\Delta$ is the Poincaré distance of Δ.

Lempert in [3] has proved that every complex geodesic in a strongly convex C^2 domain admits a $C^{0,1/2}$-extension to $\overline{\Delta}$. Our second main result is the generalisation of this theorem to the strongly pseudoconvex case:

(*) Nella seduta dell'8 marzo 1986.

THEOREM 0.2. *Let* $D \subset\subset \mathbf{C}^n$ *be a strongly pseudoconvex* C^2 *domain. Then every complex geodesic* (*either for* c_D *or for* k_D) *admits a* $C^{0,1/2}$ *-extension to* $\overline{\Delta}$.

The proof is essentially based on the estimates obtained in Part 1.

1. An easy computation yields the following lemma:

LEMMA 1.1. *Let* B_r *be the euclidean ball of radius* r *in* $\mathbf{C}^n$. *Then*

$$\forall z \in B_r \qquad \frac{1}{2}\log(2r) - \frac{1}{2}\log d(z, \partial B_r) \geq k_{B_r}(0, z) = c_{B_r}(0, z) \geq$$

$$\geq \frac{1}{2}\log r - \frac{1}{2}\log d(z, \partial B_r).$$

The upper estimate holds under weaker hypothesis than those of Theorem 0.1:

PROPOSITION 1.2. *Let* $D \subset\subset \mathbf{C}^n$ *be a* C^2 *domain and let* $z_0 \in D$. *Then there exists a positive constant* $c_1 = c_1(z_0, D)$ *such that*

$$\forall z \in D \qquad c_1 - \frac{1}{2}\log d(z, \partial D) \geq k_D(z_0, z) \geq c_D(z_0, z).$$

Proof. Since D is a C^2 domain, ∂D admits regular tubular neighbourhoods of given radius $\varepsilon > 0$, for ε small enough; let U_ε be such a neighbourhood, of radius $\varepsilon < 1$. Put

$$c_1 = \sup\{k_D(z_0, w) \mid w \in D \setminus U_{\varepsilon/4}\} + \max\left\{0, \frac{1}{2}\log \operatorname{diam}(D)\right\}.$$

We have two possibilities:

(i) $z \in U_{\varepsilon/4} \cap D$. Let $x \in \partial D$ be such that $\| x - z \| = d(z, \partial D)$. Since $U_{\varepsilon/2}$ is a tubular neighbourhood, there exists $\lambda \in \mathbf{R}$ such that $w = \lambda(x - z) \in U_{\varepsilon/2} \cap D$ and the Euclidean ball B of centre w and radius $\varepsilon/2$ is contained in $U_\varepsilon \cap D$ and tangent to ∂D in x. Therefore Lemma 1.1 yields

$$c_D(z_0, z) \leq k_D(z_0, z) \leq k_D(z_0, w) + k_D(w, z) \leq k_D(z_0, w) + k_B(w, z) \leq$$

$$\leq k_D(z_0, w) + \frac{1}{2}\log \varepsilon - \frac{1}{2}\log d(z, \partial B) \leq$$

$$\leq k_D(z_0, w) - \frac{1}{2}\log d(z, \partial D) \leq c_1 - \frac{1}{2}\log d(z, \partial D),$$

because $w \notin U_{\varepsilon/4}$ (and $\varepsilon < 1$).

(ii) $z \in \mathbf{D} \setminus \mathbf{U}_{\varepsilon/4}$. Then

$$c_D(z_0, z) \leq k_D(z_0, z) \leq c_1 - \frac{1}{2} \log \operatorname{diam}(D) \leq c_1 - \frac{1}{2} \log d(z, \partial D),$$

because $d(z, \partial D) \leq \operatorname{diam}(D)$, q.e.d.

