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The energy method for a class of hyperbolic equations

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Equazioni a derivate parziali. — *The energy method for a class of hyperbolic equations.* Nota (*) di ENRICO JANNELLI, presentata dal Corrisp. E. DE GIORGI.

RIASSUNTO. — In questa nota viene introdotto un nuovo metodo per ottenere espressioni esplicite dell'energia della soluzione dell'equazione iperbolica

(*) $$\left(\frac{\partial}{\partial t}\right)^m u + \sum_{\substack{|\nu|+j\leq m \\ j\leq m-1}} a_{\nu,j}(t) \left(\frac{\partial}{\partial x}\right)^{\nu} \left(\frac{\partial}{\partial t}\right)^{j} u = 0 \, .$$

Stimando opportunamente queste espressioni si ottengono nuovi risultati di buona positura negli spazi di Gevrey per l'equazione (*) quando questa è debolmente iperbolica

§ 1. INTRODUCTION

Let us consider the following Cauchy problem

(1) $$\begin{cases} Lu = \left(\dfrac{\partial}{\partial t}\right)^m u + \displaystyle\sum_{\substack{|\nu|+j\leq m \\ j\leq m-1}} a_{\nu,j}(t) \left(\dfrac{\partial}{\partial x}\right)^{\nu} \left(\dfrac{\partial}{\partial t}\right)^{j} u = 0 \qquad on \quad \mathbf{R}^n_x \times [0,T] \\ u(0,x) = \varphi_1(x) \\ \vdots \\ \left(\dfrac{\partial}{\partial t}\right)^{m-1} u(0,x) = \varphi_m(x) \, . \end{cases}$$

We assume that equation (1) is hyperbolic, in the sense that the principal symbol

(2) $$P_m(t\,;\tau\,,\xi) = \tau^m + \sum_{\substack{|\nu|+j=m \\ j\leq m-1}} a(t)\, \xi^{\nu}\, \tau^{j} \qquad (t\,;\xi) \in [0,T] \times \mathbf{R}^n_{\xi}$$

(*) Pervenuta all'Accademia il 5 ottobre 1985.

has only real roots $\tau_h(t\,;\,\xi)$. More precisely, if we define

$$\lambda = \inf_{\substack{|\xi|=1 \\ t\in[0;T] \\ i\neq j}} |\tau_i(t\,;\,\xi) - \tau_j(t\,;\,\xi)| \tag{3}$$

then we shall say that equation (1) is *strictly hyperbolic* if $\lambda > 0$ and *weakly hyperbolic* if $\lambda = 0$.

Throughout this work we shall suppose that $a_{\nu,j}(t)$ belong to $L^\infty([0\,,T])$ if $|\nu| + j \leq m - 1$; as regards the coefficients $a_{\nu,j}(t)$ of the principal part ($|\nu| + j = m$), we suppose for the moment that they belong to $C^1([0\,,T])$.

We want to obtain suitable « a priori » estimates on $u(t\,,x)$, depending on the coefficients $a_{\nu,j}(t)$ and on the initial data $\varphi_1(x)\ldots\varphi_m(x)$.

To this aim, we introduce the Fourier transform of $u(t\,,x)$ (with respect to x) and of $\varphi_h(x)$ $(h = 1\ldots m)$

$$v(t\,;\,\xi) = \int_{\mathbf{R}^n_x} u(t\,;\,x)\, e^{-i\langle\xi,x\rangle}\, dx \quad ; \tag{4}$$

$$\widehat{\varphi}_h(\xi) = \int_{\mathbf{R}^n_x} \varphi_h(x)\, e^{-i\langle\xi,x\rangle}\, dx \qquad h = 1\ldots m\,. \tag{5}$$

