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Automorphism groups of the classical domains. II

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Geometria. — *Automorphism groups of the classical domains.* II. Nota di MARCO ABATE, presentata (*) dal Corrisp. E. VESENTINI.

RIASSUNTO. — In questa Nota vengono determinati, con un nuovo metodo elementare, i gruppi di automorfismi del terzo e del quarto dominio classico. Gli strumenti utilizzati sono quelli già introdotti nella precedente Nota, ove erano stati usati per determinare il gruppo degli automorfismi del primo dominio classico.

§ 2. We begin by considering the domains of type III in É. Cartan's realisation, that is the Siegel disks

$$D = \{Z \in \mathbf{S}_n \mid \| Z \| < 1\}$$

where $\mathbf{S}_n$ is the space of $n \times n$ symmetric complex matrices.

For every $Z \in \mathbf{S}_n$, the non-negative square roots $\lambda_1, \ldots, \lambda_n$ of eigenvalues of ZZ^* are called, as in Note I, the *modules* of Z. It is well known (see e.g. [6]) that for every $Z \in \mathbf{S}_n$ there exists a unitary matrix $U \in \mathbf{U}(n)$ so that

$$U Z {}^tU = \operatorname{diag}(\lambda_1, \ldots, \lambda_n) .$$

The mappings $Z \mapsto UZ {}^tU$ with $U \in \mathbf{U}(n)$ are elements of the isotropy group at the origin K, which will be called *unitary automorphisms*. As in Note I we have

PROPOSITION 3. *Let* $L \in K$. *Then* L *is linear and preserves the modules.*

Also the analogues of Lemma 1 and Corollary 1 hold in this case. Moreover, the Lemma 2 takes now the following form:

LEMMA 3. $\forall Z \in \mathbf{S}_n$ *with rank* $1 \ \exists u \in \mathbf{C}^n$ *with* $|u| = 1$ *and* $\alpha > 0$ *so that*

$$Z = \alpha u \otimes u .$$

So we have

(*) Nella seduta del 22 novembre 1985.

PROPOSITION 4. *Let* $L \in K$. *Then* $\exists\, U \in \mathbf{U}(n)$ *such that for every diagonal matrix* Z *we have* $L(Z) = UZ\,{}^tU$.

Proof. Let E_{ij} be the symmetric elementary matrix (i.e. the matrix with 1 in the (i,j) and (j,i) entries and 0 elsewhere), and let $A_{ij} = L(E_{ij})$ for $i, j = 1, \dots, n$. Then, as in Proposition 2, we find $u^j \in \mathbf{C}^n$ with $|u^j| = 1$ such that $A_{jj} = u^j \otimes u^j$ $\forall j = 1, \dots, n$. Moreover, $Tr(E_{hh} E^*_{kk}) = 0$ $\forall h \neq k$ implies $(u^h, u^k)^2 = Tr(A_{hh} A^*_{kk}) = 0$, so that $\{u^j\}$ is an orthonormal basis of $\mathbf{C}^n$. Therefore $U = (u^\mu_\nu)$ is the matrix we are looking for, q.e.d.

Now we can prove

THEOREM 2. *For each* $L \in K$ *there exists* $U \in \mathbf{U}(n)$ *so that*

$$L(Z) = U\,Z\,{}^tU\,.$$

Proof. By Proposition 4, we can suppose that

$$A_{jj} = E_{jj} \qquad \forall j = 1, \dots, n, \tag{1}$$

so that, by Corollary 1,

$$A_{ij,hh} = 0 \qquad \forall 1 \leq i \neq j \leq n \qquad \forall h = 1, \dots, n. \tag{1'}$$

Let $u \in \mathbf{C}^n$ with $|u| = 1$. By Lemma 3 and Proposition 3 there exists $v \in \mathbf{C}^n$ with $|v| = 1$ so that $L(u \otimes u) = v \otimes v$. If we write this componentwise, (1) and (1′) yield

$$\begin{cases} u_h^2 = v_h^2 & (h = 1, \dots, n), \\ \sum\limits_{\mu \leq \nu} u_\mu u_\nu\, A_{\mu\nu,hk} = v_h v_k & (1 \leq h < k \leq n). \end{cases}$$

