MARCO ABATE

Automorphism groups of the classical domains. II

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1985_8_79_5_127_0>
Geometria. — Automorphism groups of the classical domains. II.
Nota di MARCO ABATE, presentata (*) dal Corrisp. E. VESENTINI.

RIASSUNTO. — In questa Nota vengono determinati, con un nuovo metodo elementare, i gruppi di automorfismi del terzo e del quarto dominio classico. Gli strumenti utilizzati sono quelli già introdotti nella precedente Nota, ove erano stati usati per determinare il gruppo degli automorfismi del primo dominio classico.

§ 2. We begin by considering the domains of type III in É. Cartan's realisation, that is the Siegel disks

\[ D = \{ Z \in S_n \mid \| Z \| < 1 \} \]

where \( S_n \) is the space of \( n \times n \) symmetric complex matrices.

For every \( Z \in S_n \), the non-negative square roots \( \lambda_1, \ldots, \lambda_n \) of eigenvalues of \( ZZ^* \) are called, as in Note I, the modules of \( Z \). It is well known (see e.g. [6]) that for every \( Z \in S_n \) there exists a unitary matrix \( U \in U(n) \) so that

\[ UZ^*U = \text{diag} (\lambda_1, \ldots, \lambda_n). \]

The mappings \( Z \mapsto UZ^*U \) with \( U \in U(n) \) are elements of the isotropy group at the origin \( K \), which will be called unitary automorphisms. As in Note I we have

**Proposition 3.** Let \( L \in K \). Then \( L \) is linear and preserves the modules.

Also the analogues of Lemma 1 and Corollary 1 hold in this case. Moreover, the Lemma 2 takes now the following form:

**Lemma 3.** \( \forall Z \in S_n \) with rank 1 \( \exists u \in \mathbb{C}^n \) with \( |u| = 1 \) and \( \alpha > 0 \) so that

\[ Z = \alpha u \otimes u. \]

So we have

Proposition 4. Let $L \in K$. Then $\exists U \in U(n)$ such that for every diagonal matrix $Z$ we have $L(Z) = UZU$.

Proof. Let $E_{ij}$ be the symmetric elementary matrix (i.e. the matrix with 1 in the $(i,j)$ and $(j,i)$ entries and 0 elsewhere), and let $A_{ij} = L(E_{ij})$ for $i,j = 1, \ldots, n$. Then, as in Proposition 2, we find $w^i \in C^n$ with $|w^i| = 1$ such that $A_{jj} = w^i \otimes w^i$ for every diagonal matrix $Z$ we have $L(Z) = UZU$.

Now we can prove

Theorem 2. For each $L \in K$ there exists $U \in U(n)$ so that $L(Z) = UZU$.

Proof. By Proposition 4, we can suppose that

$A_{jj} = E_{jj} \forall j = 1, \ldots, n$,

so that, by Corollary 1,

$A_{ij,hh} = 0 \forall 1 \leq i \neq j \leq n \forall h = 1, \ldots, n$.

Let $u \in C^n$ with $|u| = 1$. By Lemma 3 and Proposition 3 there exists $v \in C^n$ with $|v| = 1$ so that $L(u \otimes u) = v \otimes v$. If we write this componentwise, (1) and (1') yield

$$
\begin{cases}
  v_h^2 = v_h^2 \\
  \sum_{\mu \leq v} u_{\mu} u_v A_{\mu \nu, hh} = v_{\mu} v_k 
\end{cases}
\quad (1 \leq h < k \leq n).
$$

Thus $\exists c_h \in \{0, 1\}$ so that

$$
v_h := (-1)^h u_h
$$

and we have

$$
\sum_{\mu \leq v} u_{\mu} u_v A_{\mu \nu, hh} = (-1)^h (-1)^h u_h u_k.
$$

By continuity, $(-1)^h (-1)^h$ is independent from $u$; hence

$$
\sum_{\mu \leq v} u_{\mu} u_v A_{\mu \nu, hh} = (-1)^h (-1)^h \sum_{\mu \leq v} u_{\mu} u_v E_{\mu \nu, hh} = \sum_{\mu \leq v} u_{\mu} u_v (-1)^\mu (-1)^\mu E_{\mu \nu, hh}
$$
and therefore
\[ A_{\mu\nu} = (-1)^{\mu} (-1)^\nu E_{\mu\nu}. \]

Then \( L \) is given by
\[ L(Z) = \text{diag } ((-1)^{\mu_1}, \ldots, (-1)^{\mu_n}) Z \text{ diag } ((-1)^{\nu_1}, \ldots, (-1)^{\nu_n}) \]
and the assertion follows, q.e.d.

For the standard determination of \( \text{Aut}(D) \) from \( K \) see \([5]\) or \([6]\).

§ 3. Now we consider the domains of type IV in É. Cartan's realisation, that is the Lie balls
\[ D = \{ z \in \mathbb{C}^n \, | \, (z, z) + \| (z, z) \|^2 - |(z, \bar{z})|^2 < 1 \}; \]
\( D \) is the unit ball for the norm
\[ \| z \| = [(z, z) + \| (z, z) \|^2 - |(z, \bar{z})|^2]^\frac{1}{2} \]
(for a direct proof that \( \| \| \) is really a norm see \([3]\)).

