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Automorphism groups of the classical domains, I

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Geometria. — *Automorphism groups of the classical domains*, I. Nota (*) di MARCO ABATE, presentata dal Corrisp. E. VESENTINI.

RIASSUNTO. — In questa Nota viene dato un nuovo metodo elementare per determinare il gruppo degli automorfismi del primo dominio classico. In una Nota successiva, con procedimenti del tutto analoghi verranno determinati i gruppi degli automorfismi del terzo e del quarto dominio classico.

INTRODUCTION

In this and in the following Note we shall construct the groups of all holomorphic automorphisms of bounded symmetric domains of type I, III and IV in E. Cartan's classification. Since these domains are homogeneous, and a transitive group of holomorphic automorphisms is easily constructed, the main point of the proof boils down to the determination of the isotropy group K of a point 0 in the domain.

Our approach – which is different from and perhaps more direct than the classical ones developed by C.L. Siegel [5], H. Klingen [1, 2] and K. Morita [3] (1) – is based on a parametrisation of the orbit of the action of the isotropy group, whose parameters are r (= rank of the domain) non negative real numbers, called modules. Using the Harish-Chandra's realisation, it would be possible to give this parametrization in a unified form for all the bounded symmetric domains; however, in these notes we have preferred to limit ourselves to case-by-case definitions.

§ 1. The present note deals with domains of type I in E. Cartan's realisation. Thus let D be the domain

$$D = \{Z \in M_{p,q} \mid \| Z \| < 1\} = \{Z \in M_{p,q} \mid I_p - ZZ^* > 0\}$$

where $M_{p,q}$ is the set of $p \times q$ complex matrices with $p \leq q$, and $\| \ \|$ is the usual operator norm. For every $Z \in M_{p,q}$, ZZ^* is hermitian positive semidefinite; we call *modules* of Z the non-negative square roots $\lambda_1, \ldots, \lambda_p$ of the p eigenvalues of ZZ^*.

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(1) Numbers in brackets refer to the bibliography at the end of Note II.

It is well known that for every $Z \in M_{p,q}$ there exist unitary matrices $U \in U(p)$, $V \in U(q)$ such that

$$UZV = \operatorname{diag}(\lambda_1, \ldots, \lambda_p).$$

The mappings $Z \mapsto UZV$ with fixed $U \in U(p)$ and $V \in U(q)$ are elements of the isotropy group at the origin K, which will be called *unitary automorphisms.* Using unitary automorphisms, the following Lemma is trivially established:

LEMMA 1. *Let* $Z \in M_{p,q}$ *and let* $\lambda_1, \ldots, \lambda_p \geq 0$ *be the modules of* Z. *Then*

(i) $\| Z \| = \max \{\lambda_1, \ldots, \lambda_p\}$;

(ii) $\operatorname{Tr}(ZZ^*) = \sum_1^p {}_j \lambda_j^2$;

(iii) *The rank of* Z *is the number of non-vanishing modules.*

Clearly, the unitary automorphisms preserve the modules. As noted by C.L. Siegel [5], this is true for all automorphisms of D:

PROPOSITION 1. *Let* $L \in K$. *Then* L *is linear and preserves the modules.*

Proof. (Siegel). The linearity is a H. Cartan theorem (see e.g. [4]). Next, for every $Z \in M_{p,q}$ let

$$p_Z(\lambda) = \det(\lambda I_p - ZZ^*) = \prod_1^p {}_j (\lambda - \lambda_j^2)$$

where $\lambda_1, \ldots, \lambda_p$ are the modules of Z. In particular, $p_Z(1)$ is a polynomial of total degree $2p$ in $\lambda_1, \ldots, \lambda_p$ and we have $p_Z(1) = 0$ iff $Z \in \partial D$. Since L is a linear mapping, $p_{L(Z)}(1)$ is again a polynomial of total degree $2p$ in $\lambda_1, \ldots, \lambda_p$; moreover $(L(\partial D) = \partial D)$ $p_{L(Z)}(1) = 0$ iff $\lambda_j = \pm 1$ for some $j = 1, \ldots, p$. Therefore $(L(0) = 0)$ $p_{L(Z)}(1) = p_Z(1)$ and, by the linearity of L, the assertion follows, q.e.d.

COROLLARY 1. *Let* $L \in K$. *Then*

(i) $\forall Z \in M_{p,q}$ *rnk* $L(Z) =$ *rnk* Z;

(ii) L *is unitary for the hermitian product* $\operatorname{Tr}(ZW^*)$.