Thus $\exists\, c_h \in \{0, 1\}$ so that

$$v_h = (-1)^{c_h} u_h$$

and we have

$$\sum_{\mu \leq \nu} u_\mu u_\nu\, A_{\mu\nu,hk} = (-1)^{c_h} (-1)^{c_k} u_h\, u_k\,.$$

By continuity, $(-1)^{c_h}(-1)^{c_k}$ is independent from u; hence

$$\sum_{\mu \leq \nu} u_\mu u_\nu\, A_{\mu\nu,hk} = (-1)^{c_h} (-1)^{c_k} \sum_{\mu \leq \nu} u_\mu u_\nu\, E_{\mu\nu,hk} =$$

$$= \sum_{\mu \leq \nu} u_\mu u_\nu\, (-1)^{c_\mu} (-1)^{c_\nu}\, E_{\mu\nu,hk}$$

and therefore

$$A_{\mu\nu} = (-1)^{c_\mu} (-1)^{c_\nu} E_{\mu\nu} .$$

Then L is given by

$$L(Z) = \operatorname{diag}((-1)^{c_1}, \ldots, (-1)^{c_n}) \, Z \operatorname{diag}((-1)^{c_1}, \ldots, (-1)^{c_n})$$

and the assertion follows, q.e.d.

For the standard determination of Aut (D) from K see [5] or [6].

§ 3. Now we consider the domains of type IV in É. Cartan's realisation, that is the Lie balls

$$D = \{z \in \mathbf{C}^n \mid (z, z) + \sqrt{(z, z)^2 - |(z, \bar{z})|^2} < 1\};$$

D is the unit ball for the norm

$$\| z \| = [(z, z) + \sqrt{(z, z)^2 - |(z, \bar{z})|^2}]^{\frac{1}{2}}$$

(for a direct proof that $\| \; \|$ is really a norm see [3]).

For $z \in \mathbf{C}^n$ the *modules* of z are the non-negative real numbers

$$\lambda_1 = \lambda_1(z) = [(z, z) + \sqrt{(z, z)^2 - |(z, \bar{z})|^2}]^{\frac{1}{2}} ,$$

$$\lambda_2 = \lambda_2(z) = [(z, z) - \sqrt{(z, z)^2 - |(z, \bar{z})|^2}]^{\frac{1}{2}} .$$

In this case, the rôle of diagonal matrices is played by the first two coordinates. In fact, we have

PROPOSITION 5. *$\forall z \in \mathbf{C}^n \; \exists \; \theta \in \mathbf{R}$ and $V \in \mathbf{O}(n)$ so that*

$$e^{i\theta} V z = \frac{1}{2} \begin{pmatrix} \lambda_1 + \lambda_2 \\ i(\lambda_1 - \lambda_2) \\ 0 \end{pmatrix}$$

where $\lambda_1 \geq \lambda_2 \geq 0$ are the modules of z.

Proof. If we identify $\mathbf{C}^n$ with $\mathbf{M}_{n,2}(\mathbf{R})$ (real matrices $n \times 2$) through the mapping $x + iy \mapsto (x, y)$ $(x, y \in \mathbf{R}^n)$ and $\{\zeta \in \mathbf{C} \mid |\zeta| = 1\}$ with $\mathbf{SO}(2)$ in the standard way, we see at once that there exist $\theta \in \mathbf{R}$ and $V \in \mathbf{O}(n)$ so that

$$e^{i\theta} V z = {}^t(\mu_1, i\mu_2, 0, \ldots, 0)$$

with $\mu_1 \geq \mu_2 \geq 0$. Then, setting

$$\lambda_1 = \mu_1 + \mu_2 \qquad \lambda_2 = \mu_1 - \mu_2$$

a straightforward computation shows that λ_1 and λ_2 are the modules of z, q.e.d.