To establish the lower estimate, we need the pseudoconvexity condition:

PROPOSITION 1.3. *Let* $D \subset\subset \mathbf{C}^n$ *be a strongly pseudoconvex* C^2 *domain and let* $z_0 \in D$. *Then there exists a constant* $c_2 = c_2(z_0, D)$ *such that*

$$\forall z \in D \qquad k_D(z_0, z) \geq c_D(z_0, z) \geq c_2 - \frac{1}{2} \log d(z, \partial D).$$

Proof. Let $\Psi : \partial D \times \widehat{D} \to \mathbf{C}$ ($\widehat{D}$ open neighbourhood of $\overline{D}$) as in Proposition 4 of [2], and define $\phi : \partial D \times \Delta \to \Delta$ by

$$\phi(x, \zeta) = \frac{1 - \overline{\Psi(x, z_0)}}{1 - \Psi(x, z_0)} \, \frac{\zeta - \Psi(x, z_0)}{1 - \overline{\Psi(x, z_0)}\, \zeta}.$$

Then $\Phi(x, z) = \Phi_x(z) = \phi(x, \Psi(x, z))$ is defined on a neighbourhood $\partial D \times D'$ of $\partial D \times \overline{D}$ (with $D' \subset\subset \widehat{D}$) and satisfies

(a) Φ is continuous and $\forall x \in \partial D$ Φ_x is a peak function for D at x;

(b) $\forall x \in \partial D$ $\Phi_x(z_0) = 0$.

$\forall \varepsilon > 0$ set $U_\varepsilon = \bigcup_{x \in \partial D} P_\varepsilon(x)$, where $P_\varepsilon(x)$ is the polydisk of centre x and radius ε. The family $\{U_\varepsilon\}$ is a basis for the neighbourhoods of ∂D; hence there exists $\varepsilon > 0$ such that $U_\varepsilon \subset\subset D'$. Then the Cauchy estimates yield $\forall x \in \partial D$ $\forall z \in P_{\varepsilon/2}(x)$:

$$|1 - \Phi_x(z)| = |\Phi_x(x) - \Phi_x(z)| \leq \left\| \frac{\partial \Phi_x}{\partial z} \right\|_{P_{\varepsilon/2}(x)} \| z - x \| \leq$$

$$\leq \frac{2\sqrt{n}}{\varepsilon} \| \Phi \|_{\partial D \times U_\varepsilon} \| z - x \| = M \| z - x \|$$

where M is independent of z and x. Then if we put

$$c_2 = \min \left\{ -\frac{1}{2} \log M, \frac{1}{2} \log (\varepsilon/2) \right\},$$

we again have two possibilities:

(i) $z \in \mathbf{D} \cap \mathbf{U}_{\varepsilon/2}$. Setting

$$j_D(z) = \sup \{ |f(z)| \mid f \in \mathrm{Hol}(D, \Delta), f(z_0) = 0 \},$$

we have

$$c_D(z_0, z) = \omega(0, j_D(z)) \geq \frac{1}{2} \log \frac{1}{1 - j_D(z)}.$$

Choose $x \in \partial D$ so that $d(z, \partial D) = \| z - x \|$. Then $j_D(z) \geq |\Phi_x(z)|$ and

$$1 - j_D(z) \leq 1 - |\Phi_x(z)| \leq |1 - \Phi_x(z)| \leq M \| z - x \| = M d(z, \partial D).$$

Therefore

$$k_D(z_0, z) \geq c_D(z_0, z) \geq \frac{1}{2} \log \frac{1}{M} - \frac{1}{2} \log d(z, \partial D) \geq$$
$$\geq c_2 - \frac{1}{2} \log d(z, \partial D).$$

(ii) $z \in \mathbf{D} \setminus \mathbf{U}_{\varepsilon/2}$. Then $d(z, \partial D) \geq \varepsilon/2$ so that

$$k_D(z_0, z) \geq c_D(z_0, z) \geq 0 \geq \frac{1}{2} \log \frac{\varepsilon}{2} - \frac{1}{2} \log d(z, \partial D) \geq$$
$$\geq c_2 - \frac{1}{2} \log d(z, \partial D),$$ q.e.d.

A result equivalent to our Proposition 1.3 has been established by Vormoor [5] in the C^∞ case.

Theorem 0.1 follows at once from Propositions 1.2 and 1.3.

2. Now we examine the boundary behaviour of complex geodesics. Since in our proof of Theorem 0.2 there will be no difference between complex geodesics for the Carathéodory distance or for the Kobayashi distance, we shall not distinguish between the two classes, and we shall simply refer to complex geodesics.

