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Soluble Groups with Many Černikov Quotients

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RIASSUNTO. — Si studiano i gruppi risolubili non di Černikov a quozienti propri di Černikov. Nel caso periodico tali gruppi sono tutti e soli i prodotti semidiretti $H \times N$ con $N$ gruppo abeliano elementare infinito e $H$ gruppo irriducibile di automorfismi di $N$ che sia infinito e di Černikov. Nel caso non periodico invece si riconduce tale studio a quello dei moduli a quozienti propri artiniani su un gruppo risolubile finito, e si fornisce una caratterizzazione di tali moduli.

§ 1. INTRODUCTION

If $X$ is a class of groups, a group $G$ is said to be just-non-$X$ if it is not in $X$ but all its proper quotients are $X$-groups.

Soluble just-non-polycyclic groups are studied in recent papers of Groves [2] and Robinson and Wilson [7], while just-infinite groups are considered in earlier papers of McCarthy [3] and Wilson [8].

Our aim here is to study soluble just-non-Černikov groups. Torsion soluble just-non-Černikov groups are described in a satisfactory way by Theorem A: such groups are precisely the semi-direct products $H \times M$, where $M$ is an infinite abelian group of prime exponent and $H$ is an irreducible infinite Černikov group of automorphisms of $M$. Theorem B reduces the study of non-torsion soluble just-non-Černikov groups to that of just-non-artinian modules over a finite soluble group, while Theorem C gives a description of such modules.

Finally in § 4 we construct many examples of just-non-Černikov groups, and we embed every Černikov group in a just-non-Černikov group.

Most of our notation is standard. In particular we refer to [5].

§ 2. TORSION SOLUBLE JUST-NON-ČERNIKOV GROUPS

We shall prove:

**THEOREM A.** A torsion soluble group $G$ is just-non-Černikov if and only if $G = H \times M$ where $M$ is an infinite abelian group of prime exponent and $H$ is

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an infinite Černikov group which acts faithfully as an irreducible group of automorphisms of $M$.

**Proof.** Let $G$ be just-non-Černikov; then the last non-trivial term $A$ of the derived series of $G$ is obviously a $p$-group (for some prime $p$). Denote by $D$ the divisible part of $A$ and by $S$ the socle of $D$. If $S \neq 1$, $D/S$ is in $\text{Min}$ and so $D$ is in $\text{Min}$, which is impossible. Therefore $D = 1$ and $A$ is reduced. It follows that $A^p = 1$, since otherwise $A/A^p$ is finite and $A$ is finite, a contradiction; thus $A$ is an infinite elementary abelian $p$-group. For each $a \in A \setminus \{1\}$, the subgroup $C_G(a)$ has infinite index in $G$, and so we have $B \leq C_G(a)$ where $B/A$ is the finite residual of $G/A$. Since $A/a^G$ is finite and $B/A$ is divisible, we obtain

$$[A, B] \leq a^G,$$

and so $1 \neq [A, B] \leq \bigcap_{a \in A \setminus \{1\}} a^G = M$.

Obviously $M$ is the unique minimal normal subgroup of $G$. Write $Q = G/M$. Then $M$ is a simple $Q$-module and, if $R$ denotes the finite residual of $Q$, we have $H^0(R, M) = 0$ since $|Q : R| < \infty$. By a theorem of Robinson ([6] Theorem B) it follows that $H^2(Q, M) = 0$ and so $G = H \times M$ for some $H \leq G$. If $C = C_G(M)$, we have $C = HM \cap C = M (H \cap C)$, and $H \cap C$ is normal in $HM = G$. Since $H \simeq G/M$ is Černikov it follows that $H \cap C = 1$ and $C_G(M) = M$. Therefore $H$ acts faithfully as an irreducible group of automorphisms of $M$.

Conversely, if $G = H \times M$ has this structure, $G$ is not a Černikov group, since $M$ is infinite. If $N$ is a non-trivial normal subgroup of $G$, we have $M$ isomorphic to $M$ and so $M \leq N$ and $G/N$ is a Černikov group.

**Remark.** If $M$ is an infinite abelian group of prime exponent $p$ and $H$ is an irreducible infinite Černikov group of automorphisms of $M$, then the finite residual $R$ of $H$ is a $p'$-group.