Then we define

$$V(t\,;\,\xi) = \begin{vmatrix} (i\,|\,\xi\,|)^{m-1}\, v \\ (i\,|\,\xi\,|)^{m-2}\, \dfrac{\partial}{\partial t}\, v \\ \vdots \\ i\,|\,\xi\,|\left(\dfrac{\partial}{\partial t}\right)^{m-2} v \\ \left(\dfrac{\partial}{\partial t}\right)^{m-1} v \end{vmatrix} \qquad m\text{ — vector ;} \tag{6}$$

$$H_{m-j}(t\,;\,\xi) = -\sum_{|\nu|=m-j} a_{\nu,j}(t)\, \frac{(i\,\xi)^\nu}{|\,\xi\,|^{m-j}}\ ; \tag{7}$$

$$A(t\,;\,\xi) = \begin{vmatrix} 0 & 1 & 0 & \ldots\ldots & 0 \\ 0 & 0 & 1 & 0\ \ldots & 0 \\ & & \ddots & & \\ 0 & \ldots & \ldots & 0 & 1 \\ H_m & H_{m-1} & \ldots & H_2 & H_1 \end{vmatrix} \qquad m\times m \text{ matrix ;} \tag{8}$$

$$\Phi(\xi) = \begin{vmatrix} \widehat{\varphi_1}(\xi) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \widehat{\varphi}_m(\xi) \end{vmatrix} \quad m\text{ — vector .} \tag{9}$$

Now problem (1) may be transformed into a family of first order ordinary differential systems, parametrized by ξ, as follows (here and in the following, if F is any function, vector or matrix, we shall denote $\frac{\partial}{\partial t}$ F by F′) :

$$\begin{cases} V'(t\,;\xi) = i\,|\,\xi\,|\,A(t\,;\xi)\,V(t\,;\xi) + B(t\,;\xi)\,V(t\,;\xi) \\ \qquad\qquad\qquad\qquad |\,\xi\,| \geq 1\ ;\ t \in [0\,,T] \\ V(0\,;\xi) = \Phi(\xi) \end{cases} \tag{10}$$

where $B(t\,;\xi)$ is a bounded $m \times m$ matrix (we remark that we confine ourselves to consider only $|\,\xi\,| \geq 1$).

In view of the Paley-Wiener theorem, we are interested in estimating the growth of $|\,V(t\,;\xi)\,|$ as $|\,\xi\,| \to +\infty$; now, as the eigenvalues of $A(t\,;\xi)$ are the roots $\tau_h\left(t\,;\frac{\xi}{|\,\xi\,|}\right)$ of the principal symbol $P_m\left(t\,;\frac{\xi}{|\,\xi\,|}\right)$, we see that, in the strictly hyperbolic case, the matrix $A(t\,;\xi)$ may be uniformly diagonalized; this is the classical method used to obtain estimates on $|\,V(t\,;\xi)\,|$ and, therefore, on $u(t\,,x)$ (see, for instance, [6]).

On the other hand, in the weakly hyperbolic case $A(t\,;\xi)$ is no longer diagonalizable; moreover, the roots $\tau_h\left(t\,;\frac{\xi}{|\,\xi\,|}\right)$ may not be regular in t.

Nevertheless, we shall perform an energy method which will be useful both in strictly and weakly hyperbolic cases, and will lead us to obtain explicit expressions for the energy of $V(t\,;\xi)$, which will be written only in terms of v, its derivatives, the dual variable ξ and the coefficients $a_{\nu,j}(t)$ of the principal part of equation (1) ($|\,\nu\,| + j = m$). This energy method, as we shall see later on, will allow us to get some new results of Gevrey well–posedness for equation (1) in the weakly hyperbolic case; these results will appear in [5].

§ 2. THE ENERGY METHOD

Let us consider problem (10). Let $Q(t\,;\xi)$ be any symmetric non negative real–valued $m \times m$ matrix of C^1-class in t.