For $u \in \mathbf{C}^p$, $v \in \mathbf{C}^q$, the (μ, ν) entry of the $p \times q$ matrix $u \otimes v$ is $(u \otimes v)_{\mu\nu} = u_\mu v_\nu$ $(\mu = 1, \ldots, p,\ \nu = 1, \ldots, q)$. The following Lemma is easily established:

LEMMA 2. *Let* $u \in \mathbf{C}^p$, $v \in \mathbf{C}^q$, $u, v \neq 0$. *Then*

(i) $u \otimes v$ *has rank* 1 *and modules* $\{|u| \cdot |v|, 0, \ldots, 0\}$;

(ii) $\forall Z \in M_{p,q}$ *with rank* $1\ \exists u \in \mathbf{C}^p$, $v \in \mathbf{C}^q$ *and* $\alpha > 0$ *with* $|u| = |v| = 1$ *such that* $Z = \alpha\, u \otimes v$.

Now we can prove the key result:

PROPOSITION 2. *Let* $L \in K$. *The* $\exists U \in U(p)$ *and* $V \in U(q)$ *such that for every diagonal matrix* Z *we have*

$$L(Z) = UZV .$$

Proof. Let E_{ij} be the usual elementary matrices, and let $A_{ij} = L(E_{ij})$ for $i = 1, \ldots\ p$ and $j = 1, \ldots, q$. By Proposition 1, each A_{jj} has modules $\{1, 0, \ldots, 0\}$; then (Lemma 2) $\exists u^j \in \mathbf{C}^p$, $v_j \in \mathbf{C}^q$ with $|u^j| = |v_j| = 1$ and $A_{jj} = u^j \otimes v_j$. Since $\mathrm{Tr}\,(E_{hh} E^*_{kk}) = 0\ \forall h \neq k$, by Corollary 1 (ii) we have

$$\forall h \neq k \qquad 0 = \mathrm{Tr}\,(A_{hh} A^*_{kk}) = (u^h, u^k)(v_h, v_k) .$$

We will prove that $(u^h, u^k) = (v_h, v_k) = 0\ \forall h \neq k$. Suppose for instance that $(v_h, v_k) \neq 0$ – and therefore that $(u^h, u^k) = 0$ – and let $Z = E_{hh} + E_{kk}$. The modules of Z are $\{1, 1, 0, \ldots, 0\}$; but, since $(u^h, u^k) = 0$, the modules different from zero of L (Z) are $1 \pm |(v_h, v_k)|$, and this is impossible by Proposition 1.

Then $\{u^j\}$ is a orthonormal basis of $\mathbf{C}^p$, and $\{v_1, \ldots, v_p\}$ is extensible to an orthonormal basis $\{v_j\}$ of $\mathbf{C}^q$. So, letting $U = (u^\mu_\nu)$ and $V = (v^\nu_\mu)$, then $U \in U(p)$, $V \in U(q)$ and $A_{jj} = UE_{jj}V$ for every $j = 1, \ldots, p$, q.e.d.

THEOREM 1. *For each* $L \in K$ *there exist* $U \in U(p)$ *and* $V \in U(q)$ *such that*

$$L(Z) = UZV$$

or (*only if* $p = q$)

$$L(Z) = U\,{}^tZ V .$$

Proof. (i) $p = q = n$. By Proposition 2, we can suppose that

$$A_{jj} = E_{jj} \qquad \forall j = 1, \ldots, n . \tag{1}$$

Since L (Corollary 2) is unitary for the hermitian product $\mathrm{Tr}\,(ZW^*)$, we have also

$$A_{ij,hh} = 0 \qquad \forall 1 \leq i \neq j \leq n \qquad \forall h = 1, \ldots, n . \tag{1'}$$

Let $u, v \in \mathbf{C}^n$ with $|u| = |v| = 1$. By Lemma 2 and Corollary 1, there exist $\tilde{u}, \tilde{v} \in \mathbf{C}^n$ with $|\tilde{u}| = |\tilde{v}| = 1$ such that $L(u \otimes v) = \tilde{u} \otimes \tilde{v}$. If we write this componentwise using (1) and (1′), we obtain

$$(2) \qquad \begin{cases} u_h v_h = \tilde{u}_h \tilde{v}_h & (h = 1, \ldots, n), \\ \sum_1^n{}_{\mu\nu} u_\mu v_\nu A_{\mu\nu,hk} = \tilde{u}_h \tilde{v}_k & (1 \leq h \neq k \leq n). \end{cases}$$