Propositions 1.2 and 1.3 yield

COROLLARY 2.1. *Let* $D \subset\subset \mathbf{C}^n$ *be a strongly pseudoconvex* C^2 *domain and let* $f: \Delta \to D$ *be a complex geodesic. Then there exist* $k_1, k_2 > 0$ *(depending only on* D *and* $f(0)$*) such that*

$$\forall \zeta \in \Delta \qquad k_1 d(f(\zeta), \partial D) \leq 1 - |\zeta| \leq k_2 d(f(\zeta), \partial D).$$

We shall need the

LEMMA 2.2. *Let* B_R *be the Euclidean ball of radius* R *and centre* 0, *and let* $f : \Delta \to B_R$ *be a holomorphic map. Then*

$$\| f'(0) \| \leq \sqrt{2R}\, d(f(0), \partial B_R)^{1/2} .$$

which is an easy consequence of the Schwarz Lemma for B_R. This is all we need for the

Proof of Theorem 0.2. Let h be a C^2 defining function for D and $z_0 \in \partial D$. Up to a linear change of coordinates, in a neighbourhood of z_0 h takes the form

$$h(z) = -2\,\mathrm{Re}\,(z^1 - z_0^1) + 2\,\mathrm{Re}\left\{\frac{1}{2}\sum_{\mu,\nu=1}^{n} \frac{\partial^2 h}{\partial z^\mu\, \partial z^\nu}(z_0)\,(z^\mu - z_0^\mu)\,(z^\nu - z_0^\nu)\right\} + \\ + L_{h,z_0}(z - z_0, z - z_0) + o(\| z - z_0 \|^2),$$

where L_{h,z_0} — the Levi form at z_0 — is strictly positive definite.

The map $\Phi_{z_0} : \mathbf{C}^n \to \mathbf{C}^n$ defined by

$$w^1 = \Phi_{z_0}^1(z) = z^1 - z_0^1 + \frac{1}{2}\sum_{\mu,\nu=1}^{n} \frac{\partial^2 h}{\partial z^\mu\, \partial z^\nu}(z_0)\,(z^\mu - z_0^\mu)\,(z^\nu - z_0^\nu),$$

$$w^j = \Phi_{z_0}^j(z) = z^j - z_0^j \qquad (j = 2, \ldots n,),$$

is a biholomorphism between a neighbourhood of z_0 and a neighbourhood of 0 Since $\bar{D}$ is compact, we can assume that

(i) there exist neighbourhoods P, P′ of 0 such that $\forall z_0 \in \partial D$ Φ_{z_0} is a biholomorphism between $P_{z_0} = P + z_0$ and P′;

(ii) there exists $c_1 > 0$ such that

$$\forall z_0 \in \partial D \quad \forall z \in P_{z_0} \quad \| d(\Phi_{z_0})(z)^{-1} \| < c_1 ;$$

(iii) there exists $c_2 > 0$ such that

$$\forall z_0 \in \partial D \quad \forall x, y \in P_{z_0} \quad \| \Phi_{z_0}(x) - \Phi_{z_0}(y) \| \leq c_2 \| x - y \| .$$

In P_{z_0} h takes the form

$$h(w) = -w^1 - \overline{w^1} + H_{z_0}(w, w) + o(\| w \|^2),$$

where H_{z_0} is a strictly positive definite hermitian form.

We can assume also that

(iv) there exists $R > 0$ such that $\forall z_0 \in \partial D$ $1/R$ is less than all eigenvalues of H_{z_0};

(v) if $B = \{w \in \mathbf{C}^n \mid -w^1 - \overline{w^1} + \| w \|^2/R^2 > 0\}$, we have

$$\forall z_0 \in \partial D \quad \Phi_{z_0}(D \cap P_{z_0}) \subset B .$$

Let $U = \bigcup_{z_0 \in \partial D} P_{z_0}$. U is a neighbourhood of ∂D; therefore $D \setminus U$ is compact, and $\sup \{k_D(z, f(0)) \mid z \in D \setminus U\} < +\infty$. Therefore $V = f^{-1}(U \cap D) \supset\supset V_\varepsilon = \{\zeta \in \Delta \mid |\zeta| > 1 - \varepsilon\}$ for some $\varepsilon > 0$.