We set (here and in the following $\langle\,,\rangle$ denotes the usual bracket in C^m)

$$E(t\,;\xi) = \langle\, Q(t\,;\xi)\,V(t\,;\xi)\ ,\ V(t\,;\xi)\,\rangle \tag{11}$$

and we call $E(t\,;\xi)$ an *energy* for $V(t\,;\xi)$. Obviously, we must carefully choose the matrix Q. In fact, deriving (11) with respect to t and taking into account (10), we easily get

$$\mathrm{E}' = \langle \mathrm{Q'V}, \mathrm{V} \rangle + i \, | \xi | \, \langle (\mathrm{QA} - \mathrm{A^*Q}) \, \mathrm{V}, \mathrm{V} \rangle + \langle (\mathrm{QB} + \mathrm{B^*Q}) \, \mathrm{V}, \mathrm{V} \rangle \tag{12}$$

where A^* is the transposed of A (which is real-valued) and B^* is the transposed conjugated of B.

Now, in view of the Paley-Wiener theorem, we require that $\mathrm{E}(t\,;\xi)$ has, if possible, the same behaviour as $\mathrm{E}(0\,;\xi)$ with respect to ξ when $|\xi| \to +\infty$; if we want to obtain this fact, it is clear from (12) that we must try to find a non-zero matrix $\mathrm{Q}(t\,;\xi)$ such that $\mathrm{QA} = \mathrm{A^*Q}$. This is just what we have obtained; more precisely, the following theorem holds:

THEOREM 1. *Given the hyperbolic operator* L *with principal symbol* P_m, *there exists a symmetric real-valued* $m \times m$ *matrix* $\mathrm{Q}(t\,;\xi)$ *having the following properties*:

(13) *the entries of* $\mathrm{Q}(t\,;\xi)$ *are polynomials of* m *variables, whose coefficients depend only on* m, *calculated in the* m*-tuple* $(\mathrm{H}_1 \ldots \mathrm{H}_m)$ *defined by* (7);

(14) $\mathrm{Q}(t\,;\xi)$ *is weakly positive defined if* L *is weakly hyperbolic, and it is strictly positive defined if* L *is strictly hyperbolic*;

(15) $\mathrm{Q}(t\,;\xi)\,\mathrm{A}(t\,;\xi) = \mathrm{A}^*(t\,;\xi)\,\mathrm{Q}(t\,;\xi)$ *for any* $(t\,;\xi)$.

Before we go on, let us consider a few examples.

EXAMPLE 1. (The case $m = 2$).

If L is a second order operator, then

$$\mathrm{A}(t\,;\xi) = \begin{pmatrix} 0 & 1 \\ \mathrm{H}_2 & \mathrm{H}_1 \end{pmatrix}. \tag{16}$$

In this case, the matrix Q given by Theorem 1 is

$$\mathrm{Q}(t\,;\xi) = \begin{pmatrix} 2\,\mathrm{H}_2 + \mathrm{H}_1^2 & -\mathrm{H}_1 \\ -\mathrm{H}_1 & 2 \end{pmatrix} \tag{17}$$

For instance, if we consider the equation

$$\mathrm{L}u = \frac{\partial^2}{\partial t^2} u - a(t) \frac{\partial^2}{\partial x^2} u - b(t) \frac{\partial^2}{\partial t \partial x} u = 0 \tag{18}$$

then we have (see (7))

$$H_1(t\,;\xi)=b(t)\frac{\xi}{|\xi|}\quad;\quad H_2(t\,;\xi)=a(t)\,. \tag{19}$$

Hence

$$Q(t\,;\xi)=\begin{vmatrix} 2\,a(t)+b^2(t) & b(t)\dfrac{\xi}{|\xi|} \\ b(t)\dfrac{\xi}{|\xi|} & 2 \end{vmatrix} \tag{20}$$

and

$$E(t\,;\xi)=\langle QV\,,V\rangle=(2\,a(t)+b^2(t))\,\xi^2\,|v|^2+2\,|v'|^2-\\ -2\,\xi\,b(t)\,\mathrm{Im}\,(v\,\overline{v}')\,. \tag{21}$$

We remark that, when $b(t)\equiv 0$, equation (18) reduces to a wave-type equation and the energy $E(t\,;\xi)$ reduces (save a multiplicative constant) to the well-known energy $E(t\,;\xi)=a(t)\,\xi^2\,|v|^2+|v'|^2$.