**Proof.** By contradiction suppose that the $p$-component $K$ of $R$ is non-trivial, and denote by $K_n$ the subgroup of the elements of order $\leq p^n$ of $K$; then $K = \bigcup_{n \in N} K_n$ and every $K_n$ is finite. By Theorem A the group $G = H \times M$ is just-non-Černikov and every $L_n = K_n \times M$ is nilpotent (see [5] Part 2, Lemma 6.34). Therefore $1 \neq Z(L_n) \triangleleft G$, so that $G/Z(L_n)$ is a Černikov group and $L_n/Z(L_n)$ is finite since it has finite exponent. Then $L'_n$ is a finite normal subgroup of $G$, so that $L'_n = 1$ and each $L_n$ is abelian. Therefore $K$ acts trivially on $M$ and $K = 1$.

In § 4 we will construct a torsion soluble just-non-Černikov group whose unique minimal normal subgroup is not a Hall subgroup.

**Example.** Let $K$ be the algebraic closure of the field $GF(p)$; then the multiplicative group $K^*$ of $K$ is a direct product of groups of Prüfer type, one
for each prime other than \( p \). Let \( H \) be an infinite subgroup of \( K^* \) satisfying the minimal condition on subgroups, and let \( A \) be the additive group of the subfield of \( K \) generated by \( H \); then \( A \) is an infinite elementary abelian \( p \)-group on which \( H \) acts faithfully and irreducibly by multiplication. The split extension \( G = H \ltimes A \) is just-non-Černikov.

This example is essentially due to Carin.

§ 3. NON-TORSION SOLUBLE JUST-NON-ČERNIKOV GROUPS

The analysis of the Fitting subgroup and of the Fitting quotient is essential in describing non-torsion soluble just-non-Černikov groups. In fact we have:

**Theorem B.** (1) Let \( G \) be a non-torsion soluble just-non-Černikov group and let \( F \) be the Fitting subgroup of \( G \). Then \( Q = G/F \) is a finite group and \( F \) is a faithful just-non-artinian \( Q \)-module.

(2) Conversely, if \( Q \) is a finite soluble group and \( F \) is a faithful just-non-artinian \( Q \)-module, every extension of \( F \) by \( Q \) is a non-torsion soluble just-non-Černikov group with Fitting subgroup \( F \).

**Proof.** (1) Let \( A \) be the last non-trivial term of the derived series of \( G \), and let \( M \) be the intersection of all non-trivial \( G \)-invariant subgroups of \( A \). If \( M \neq 1 \), \( M \) is a minimal normal subgroup of \( G \) and \( G/C_G(M) \) is an irreducible locally finite group of automorphisms of \( M \), so that \( M \) is torsion (see [5] Part 1, Lemma 5.26), which is impossible. Therefore \( M = 1 \). Denote by \( B/A \) the finite residual of the Černikov group \( G/A \). If \( H \) is a non-trivial \( G \)-invariant subgroup of \( A \), the torsion group \( B/C_B(A/H) \) is finite (see [5] Part 1, Theorem 3.29.2) and so \( C_B(A/H) = B \) since \( B/A \) is divisible; it follows that \( [A, B] \leq M = 1 \), and so \( A \leq Z(B) \) and \( B \) is nilpotent. Therefore \( G \) is nilpotent-by-finite; it follows that \( F \) is a torsion-free nilpotent group and \( Q \) is finite.

The group \( F/Z(F) \) is Černikov, so that \( F' = 1 \) and \( F \) is abelian. Then \( C_G(F) = F \) and \( F \) is a faithful \( Q \)-module. If \( K \) is a non-trivial \( Q \)-submodule of \( F \), we have \( K \triangleleft G \) and obviously \( F/K \) is an Černikov \( Q \)-module. Finally the \( Q \)-module \( F \) is non-artinian since \( G \) has no minimal normal subgroups.

(2) Let \( G \) be an extension of \( F \) by \( Q \). Then \( C_G(F) = F \) and so, if \( N \) is a non-trivial normal subgroup of \( G \), we have that \( N \cap F \) is a non-trivial \( Q \)-submodule of \( F \) and \( F/N \cap F \) is an Černikov \( Q \)-module. Thus \( G/N \cap F \) is in Min-\( n \), and so \( F/N \cap F \) is in Min (see [5] Part 1, Theorem 5.21); it follows that \( G/N \) is Černikov and \( G \) is a just-non-Černikov group. Since \( Q \) is finite, we obtain that \( F \) is torsion-free by Dietzmann's Lemma.

By (1) the Fitting subgroup \( F(G) \) of \( G \) is abelian, so that \( F(G) = C_G(F) = F \) and \( F(G) = F \).