Let us fix now two distinct indices α, β. By Corollary 1, $rnk\, A_{\alpha\beta} = rnk\, A_{\beta\alpha} = 1$, so that

$$(3) \qquad A_{\alpha\beta,\alpha\beta}\, A_{\alpha\beta,\beta\alpha} = A_{\beta\alpha,\alpha\beta}\, A_{\beta\alpha,\beta\alpha} = 0 .$$

Now we choose u and v so that $u_\gamma, v_\gamma \neq 0$ iff $\gamma = \alpha, \beta$. Then (2) yields $\tilde{u}_\alpha, \tilde{v}_\alpha, \tilde{u}_\beta, \tilde{v}_\beta \neq 0$ and

$$(4) \qquad \begin{cases} \tilde{u}_\alpha\, \tilde{v}_\beta = u_\alpha\, v_\beta\, A_{\alpha\beta,\alpha\beta} + u_\beta\, v_\alpha\, A_{\beta\alpha,\alpha\beta} \\ \tilde{u}_\beta\, \tilde{v}_\alpha = u_\alpha\, v_\beta\, A_{\alpha\beta,\beta\alpha} + u_\beta\, v_\alpha\, A_{\beta\alpha,\beta\alpha} \end{cases}$$

so that

$$(5') \qquad A_{\alpha\beta,\beta\alpha} = A_{\beta\alpha,\alpha\beta} = 0 \quad \text{and} \quad A_{\alpha\beta,\alpha\beta}, A_{\beta\alpha,\beta\alpha} \neq 0$$

or

$$(5'') \qquad A_{\alpha\beta,\alpha\beta} = A_{\beta\alpha,\beta\alpha} = 0 \quad \text{and} \quad A_{\alpha\beta,\beta\alpha}, A_{\beta\alpha,\alpha\beta} \neq 0 .$$

Let $\gamma \neq \alpha, \beta$; using once again the fact that $rnk\, A_{\alpha\beta} = rnk\, A_{\beta\alpha} = 1$, we obtain

$$(6) \qquad \begin{cases} \tilde{u}_\alpha \tilde{v}_\gamma = u_\alpha v_\beta\, A_{\alpha\beta,\alpha\gamma} \\ \tilde{u}_\beta \tilde{v}_\gamma = u_\beta v_\alpha\, A_{\beta\alpha,\beta\gamma} \end{cases} \quad \text{and} \quad \begin{cases} \tilde{u}_\gamma \tilde{v}_\alpha = u_\beta v_\alpha\, A_{\beta\alpha,\gamma\alpha} \\ \tilde{u}_\gamma \tilde{v}_\beta = u_\alpha v_\beta\, A_{\alpha\beta,\gamma\beta} \end{cases}$$

if we have (5′), or the analogous (with $A_{\alpha\beta}$ and $A_{\beta\alpha}$ interchanged) if we have (5″). Now, if we take $u = v$, by (2) we have

$$(|\tilde{u}_\alpha|\,|\tilde{v}_\alpha| + |\tilde{u}_\beta|\,|\tilde{v}_\beta|)^2 = (|u_\alpha|^2 + |u_\beta|^2)^2 = 1 \geq$$
$$\geq (|\tilde{u}_\alpha|^2 + |\tilde{u}_\beta|^2)(|\tilde{v}_\alpha|^2 + |\tilde{v}_\beta|^2) .$$

Hence we find a $\lambda > 0$ such that $|\tilde{u}_\alpha| = |u_\alpha|/\lambda$ and $|\tilde{u}_\beta| = |u_\beta|/\lambda$. Then (6) implies that

$$|u_\beta|\,|A_{\alpha\beta,\alpha\gamma}| = |u_\alpha|\,|A_{\beta\alpha,\beta\gamma}|$$

and this is possible iff $A_{\alpha\beta,\alpha\gamma} = A_{\beta\alpha,\beta\gamma} = 0\ \forall \gamma \neq \alpha, \beta$. Therefore (by (6)) $\tilde{v}_\gamma = 0$ for every u_α, u_β, that is $A_{\alpha\beta,\delta\gamma} = A_{\beta\alpha,\delta\gamma} = 0\ \forall \delta = 1, \ldots, n$. By iteration of this argument $\forall \alpha, \beta, \gamma$, we are left with only two possibilities:

$$(7') \qquad \begin{cases} A_{\alpha\beta} = \lambda_{\alpha\beta}\, E_{\alpha\beta} \\ A_{\beta\alpha} = \lambda_{\beta\alpha}\, E_{\beta\alpha} \end{cases}$$

from (5′), or

$$(7'') \qquad \begin{cases} A_{\alpha\beta} = \lambda_{\alpha\beta}\, E_{\beta\alpha} \\ A_{\beta\alpha} = \lambda_{\beta\alpha}\, E_{\alpha\beta} \end{cases}$$

from (5″), with $\lambda_{\alpha\beta}\,\lambda_{\beta\alpha} = 1$ and $|\lambda_{\alpha\beta}| = |\lambda_{\beta\alpha}| = 1$.

If $n = 2$, the argument is complete. If $n > 2$, choose three distinct indices α, β, γ. Then, checking the conservation of the modules of matrices with only the entries with both indices in $\{\alpha, \beta, \gamma\}$ different from zero, we obtain:

- that it is impossible to have (7′) for α, β and (7″) for β, γ;
- that $\lambda_{\alpha\beta}\,\lambda_{\beta\gamma}\,\lambda_{\gamma\alpha} = 1$.

Since this is true for all α, β, γ, we can write $\lambda_{\alpha\beta} = \lambda_\alpha/\lambda_\beta$ with $|\lambda_\alpha| = 1$ $\forall \alpha, \beta = 1, \ldots, n$; therefore L takes the form

$$L(Z) = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)\, Z \operatorname{diag}(1/\lambda_1, \ldots, 1/\lambda_n)$$

or

$$L(Z) = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)\, {}^tZ \operatorname{diag}(1/\lambda_1, \ldots, 1/\lambda_n)$$

and the assertion follows in this case.

(ii) $p \le q$. By induction on $\max\{p, q\}$. For $p = q = 1$ the assertion is obvious; so, let $\max\{p, q\} > 1$. By Proposition 2, we can assume that $A_{jj} = E_{jj}$ $\forall j = 1, \ldots, p$; then, proceeding as in (i), we see that L maps the subspace of the matrices of the form $(Z_1, 0)$, with $Z_1 \in M_{p,p}$, into itself. By orthogonality, L maps also the subspace of the matrices of the form $(0, Z_2)$ with $Z_2 \in M_{p,q-p}$ into itself. By (i) and the inductive hypothesis, up to a unitary automorphism, we can suppose that L is of the form

$$L(Z_1, Z_2) = (Z_1, UZ_2) \text{ (or, if } q = 2p,\ L(Z_1, Z_2) = (Z_1, U{}^tZ_2))$$

or

$$L(Z_1, Z_2) = ({}^tZ_1, UZ_2) \text{ (or, if } q = 2p,\ L(Z_1, Z_2) = ({}^tZ_1, U{}^tZ_2))$$

for some $U \in U(p)$. Now, checking the conservation of the modules of the matrix $Z = (Z_1, Z_2)$ given by

$$Z_1 = E_{hh} \qquad Z_2 = \begin{cases} E_{hh} & \text{if } h \le q - p \\ E_{h,q-p} & \text{if } h > q - p \end{cases}$$

for all $h = 1, \ldots, p$, we obtain that U is a diagonal matrix.

Next, checking the conservation of the modules of the matrix $Z = (Z_1, Z_2)$ given by

$$Z_1 = E_{hh} + E_{kk} + E_{hk} + E_{kh}$$

$$Z_2 = \begin{cases} E_{hh} + E_{kk} + E_{hk} + E_{kh} & \text{if } h, k \le q - p \\ E_{hh} + E_{k,q-p} + E_{h,q-p} + E_{kh} & \text{if } h \le q - p < k \\ 2\,(E_{h,q-p} + E_{k,q-p}) & \text{if } q - p < h, k \end{cases}$$

for all $h, k = 1, \ldots, p$, $h \neq k$, we obtain that $U = e^{i\theta} I_p$ for some $\theta \in \mathbf{R}$. Finally, checking the conservation of the modules of other suitable matrices, we can also conclude that there are no transpositions. Hence L is a unitary automorphism, q.e.d.

For the standard determination of Aut (D) from K, see for instance [1] and [2].