EXAMPLE 2 (the case $m=3$).

If L is a third order operator, then

$$A(t\,;\xi)=\begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ H_3 & H_2 & H_1 \end{vmatrix}\,. \tag{22}$$

In this case, the matrix Q given by Theorem 1 is

$$Q(t\,;\xi)=\begin{vmatrix} H_2^2-2\,H_1H_3 & H_1H_2+3\,H_3 & -H_2 \\ H_1H_2+3\,H_3 & 2\,H_1^2+2\,H_2 & -2\,H_1 \\ -H_2 & -2\,H_1 & 3 \end{vmatrix}\,. \tag{23}$$

For instance, if we consider the equation

$$Lu=\frac{\partial^3}{\partial t^3}u-a(t)\frac{\partial^3}{\partial^2 t\,\partial x}u-b(t)\frac{\partial^3}{\partial t\,\partial^2 x}u-c(t)\frac{\partial^3}{\partial x^3}u=0 \tag{24}$$

then we have

$$H_1(t\,;\xi)=a(t)\frac{\xi}{|\xi|}\;;\;H_2(t\,;\xi)=b(t);\;H_3(t\,;\xi)=c(t)\frac{\xi}{|\xi|}\,. \tag{25}$$

Hence

$$\begin{aligned}(26)\qquad E(t\,;\xi) &= \langle QV , V \rangle = \\ &= (b^2(t) - 2\,a(t)\,c(t))\,\xi^4\,|\,v\,|^2 + \\ &+ 2\,(a^2(t) + b(t))\,\xi^2\,|\,v'\,|^2 + 3\,|\,v''\,|^2 - \\ &- 2\,(a(t)\,b(t) + 3\,c(t))\,\xi^3\,\mathrm{Im}\,(v\bar{v}') + \\ &+ 2\,b(t)\,\xi^2\,\mathrm{Re}\,(v\bar{v}'') + 4\,a(t)\,\xi\,\mathrm{Im}\,(v'\,\bar{v}'')\,.\end{aligned}$$

It is clear that in the strictly hyperbolic case, when Q is strictly positive defined, any estimate regarding $E(t\,;\xi)$ obviously corresponds to an estimate on $V(t\,;\xi)$.

The situation is much more complicated in the weakly hyperbolic case, due to possible degenerations of $Q(t\,;\xi)$; however, we shall take advantage, in this case, of the fact that $Q(t\,;\xi)$ has the same regularity (in t) of the coefficients $a_{\nu,j}(t)$ of the principal part: in fact, if $a_{\nu,j}(t) \in C^{p,\alpha}([0\,,T])$, so it is for $H_k(t\,;\xi)$ $(k = 1 \ldots m)$ and then for $Q(t\,;\xi)$. In this sense, we see that the loss of regularity of the characteristic roots $\tau_h\left(t\,;\frac{\xi}{|\xi|}\right)$ is *not* an intrinsic difficulty.

When L is weakly hyperbolic, we shall consider some perturbed energies $E_\varepsilon(t\,;\xi) = \langle (Q + \Gamma_\varepsilon)\,V , V\rangle$, where Γ_ε is a suitable diagonal matrix with constant coefficient such that $Q + \Gamma_\varepsilon$ is strictly positive defined, and a key-point of our construction will be made up by the evaluation (as a function of ε) of the ratio $\langle Q'\,V , V\rangle / \langle (Q + \Gamma_\varepsilon)\,V , V\rangle$, to estimate which we shall use the regularity in t of $Q(t\,;\xi)$ together with some consequences of the following

