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Traces of functions in Bergman weighted spaces on tubular domains

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Geometria. — *Traces of functions in Bergman weighted spaces on tubular domains.* Nota di UMBERTO SAMPIERI (*), presentata (**) dal Corrisp. E. VESENTINI.

RIASSUNTO. — Si dà una caratterizzazione completa per tracce di funzioni olomorfe a quadrato sommabile per particolari misure su domini tubolari.

INTRODUCTION

Let Ω be an open, convex, sharp, homogeneous cone in $\mathbf{R}^n$ and let $D = \{z \in \mathbf{C}^n : \operatorname{Im}(z) \in \Omega\}$ be the associated tubular domain.

In a forthcoming paper ([2]) we proved, in the more general setting of Siegel domains, that, chosen a point e in Ω, it is possible to introduce a Lie group structure on D such that ie is the identity element, left translations are holomorphic automorphisms and $S(e) = \{x + \mathrm{ie},\ x \in \mathbf{R}^n\}$ is a Lie subgroup homomorphic to $\mathbf{R}^n$.

We also introduced a family of Bergman weighted spaces F_h, h in $(0, \infty)$ by setting:

(1) $$F_h = \{F : D \to \mathbf{C},\ \text{holomorphic and such that:}$$

(2) $$\| F \|_h^2 = \int_D | F(z) |^2 \Phi_\Omega (\operatorname{Im}(z))^{1-h} \, dm(z)$$

where Φ_Ω is the characteristic function for Ω.

In this note we give a complete characterization of traces of functions in F_h on $S(e)$ by proving the following:

THEOREM: *Let g be an analytic function on* $\mathbf{R}^n$. *Then there exists a function* F *in* F_h *such that* $F(x + \mathrm{ie}) = g(x)$ *for each* x *in* $\mathbf{R}^n$ *if and only if:*

i) $g \in L^2(\mathbf{R}^n, dm)$

ii) $\widehat{g}(x) = 0$ *for each* x *in* $\mathrm{R}^n \setminus \Omega^*$

iii) $\widehat{g}(x)\, e^{2\pi \langle x,e\rangle}\, \Phi_{\Omega^*}(x)^{h/2} \in L^2(\Omega^*, dm)$.

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Moreover for each z in D we have:

$$\text{(3)} \qquad F(z) = \int_{\Omega^*} e^{2\pi i \langle z,\zeta\rangle} e^{2\pi \langle \zeta, e\rangle} \widehat{g}(\zeta)\, dm(\zeta).$$

We observe that for $h = 1$, i.e. when F_h is the ordinary Bergman space, condition iii) may be rephrased as:

iii') $\widehat{g}(x)\, e^{2\pi \langle x,e\rangle} \in L^2(\Omega^*, d\mu)$

where $d\mu$ is the Haar measure for Ω^*.

LEMMATA. *We recall that, denoting by* $Aut(\Omega)$ *the subgroup of* $GL(n, \mathbf{R})$ *of all linear transformations preserving* Ω, *we have:*

$$\text{(4)} \qquad \Phi_\Omega(Ay) \det(A) = \Phi_\Omega(y)$$

for each y *in* Ω.

The following lemmata hold:

LEMMA 1. *There exists a constant* $0 \leq c(\Omega) < 1$ *such that for each* h *in* $(C(\Omega), \infty)$ *there exists* $1(h)$ *in* $\mathbf{R}^+$ *such that:*

$$\text{(5)} \qquad \int_\Omega e^{-\langle x,y\rangle} \Phi_\Omega(x)^{1-h}\, dm(x) = 1(h)\, \Phi_{\Omega^*}(y)^h$$

for each y *in* Ω^* *where* Ω^* *is the dual cone* (*see for proof* [1], pp. 22).

LEMMA 2. *Let* h *be in* $(0, \infty)$. *Then the* C^∞ *function* $f_h : \Omega^* \to \mathbf{R}^+$:

$$\text{(6)} \qquad f_h(x) = e^{-2\pi\langle e,x\rangle} \Phi_{\Omega^*}(x)^{-h}$$

is bounded.