From Theorem B it follows that it is sufficient to study faithful just-non-artinian modules over finite soluble groups. We recall that a \( Q \)-module \( A \) is said to have finite \( Q \)-rank \( r \) if every finitely generated \( Q \)-submodule of \( A \) can be...
generated by $s \leq r$ elements and $r$ is the least positive integer with this property, while $A$ is said to have finite total $Q$-rank if the sum of the $Q$-rank of $A/T$ and of the $Q$-ranks of the $A_p$ (for all primes $p$) is finite (where $T$ is the torsion subgroup of $A$ and $A_p$ is the $p$-component of $A$). Here $\mathcal{F}$ denotes the field of rational numbers.

**Theorem C.** Let $Q$ be a finite soluble group and let $A$ be a faithful $Q$-module. Then $A$ is a just-non-artinian $Q$-module if and only if $A$ is $\mathbb{Z}$-torsion-free, $A \otimes_{\mathbb{Z}} \mathcal{F}$ is a simple $Q$-module and the $Q$-sections of $A$ have finite total $Q$-rank.

**Proof.** Let $A$ be just-non-artinian. Since $Q$ is finite, the abelian group $A$ is obviously torsion-free. Let $M$ be a non-trivial $Q$-submodule of $A \otimes_{\mathbb{Z}} \mathcal{F}$; then $M^* = \{a \in A | a \otimes 1 \in M\}$ is a non-trivial $Q$-submodule of $A$ and there is an exact sequence

$$M^* \otimes_{\mathbb{Z}} \mathcal{F} \rightarrow A \otimes_{\mathbb{Z}} \mathcal{F} \rightarrow A/M^* \otimes_{\mathbb{Z}} \mathcal{F}$$

(see [1] Theorem 60.6). The abelian group $A/M^*$ is in Min, since it is an artinian $Q$-module (see [5] Part 1, Theorem 5.21), and so

$$(A \otimes_{\mathbb{Z}} \mathcal{F})/\text{Im} x \simeq A/M^* \otimes_{\mathbb{Z}} \mathcal{F} = 0 \text{ and } M = A \otimes_{\mathbb{Z}} \mathcal{F} \text{ since } \text{Im} x \leq M.$$ 

If $x \in A \setminus \{0\}$, the $Q$-submodule $xQ$ of $A$ generated by $x$ is a free abelian group of finite rank and, as above, the abelian group $A/xQ$ is in Min, so that the torsion-free abelian group $A$ has finite (total) rank. It follows that $A$ has finite (total) $Q$-rank since each finitely generated $Q$-submodule is finitely generated as a subgroup of $A$. Let $U/V$ be a $Q$-section of $A$ with $V \neq 0$. Then $A/V$ is an abelian group in Min and so $U/V$ has finite total rank as an abelian group, and hence also finite total $Q$-rank.

Conversely let $B$ be a non-trivial $Q$-submodule of $A$. We have $0 \neq B \simeq B \otimes_{\mathbb{Z}} \mathcal{F} \leq B \otimes_{\mathbb{Z}} \mathcal{F}$ since $B$ is torsion free, and so $B \otimes_{\mathbb{Z}} \mathcal{F} \neq 0$. There is an exact sequence

$$B \otimes_{\mathbb{Z}} \mathcal{F} \rightarrow A \otimes_{\mathbb{Z}} \mathcal{F} \rightarrow A/B \otimes_{\mathbb{Z}} \mathcal{F}$$

and so $A/B \otimes_{\mathbb{Z}} \mathcal{F} = 0$ since $\text{Im} x$ is a non-trivial $Q$-submodule of the simple $Q$-module $A \otimes_{\mathbb{Z}} \mathcal{F}$. If $T/B$ is the torsion subgroup of $A/B$, it follows that $A/T \otimes_{\mathbb{Z}} \mathcal{F} = 0$, and so $A/T = 0$ since it is torsion-free. Therefore $A/B$ is a torsion group and it has finitely many non-trivial primary components since it has finite total $Q$-rank. Let $H = \langle x_1, \ldots, x_n \rangle$ be a finitely generated subgroup of $A$; since $H^Q = x_1 Q \cdot \ldots \cdot x_n Q$ is a finitely generated $Q$-submodule of $A$, there exist $y_1, \ldots, y_r$ in $A$ such that $H^Q = y_1 Q \cdot \ldots \cdot y_r Q$ where $r$ is the $Q$-rank of $A$. If $|Q| = q$, then $H^Q$ is a free abelian group of rank $\leq r q$. Therefore $A$ has finite rank as an abelian group, and so the abelian group $A/B$ is in Min. Since $A$ is torsion-free, it follows that $A$ is a just-non-artinian $Q$-module. \[\blacksquare\]
REMARK 1. Every soluble non-torsion just-non-Černikov group $G$ is a residually finite minimax group.