LEMMA 1 ([5]). *Let* $f(t) : [0\,,T] \to \mathbf{R}$ *be a* $C^{1,\alpha}$ *function, with* $f(t) \geq 0$. *Then there exists a constant* $C = C(\|f\|_{1,\alpha})$ *such that*

$$(27)\qquad |\,f'(t)\,| \leq C\left[\frac{f(t)}{t\,(T-t)}\right]^{\alpha/1+\alpha}\qquad \forall t \in (0, T)\,.$$

On the other hand, if the coefficients $a_{\nu,j}(t)$ of the principal part of (1) are only hölder continuous, we shall consider some suitable perturbations $Q_\varepsilon(t\,;\xi)$ of $Q(t\,;\xi)$ in such a way that $Q + \Gamma_\varepsilon$ is regular in t and strictly positive defined; obviously, we shall estimate the error terms so introduced.

By means of the techniques exposed here, we get the following

THEOREM 2 ([5]). *Let us consider the operator* L *in* (1). *We suppose that*

(28) L *is weakly hyperbolic, i.e.* $\lambda = 0$, *where* λ *is defined by* (3).

Let k be the greatest multiplicity of the roots $\tau_h(t\,;\xi)$ of the principal symbol $P_m(t\,;\tau\,,\xi)$ defined by (2). *Obviously, k is an integer such that $2 \leq k \leq m$. Then*

i) *if the coefficients $a_{\nu,j}(t)$ of the principal part of* L $(|\nu| + j = m)$ *belong to* $C^{0,\alpha}([0\,,T])$, *the Cauchy problem* (1) *is well-posed in the Gevrey spaces* $\gamma^{(s)}_{loc}$ *for*

$$1 \leq s < 1 + \frac{\alpha}{(\alpha+1)(k-1)+1-\alpha}\;; \tag{29}$$

ii) *if the coefficients $a_{\nu,j}(t)$ of the principal part of* L $(|\nu| + j = m)$ *belong to* $C^{1,\alpha}([0\,,T])$, *the Cauchy problem* (1) *is well-posed in the Gevrey spaces* $\gamma^{(s)}_{loc}$ *for*

$$1 \leq s < 1 + \frac{1+\alpha}{2(k-1)}\,. \tag{30}$$

We recall that a function $f(x) : \mathbf{R}^n_x \to \mathbf{R}$ belongs to $\gamma^{(s)}_{loc}$ if for any K compact subset of $\mathbf{R}$ there exist Λ_k and A_k such that

$$|D^\alpha f(x)| \leq \Lambda_k A_k^{|\alpha|}(|\alpha|\,!)^s \qquad x \in K\,. \tag{31}$$

We point out that the results of Theorem 2, in the cases $a_{\nu,j}(t) \in C^{0,1}([0\,,T])$ or $a_{\nu,j}(t) \in C^{1,1}([0\,,T])$ $(|\nu| + j = m)$, have been already obtained, using quite different techniques, by T. Nishitani, who has considered in [8] the case of coefficients depending also on x.

Moreover, the results of Theorem 2 generalize some previous results regarding second order equations, which have been obtained in [2].

We remark that, if we suppose in Theorem 2 that $k = 1$ (i.e. strict hyperbolicity), then by (29) we get that problem (1) is well-posed in $\gamma^{(s)}_{loc}$ for $1 \leq s < 1/(1-\alpha)$; this result has already been proved in the general case of regularly hyperbolic systems with coefficients depending on x and t (see [3] and [4]), while the same result had been previously obtained in some particular cases at first in [1] and then in [7].

Let us also observe that we deduce by (30) that, if the coefficients $a_{\nu,j}(t)$ of the principal part belong to $C^{1,1}([0\,,T])$, then problem (1) is well-posed in $\gamma^{(s)}_{loc}$ for $1 \leq s < k/(k-1)$, and this result is not improvable, because, if $s > k/(k-1)$, problem (1) is not well-posed in $\gamma^{(s)}_{loc}$ even in the case of constant coefficients.

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