Proof. Chosen a point x in Ω^* let us consider the C^∞ function $s : \mathbf{R}^+ \to \mathbf{R}^+$,

$$\text{(7)} \qquad s(t) = f_h(t\, x).$$

By computing its first derivative we deduce that it takes its maximum for $t = (nh)/2\pi \langle e, x\rangle$ and consequently:

$$\text{(8)} \qquad s(t) \leq e^{-nh} (nh/2\pi \langle e, x\rangle)^{nh}\, \Phi_{\Omega^*}(x)^{-h}$$

for each t in $\mathbf{R}^+$. Hence to prove Lemma 2 we only need to show that:

$$k(x) = (\langle e, x\rangle)^{-nh} \Phi_{\Omega^*}(x)^{-h} \tag{9}$$

is bounded on $B = \Omega^* \cap \{x \text{ in } \mathbf{R}^n : \| x \| = 1\}$.

This is an immediate consequence of the fact that, since Φ_{Ω^*} is unbounded on the boundary of Ω^* and there exists a constant a such that $\langle e, x\rangle \geq a \| x \|$ for each x in $\overline{\Omega^*}$, the function k can be extended to a continuous function on the compact set $\overline{B}$.

The affine group of transformations of D onto itself is given by:

$$\mathrm{Aff}(D) = \{z \to Az + b\,,\quad A \in \mathrm{Aut}(\Omega)\,,\quad b \in \mathbf{R}^n\}\,. \tag{10}$$

We shall denote by $\mathrm{Aff}_0(D)$ the identity connected component of $\mathrm{Aff}(D)$.

Let us introduce the closed subspace of $L^2(\mathbf{R}^n)$:

$$H^2(\Omega) = \{f \text{ in } L^2(\mathbf{R}^n) : f(x) = 0 \text{ for each } x \text{ in } \mathbf{R}^n \setminus \Omega^*\} \tag{11}$$

and the unitary representation $R_0 : \mathrm{Aff}_0(D) \to L(L^2(\mathbf{R}^n))$:

$$(R_0(A, b)f)(x) = \sqrt{\det(A)}\, f(Ax + b)\,. \tag{12}$$

LEMMA 3. $H^2(\Omega)$ *is an invariant subspace of* R_0.

A straightforward calculation shows in fact that:

$$\widehat{R_0(A, b) f}(x) = (\det(A))^{-1/2}\, e^{2\pi i\langle {}^tA^{-1}x, b\rangle}\, \hat{f}({}^tA^{-1} x) \tag{13}$$

and so Lemma 3 is proved.

It is well known that the unitary representations $R_h : \mathrm{Aff}_0(D) \to \mathscr{L}(F_h)$, h in $(c(\Omega), \infty)$:

$$(R_h(A, b) F)(z) = (\det(A))^{h+1/2}\, F(Az + b) \tag{14}$$

are irreducible.

LEMMA 4. *The linear operators* $L_h : H^2(\Omega) \to F_h$:

$$L_h(f)(z) = \int_{\Omega^*} e^{2\pi i\langle z, x\rangle}\, \Phi_{\Omega^*}(x)^{-h/2} \hat{f}(x)\, dm(x) \tag{15}$$

are such that:

i) $\| Lf \|_h = \| f \|_{L^2(\mathbf{R}^n)}$

ii) $L_h \circ R_0 = R_h \circ L_h$.