The mappings $z \mapsto e^{i\theta} Vz$ are elements of the isotropy group at the origin $K = K(n)$ of D, which will be called *orthogonal automorphisms*.

Now we have

PROPOSITION 6. *Let* $L \in K$. *Then* L *is linear and preserves the modules.*

Proof. Exactly as in Proposition 1, using the polynomials

$$p_z(\lambda) = \lambda^2 - 2(z, z)\lambda + |(z, \bar{z})|^2 = (\lambda - \lambda_1(z)^2)(\lambda - \lambda_2(z)^2)$$

q.e.d.

PROPOSITION 7. *Let* $L \in K$. *Then* $\exists\, \theta \in \mathbf{R}$ *and* $V \in \mathbf{O}(n)$ *so that* $\forall z_1$, $z_2 \in \mathbf{C}$

$$L\begin{pmatrix} z_1 \\ z_2 \\ 0 \end{pmatrix} = e^{i\theta} V \begin{pmatrix} z_1 \\ z_2 \\ 0 \end{pmatrix}.$$

Proof. Let $e_1 = \frac{1}{2}{}^t(1, i, 0, \ldots, 0)$, $e_2 = \frac{1}{2}{}^t(1, -i, 0, \ldots, 0)$ and $a_j = L(e_j)$ $(j = 1, 2)$.

By Proposition 5 and 6, there exist $\phi, \psi \in \mathbf{R}$ and $u^1, u^2, w^1, w^2 \in \mathbf{R}^n$ with $|u^1| = |u^2| = |w^1| = |w^2| = 1$, $(u^1, u^2) = (w^1, w^2) = 0$ so that

$$a_1 = \tfrac{1}{2} e^{i\phi}(u^1 + iu^2) \qquad a_2 = \tfrac{1}{2} e^{i\psi}(w^1 - iw^2).$$

By Proposition 6, $L \in \mathbf{U}(n)$; therefore $(a_1, a_2) = 0$. Using this and the conservation of the modules of $e_1 + e_2$, we find

$$(u^1, w^1) = (u^2, w^2) \qquad (u^1, w^2) = -(u^2, w^1)$$

$$(u^1, w^j)^2 + (u^2, w^j)^2 = 1 \qquad (j = 1, 2).$$

Then $\exists\, \eta \in \mathbf{R}$ so that

$$a_2 = \tfrac{1}{2} e^{i\eta}(u^1 - iu^2).$$

Now, if we set

$$v^1 = \cos\left(\frac{\phi - \eta}{2}\right) u^1 - \sin\left(\frac{\phi - \eta}{2}\right) u^2 \qquad \theta = \frac{\phi + \eta}{2}$$

$$v^2 = \sin\left(\frac{\phi - \eta}{2}\right) u^1 + \cos\left(\frac{\phi - \eta}{2}\right) u^2$$

we have

$$a_1 = \tfrac{1}{2} e^{i\theta} (v^1 + iv^2) , \qquad a_2 = \tfrac{1}{2} e^{i\theta} (v^1 - iv^2) ,$$

and the assertion follows, q.e.d.

THEOREM 3. *For each* $L \in K$ *there exist* $\theta \in \mathbf{R}$ *and* $V \in \mathbf{O}(n)$ *so that*

$$L(z) = e^{i\theta} Vz .$$

Proof. By induction on n. For $n = 1 , 2$ the assertion is obvious; so, let $n > 2$. Then, by Proposition 7, we can assume L of the form

$$L = \begin{pmatrix} I_2 & 0 \\ 0 & L' \end{pmatrix}$$

with $L' \in K(n-2)$. By induction, up to an orthogonal automorphism, we can suppose that $L' = e^{i\theta} I_{n-2}$, for some $\theta \in \mathbf{R}$. But the conservation of the modules of $z_0 = {}^t(1, 1, 1, 0, \ldots, 0)$ implies that $e^{i\theta} = \pm 1$, and the proof is complete, q.e.d.

Finally, for the standard determination of Aut (D) from K, see [3].

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