For \( z \in \mathbb{C}^n \) the modules of \( z \) are the non-negative real numbers
\[ \lambda_1 = \lambda_1(z) = [(z, z) + \| (z, z) \|^2 - |(z, \bar{z})|^2]^\frac{1}{2}, \]
\[ \lambda_2 = \lambda_2(z) = [(z, z) - \| (z, z) \|^2 - |(z, \bar{z})|^2]^\frac{1}{2}. \]

In this case, the rôle of diagonal matrices is played by the first two coordinates. In fact, we have

**Proposition 5.** \( \forall z \in \mathbb{C}^n \exists \theta \in \mathbb{R} \) and \( V \in O(n) \) so that
\[ e^{i\theta} Vz = \frac{1}{2} \begin{pmatrix} \lambda_1 + \lambda_2 & i(\lambda_1 - \lambda_2) \\ i(\lambda_1 - \lambda_2) & 0 \end{pmatrix} \]
where \( \lambda_1 \geq \lambda_2 \geq 0 \) are the modules of \( z \).

**Proof.** If we identify \( \mathbb{C}^n \) with \( M_{n,2}(\mathbb{R}) \) (real matrices \( n \times 2 \)) through the mapping \( x + iy \mapsto (x, y) \) \( (x, y \in \mathbb{R}^n) \) and \( \{ z \in \mathbb{C} \, | \, |z| = 1 \} \) with \( SO(2) \) in the standard way, we see at once that there exist \( \theta \in \mathbb{R} \) and \( V \in O(n) \) so that
\[ e^{i\theta} Vz = \begin{pmatrix} \mu_1, i\mu_2, 0, \ldots, 0 \end{pmatrix} \]
with $\mu_1 \geq \mu_2 \geq 0$. Then, setting

$$\lambda_1 = \mu_1 + \mu_2 \quad \lambda_2 = \mu_1 - \mu_2$$

a straightforward computation shows that $\lambda_1$ and $\lambda_2$ are the modules of $z$, q.e.d.

The mappings $z \mapsto e^{i\theta} V z$ are elements of the isotropy group at the origin $K = K(n)$ of $D$, which will be called orthogonal automorphisms.

Now we have

PROPOSITION 6. Let $L \in K$. Then $L$ is linear and preserves the modules.

Proof. Exactly as in Proposition 1, using the polynomials

$$p_2(\lambda) = \lambda^2 - 2(z, z) \lambda + |(z, \bar{z})|^2 = (\lambda - \lambda_1(z))^2 (\lambda - \lambda_2(z))^2$$

q.e.d.

PROPOSITION 7. Let $L \in K$. Then $\exists \theta \in \mathbb{R}$ and $V \in O(n)$ so that $\forall z_1, z_2 \in \mathbb{C}$

$$L \begin{pmatrix} z_1 \\ z_2 \\ 0 \end{pmatrix} = e^{i\theta} V \begin{pmatrix} z_1 \\ z_2 \\ 0 \end{pmatrix}.$$ 

Proof. Let $e_1 = \frac{1}{2}(1, i, 0, \ldots, 0), e_2 = \frac{1}{2}(1, -i, 0, \ldots, 0)$ and $a_j = L(e_j) (j = 1, 2)$.

By Proposition 5 and 6, there exist $\phi, \psi \in \mathbb{R}$ and $u^1, u^2, v^1, v^2 \in \mathbb{R}^n$ with $|u^1| = |u^2| = |v^1| = |v^2| = 1$, $(u^1, u^2) = (v^1, v^2) = 0$ so that

$$a_1 = \frac{1}{2} e^{i\psi} (u^1 + iu^2) \quad a_2 = \frac{1}{2} e^{i\phi} (v^1 - iv^2).$$

By Proposition 6, $L \in U(n)$; therefore $(a_1, a_2) = 0$. Using this and the conservation of the modules of $e_1 + e_2$, we find

$$(u^1, w^1) = (u^2, w^2) \quad (u^1, w^2) = -(u^2, w^1)$$

$$(u^1, w^j)^2 + (u^2, w^j)^2 = 1 \quad (j = 1, 2).$$

Then $\exists \gamma \in \mathbb{R}$ so that

$$a_2 = \frac{1}{2} e^{i\gamma} (u^1 - iu^2).$$
Now, if we set
\[ v^1 = \cos \left( \frac{\phi - \eta}{2} \right) u^1 - \sin \left( \frac{\phi - \eta}{2} \right) u^2 \]
\[ v^2 = \sin \left( \frac{\phi - \eta}{2} \right) u^1 + \cos \left( \frac{\phi - \eta}{2} \right) u^2 \]
we have
\[ a_1 = \frac{1}{2} e^{i\theta} (v^1 + iv^2), \quad a_2 = \frac{1}{2} e^{i\theta} (v^1 - iv^2), \]
and the assertion follows, q.e.d.

**Theorem 3.** For each \( L \in K \) there exist \( \theta \in \mathbb{R} \) and \( V \in O(n) \) so that
\[ L(z) = e^{i\theta} Vz. \]

**Proof.** By induction on \( n \). For \( n = 1, 2 \) the assertion is obvious; so, let \( n > 2 \). Then, by Proposition 7, we can assume \( L \) of the form
\[ L = \begin{pmatrix} I_2 & 0 \\ 0 & L' \end{pmatrix} \]
with \( L' \in K(n-2) \). By induction, up to an orthogonal automorphism, we can suppose that \( L' = e^{i\theta} I_{n-2} \), for some \( \theta \in \mathbb{R} \). But the conservation of the modules of \( z_0 = (1, 1, 1, 0, \ldots, 0) \) implies that \( e^{i\theta} = \pm 1 \), and the proof is complete, q.e.d.

Finally, for the standard determination of Aut(D) from K, see [3].

**References**


10. — RENDICONTI 1985, vol. LXXIX, fasc. 5