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On a fixed-point theorem of Cellina

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Analisi funzionale. — *On a fixed-point theorem of Cellina.* Nota di STEPHEN SIMONS, presentata (*) dal Corrisp. R. CONTI.

RIASSUNTO. — In questa nota mostriamo come un teorema di esistenza per funzionali lineari porti un nuovo teorema di punto fisso che generalizza un teorema di punto fisso di Cellina.

§ 1. Let X be a (not necessarily locally convex) real Hausdorff topological vector space with topological dual X^*. Let K be a non-empty compact, convex subset of X. For all $f \in X^*$, let

$$Mf = \{x : x \in K, f(x) = \max f(K)\} .$$

We shall use the following existence theorem.

THEOREM 1. *Let* $T : K \to 2^X$ *satisfy* (1.1) *and* (1.2) *below.*

(1.1) *For all* $x \in K$, Tx *is nonempty and convex.*

(1.2) *For all* $f \in X^*$, $T^{-1}f$ *is open in* X.

Then

$$\exists x \in K \quad \text{and} \quad f \in X^* \quad \text{such that} \quad f \in Tx \text{ and } x \in Mf .$$

Comments on Theorem 1. A number of authors have discussed Theorem 1 or generalizations thereof. It appears, essentially as stated above, in [5, Theorem 4.5], with a proof based indirectly on the KKM theorem. In [4, Théorème 2.1, p. 329], Granas and Liu give a related result for acyclic maps that depends on a Lefschetz fixed-point theorem for multifunctions. In [3, Theorem 8, p. 526], Fan gives a generalization to concave functions, with a weakening of the compactness condition on K; Fan uses a result of Allen that, in turn, depends on the KKM theorem. In [6, Theorem 0.1], Simons gives a generalization to quasiconcave functions, with a proof based on Brouwer's fixed-point theorem for a simplex. In [1, Theorem 1], Bellenger gives a result that simultaneously generalizes [3, Theorem 8] and [6, Theorem 0.1].

(*) Nella seduta dell'11 gennaio 1986.

§ 2. In this section, we shall discuss an extension of a fixed-point theorem of Cellina and also a new fixed-point theorem. We shall suppose that X, X^* and K are as in § 1 and, in addition, X is locally convex. We suppose that $P : K \to 2^X$ and

(2.1) for all $x \in K$, Px is non-empty, closed and convex.

THEOREM 2. (*a*) *Suppose that,*

(2.2) for all $f \in X^*$, $\{x : x \in K, f(x) \geq \inf f(Px)\}$ is closed and $\supset Mf$.

Then

(2.3) $$\exists x \in K \text{ such that } Px \ni x .$$

(*b*) *Suppose that,*

for all $f \in X^*$, $\{x : x \in K, f(x) \leq \sup f(Px)\}$ *is closed and* $\supset Mf$.

Then (2.3) is true.

Proof. (*a*) *For all* $x \in K$, *let*

$$Tx = \{f : f \in X^*, f(x) < \inf f(Px)\} .$$

If (2.3) is false then, from the separation theorem for locally convex spaces, for all $x \in K$, $Tx \neq \varnothing$. It follows easily that (1.1) is satisfied. If $f \in X^*$ then

$$T^{-1}f = \{x : x \in K, f(x) < \inf f(Px)\} .$$

From (2.2), $T^{-1}f$ is open in K. Thus (1.2) is satisfied. From Theorem 1,

$$\exists x \in K \text{ and } f \in X^* \text{ such that } f(x) < \inf f(Px) \text{ and } x \in Mf .$$

which contradicts (2.2). This contradiction completes the proof of (*a*).

(*b*) The proof of this is similar to the proof of (*a*), except with

$$Tx = \{f : f \in X^*, f(x) > \sup f(Px)\} .$$

§ 3. In [2, Theorem 3, p. 31], Cellina gives the following fixed-point theorem that generalizes both a result of Glebov on partially closed maps and a result of Fan on upper demicontinuous maps.

THEOREM 3. *Suppose that* P *satisfies* (3.1), (3.2) *and* (3.3) *below.*

(3.1) For all $x \in K$, Px is non-empty, compact and convex.

$$(3.2) \quad \begin{cases} \text{If } f \in X^* \setminus \{0\},\ x \in K \text{ and } f(x) < \inf f(Px) \text{ then} \\ \exists \text{ a neighbourhood V of } x \text{ such that } \sup f(V) < \inf f(P(V)) . \end{cases}$$

$$(3.3) \qquad \text{For all } x \in \delta K , \quad x + R^+ (K - x) \text{ meets } Px .$$

Then (2.3) is true.

Comments on Theorems 2 and 3. Clearly, (3.1), (3.2) and (3.3) imply (2.1) and (2.2), so Theorem 3 can be deduced from Theorem 2 (*a*). By virtue of (3.3), Theorem 3 could be described as an "inward" fixed-point theorem. Correspondingly, Theorem 2 (*a*) could be described as a "same side" fixed-point theorem and Theorem 2 (*b*) as an "opposite side" fixed-point theorem.

Finally, we note the following result, typical of the kind of result that can be deduced from Theorem 2 ((*a*) or (*b*) !), but not from Theorem 3.

COROLLARY 4. *Suppose that* P *is upper semicontinuous and satisfies* (2.1) *and for all* $f \in X^*$ *and* $x \in Mf$, $f(Px) = \{\max f(K)\}$.

Then (2.3) is true.

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