Proof. The Fitting subgroup $F$ of $G$ is torsion-free abelian and $|G:F| < \infty$, so that, if $x \in F \setminus \{1\}$, $x^G$ is a free abelian group of finite rank and $F/x^G$ is in Min; then $F$ is a minimax group and it is residually finite (see [4] Lemma 2.21). It follows that $G$ is a residually finite minimax group.

REMARK 2. Let $G$ be a group with $Z(G) \neq 1$. Then $G$ is just-non-Černikov if and only if it is isomorphic with a non-trivial subgroup of $\mathcal{A}_\pi$ (the additive group of all rational numbers whose denominators are $\pi$-numbers) for some finite set $\pi$ of prime numbers.

Proof. Suppose $G$ just-non-Černikov. Then $G/Z(G)$ is a Černikov group and so $G'$ is Černikov (see [5], Part 1, Theorem 4.23). Therefore $G' = 1$ and $G$ is a torsion-free abelian group of rank 1, and so it is isomorphic with a subgroup $G^*$ of $\mathcal{A} (+)$. If $m/n \in G^* \setminus \{0\}$, the set $\pi$ of the prime numbers which either divide $n$ or are orders of elements of $G^*/\langle m/n \rangle$ is finite, since $G^*/\langle m/n \rangle$ is in Min. It is easily proved that $G^* \leq \mathcal{A}_{\pi}$.

EXAMPLE. If $\pi$ is a finite set of prime numbers and $\alpha$ is the automorphism $x \mapsto -x$ of $\mathcal{A}_{\pi}$, the group $\langle \alpha \rangle \times \mathcal{A}_{\pi}$ is a soluble non-torsion just-non-Černikov group which is non-abelian.

§ 4. WREATH PRODUCTS AND JUST-NON-ČERNIKOV GROUPS

In this section we give methods to construct many examples of just-non-Černikov groups.

4.1. Let $H$ be a non-abelian just-non-Černikov group, and let $K$ be a finite group. Then $G = H \wr K$ is just-non-Černikov.

Proof. If $N$ is a non-trivial normal subgroup of $G$, the intersection of $N$ with each component $H^i$ of the base group $B$ is non-trivial, since $Z(H) = 1$. Therefore every $H^i(N \cap B)/N \cap B$ is a Černikov group and so $B/N \cap B$ is Černikov. It follows that $G/N$ is a Černikov group since $|G:B| < \infty$.

4.2. Let $H$ be a non-abelian simple group, and let $K$ be an infinite Černikov group. Then $G = H \wr K$ is just-non-Černikov.

Proof. Each non-trivial normal subgroup $N$ of $G$ contains the base group of $G$, and so $G/N$ is Černikov. Moreover $G$ is not a Černikov group since $K$ is infinite.

EXAMPLE. Let $H$ be a soluble torsion just-non-Černikov group whose unique minimal normal subgroup has exponent $p$, and let $K$ be a finite soluble group whose order is divisible by $p$. Then $G = H \wr K$ is a soluble torsion
just-non-Černikov group whose unique minimal normal subgroup is not a Hall subgroup.

By 4.2 it follows that every Černikov group is a subgroup and a quotient of a just-non-Černikov group.

4.3. Let $G = H \wr K$ be a just-non-Černikov group with $H$ non-simple. Then $H$ is just-non-Černikov and $K$ is finite.

Proof. Let $B$ be the base group and denote by $L$ a proper non-trivial normal subgroup of $H$; then $L \triangleleft B$ and $L^G = L^K$. Since $G/L^G$ is Černikov, also $B/L^K \cong \text{Dr } H^k/L^k$ is Černikov, and from $H^k/L^k \neq 1$ it follows that $K$ is finite. We have $H \bigcap L^K = L$ and so $H/L \cong HL^K/L^K \leq G/L^K$ is a Černikov group. Finally $H$ is not a Černikov group since $G$ is not Černikov.

Our last result is about the cardinality of soluble just-non-Černikov groups

4.4. A soluble just-non-Černikov group $G$ is countable.

Proof. If $G$ is non-torsion the result follows from Remark 1. Suppose that $G$ is a torsion group and so $G = H \times M$ as in Theorem A. Then $H$ is countable since it is Černikov and $M$ is countable because it is an irreducible (and so cyclic) $\mathbb{Z}_pH$-module.

REFERENCES