Proof. Since:

$$\mathrm{L}_h(f)(x+iy) = e^{2\pi i\langle x,s\rangle}\left(e^{-2\pi\langle y,s\rangle}\,\Phi_{\Omega^*}(s)^{-h/2}\widehat{f}(s)\right)\mathrm{d}m(s) \tag{16}$$

by Plancherel theorem we get:

$$\int_{\mathbf{R}^n} |\,\mathbf{L}_h(f)(x+iy)\,|^2\,\mathrm{d}m(x) = \int_{\Omega^*} e^{-4\pi\langle y,s\rangle}\,\Phi_{\Omega^*}(s)^{-h}\,|\widehat{f}(s)|^2\,\mathrm{d}m(s)\,. \tag{17}$$

Therefore by Fubini theorem and remembering (5) we get, up to a multiplicative constant, i). Part ii) is an easy consequence of (13) and (4). In fact, given (A, b) in $\mathrm{Aff}_0(\mathrm{D})$, we have:

$$\mathrm{L}_h(\mathrm{R}_0(\mathrm{A}, b)f)(z) = \int_{\Omega^*} e^{2\pi i\langle z,s\rangle}\,\Phi_{\Omega^*}(s)^{-h/2}\,\mathrm{R}_0(\widehat{\mathrm{A},b)f}(s)\,\mathrm{d}m(s) = \tag{18}$$

$$= \int_{\Omega^*} e^{2\pi i\langle z,s\rangle}\,\Phi_{\Omega^*}(s)^{-h/2}\,(\det(\mathrm{A}))^{-1/2}\,e^{2\pi i\langle {}^t\mathrm{A}^{-1}s,b\rangle}\widehat{f}({}^t\mathrm{A}^{-1}s)\,\mathrm{d}m(s)\,.$$

Changing variable we get identity ii).

LEMMA 5. *$\mathrm{L}_h : \mathrm{H}^2(\Omega) \to \mathrm{F}_h$ is a surjective isometry.*

Proof. Recalling the previous Lemma by i) $\mathrm{Im}(\mathrm{L}_h)$ is a closed subspace of F_h, which is invariant under R_h by ii). We conclude remembering that R_h is irreducible. ■

Proof of the Theorem. Let g be an analytic function on $\mathbf{R}^n$ and suppose that there exists a (necessarily unique) F in F_h such that:

$$g(x) = \mathrm{F}(x+ie) \qquad \text{for each } x \text{ in } \mathbf{R}^n\,. \tag{19}$$

Consequently, by Lemma 5, there exists a unique f in $\mathrm{H}^2(\Omega)$ such that $\mathrm{F} = \mathrm{L}_h(f)$ and so:

$$g(x) = \int_{\Omega^*} e^{2\pi i\langle x,s\rangle}\left(e^{-2\pi\langle e,s\rangle}\,\Phi_{\Omega^*}(s)^{-h/2}\widehat{f}(s)\right)\mathrm{d}m(s)\,. \tag{20}$$

Therefore, in consequence of Lemma 2, g is in $\mathrm{L}^2(\mathbf{R}^n)$ and:

$$\widehat{g}(x) = e^{-2\pi\langle e,x\rangle}\,\Phi_{\Omega^*}(x)^{-h/2}\widehat{f}(x) \tag{21}$$

and that proves that g satisfies also conditions ii) and iii).

Moreover (3) is an immediate consequence of the identity $F = L_h(f)$ and of (21).

Conversely suppose that g satisfies conditions i), ii), iii). Then:

$$| e^{2\pi i\langle z,s\rangle} e^{2\pi\langle e,s\rangle} \hat{g}(s) | = e^{-2\pi\langle y,s\rangle} \Phi_{\Omega*}(s)^{-h/2} | \Phi_{\Omega*}(s)^{h/2} e^{2\pi\langle e,s\rangle} \hat{g}(s) | . \tag{22}$$

Therefore, by Lemma 1 and condition iii), we can deduce that the integral (3) converges absolutely and defines an holomorphic function on D. It is then easy to verify that F belongs to F_h and that $g(x) = F(x + ie)$ for each x in $\mathbf{R}^n$. ■

References

[1] Gindikin (1964) – *Analysis in homogeneous domains.* « Russian Math. Surveys », *19*, 3-92.

[2] Sampieri (to appear) – *Lie group structures and reproducing kernels on homogeneous Siegel domains,* to appear, in « Annali di Matematica Pura e Applicata ».