Again by compactness (and by Corollary 2.1) we can choose ε so that, setting $\forall \zeta_0 \in \Delta$ $\Delta(\zeta_0) = \{\zeta \in \Delta \mid |\zeta - \zeta_0| < 1 - |\zeta_0|\}$;

(vi) $\forall \zeta_0 \in V_\varepsilon$ $\exists!\, z_0 \in \partial D$ such that $d(f(\zeta_0), \partial D) = \| f(\zeta_0) - z_0 \|$;

(vii) $\forall \zeta_0 \in V_\varepsilon$ $f(\Delta(\zeta_0)) \subset P_{z_0}$, where $z_0 \in \partial D$ is given by (vi).

Indeed, if we choose ε so that (Corollary 2.1) $f(V_\varepsilon)$ is contained in a regular tubular neighbourhood of ∂D, we have (vi). For (vii), since f is bounded, $\lim_{r \to 1} f(re^{i\theta})$ exists for almost all the $e^{i\theta} \in \mathbf{S}^1$; let T be the set of all such points. If $e^{i\theta} \in T$, the points on the radius $r \mapsto re^{i\theta}$ satisfy (vii) for r close to 1; by continuity and Corollary 2.1, this happens in a set of the form $\{\rho e^{i\varphi} \mid \rho_\theta < \rho < 1 \mid |\varphi - \theta| < \delta_\theta\}$, for some $\rho_\theta, \delta_\theta > 0$. Now, the sets $\{e^{i\varphi} \mid |\varphi - \theta| < \delta_\theta\}$ $(e^{i\theta} \in T)$ form an open covering of $\mathbf{S}^1$; let $e^{i\theta_1}, \ldots, e^{i\theta_s} \in T$ such that

$$\mathbf{S}^1 \subset \bigcup_{j=1}^{s} \{e^{i\varphi} \mid |\varphi - \theta_j| < \delta_{\theta_j}\} .$$

Then the minimum between ε chosen as in (vi) and $1 - \max\{\rho_{\theta_j} \mid j = 1, \ldots, s\}$ is as we want.

Now, let $\zeta_0 \in V_\varepsilon$ and define $g : \Delta \to B$ by

$$g(\zeta) = \Phi_{z_0}(f(\zeta_0 + (1 - |\zeta_0|)\zeta)),$$

where $z_0 \in \partial D$ is given by (vi). By (v) and (vii), g is well defined. Then Lemma 2.2 yields

$$\| g'(0) \| \leq \sqrt{2R}\, d(\Phi_{z_0}(f(\zeta_0)), \partial B)^{1/2} .$$

Since $\Phi_{z_0}(z_0) = 0 \in \partial B$, by (iii) we have

$$\| g'(0) \| \leq \sqrt{2Rc_2}\, \| f(\zeta_0) - z_0 \|^{1/2} = \sqrt{2Rc_2}\, d(f(\zeta_0), \partial D)^{1/2} .$$

Being

$$g'(0) = d(\Phi_{z_0})(f(\zeta_0))((1 - |\zeta_0|) f'(\zeta_0)),$$

(ii) yields

$$\| g'(0) \| \geq \frac{1}{c_1}(1 - |\zeta_0|) \| f'(\zeta_0) \| .$$

Using the Corollary 2.1, we obtain

$$\forall \zeta_0 \in V_\varepsilon \quad \| f'(\zeta_0) \| \leq c_1 \sqrt{\frac{2 \, R c_2}{k_1}} \, \frac{1}{\sqrt{1 - |\zeta_0|}}$$

Since $\Delta \setminus V_\varepsilon$ is compact, we conclude that there exists $M > 0$ such that

$$\forall \zeta \in \Delta \quad \| f'(\zeta) \| \leq \frac{M}{\sqrt{1 - |\zeta|}} .$$

By the Hardy-Littlewood Theorem (see Duren [1], Theorem 5.1), this is equivalent to our assertion, q.e.d.

References

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